

# The pressure of hot QCD

**York Schröder**

Grupo de Física de Altas Energías, Departamento de Ciencias Básicas,  
Universidad del Bío-Bío, Casilla 447, Chillán, Chile

E-mail: [yschroeder@ubiobio.cl](mailto:yschroeder@ubiobio.cl)

**Abstract.** When heated and/or compressed, strongly interacting matter exhibits a rich phase structure. In this talk, I will concentrate on its behavior under variations of the temperature, which is most relevant for phenomenological applications such as in cosmology, heavy-ion collisions, and astrophysics.

In particular, effective field theory methods can be used to combine lattice and continuum calculations, in order to obtain high-precision results for the relevant thermodynamic quantities such as the QCD pressure and equation of state. I will discuss the current status of this systematic approach to QCD thermodynamics, and point out the remaining (technical) problems.

## 1. Introduction

Quantum Chromodynamics (QCD) is the theory that describes the strong interactions in nature, which binds quarks into hadrons, such as protons or pions. Due to its nature of being a non-Abelian gauge theory, its equations are however extremely difficult to solve. While a large body of experimental information on details of the hadron spectrum exists [1], first-principle theoretical predictions of these properties are rather sparse.

In the quest to check the theory of QCD versus its observed manifestations, we can identify three main types of approaches. First, one might just solve its equations. This is the realm of lattice QCD, where large-scale computer simulations are used to numerically approximate observables such as the hadron mass spectrum. As a result of worldwide efforts, multi-million-dollar investments, teraflop speeds, hardware development as well as algorithmic advances, the punchline of this line of investigation is that, starting from the (fractionally charged, partonic quark- and gluon-) QCD degrees of freedom, one indeed postdicts exactly the observed (integer-charged) low-lying hadron spectrum (see, e.g. [2]).

A second line of approaches centers around considering phenomenological models that are in some sense 'close' to QCD, but mathematically simpler to treat. Typical modifications include a change of space-time dimension, different symmetry groups, or modified particle content of the theory. These models are most successful when applied to specific narrow problems, but mostly lack generality.

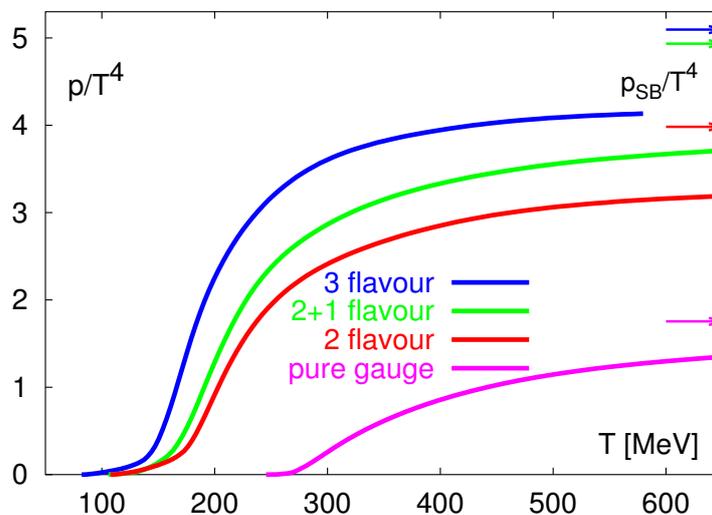
Third, one might consider 'extreme' circumstances in which the QCD equations simplify, or at least allow for an application of systematically improvable approximation methods such as weak-coupling expansions. Such 'extreme' conditions can be realized e.g. when some parameters assume extreme values, such as the very high energies realized at particle colliders such as currently at the Large Hadron Collider at CERN. Indeed, the higher the collision energy, the



cleaner the underlying interaction vertices can be seen, such that from the jet structure of the events one can map out precisely the underlying details of the theory. Of course, nowadays QCD contributions are not signals, but constitute (large and therefore important) background in searches for new particles.

Another such 'extreme' regime of QCD can be reached by heating the system up to very large temperatures  $T$ , where fundamental questions such as confinement and chiral symmetry breaking can be addressed. This is the realm of thermal field theory [3], in which problems that are phenomenologically relevant for e.g. cosmology (such as dark matter relic abundance), heavy-ion collisions (such as hydrodynamic expansion of a plasma) and for astrophysics (such as the properties of compact stars) are studied. Furthermore, due to asymptotic freedom, the high-temperature limit of QCD turns out to be tractable with analytic methods, namely weak-coupling expansions. In this limit, while staying strictly within QCD (as opposed to modelling some of its behavior), we hence gain the possibility of systematic improvements of theoretical approximations, whenever they are needed.

The thermodynamic properties QCD, such as certain features of its equilibrium phase diagram, can be studied in lattice Monte-Carlo simulations. One of the key results of thermal lattice QCD is the determination of the deconfinement temperature  $T_c$  at vanishing baryon density. These results basically confirm the expected transition from a hadron gas at low temperatures to a quark-gluon plasma above  $T_c$ , tracking the liberation of degrees of freedom via the QCD pressure, and confirming an approach to the asymptotic limit of an ideal gas, see Fig. 1.



**Figure 1.** Lattice results for the scaled QCD pressure, for different flavor content of the theory [4]. The small arrows denote the respective values for an ideal (Stefan-Boltzmann) gas.

Further explorations of features of the equilibrium phase diagram at various values of temperatures as well as baryon-chemical potentials  $\mu$  turn out to be more difficult, and typically involve some degree of modeling. Of particular interest are the equation of state (EoS), phase transition lines, their respective order, and locations of critical points, as well as medium properties such as spectral functions or correlation lengths. These questions are being actively pursued in a worldwide effort in order to obtain information relevant for early-universe cosmology (high  $T$ , low  $\mu$ ), heavy-ion physics (intermediate  $T$  and  $\mu$ ) as well as astrophysics (low  $T$ , intermediate  $\mu$ ).

As has already been mentioned above, while hot QCD is a strongly coupled system near  $T_c$  where lattice simulations are needed, at very high temperature asymptotic freedom guarantees the gauge coupling  $g$  to become small, such that weak-coupling approaches can be used. In particular, effective field theory methods turn out to be most powerful in order to combine lattice and continuum calculations [5, 6], in order to obtain high-precision results for the relevant thermodynamic quantities such as the QCD pressure and equation of state [7]. These methods alleviate the notorious infrared problem of finite-temperature field theory [8], to an extent that higher orders in the weak-coupling expansion can now be systematically included. I will discuss some key features of this systematic approach to QCD thermodynamics using the best of both – lattice and continuum – approaches, and point out the remaining (technical) problems [9].

## 2. Effective field theory approach

As becomes apparent from the lattice data of Fig. 1, the task is to explain the deviation of the value of the QCD pressure of about 20% at  $T \sim 4T_c$  from its ideal value, realized in the theoretical limit  $T \rightarrow \infty$ . In the imaginary-time path integral formalism at finite temperature, the pressure can be straightforwardly computed as the logarithm of the partition function in the thermodynamic limit,

$$p(T) = \lim_{V \rightarrow \infty} \frac{T}{V} \ln \int \mathcal{D}[A_\mu^a, \psi, \bar{\psi}] \exp \left( - \frac{1}{\hbar} \int_0^{\hbar/T} d\tau \int d^{3-2\epsilon} x \mathcal{L}_{\text{QCD}}^{\text{E}} \right). \quad (1)$$

Due to the above-mentioned infrared problems, the loop expansion of  $p(T)$  in the gauge coupling  $g$  turns out to be non-trivial, since it contains odd powers of the coupling and even logarithms thereof. After pioneering work in the seventies and eighties [10, 11, 12], the root cause of this non-analytic (in  $\alpha_s = g^2/4\pi$ ) behavior has been well understood as coming from the dynamically generated scales in the thermal system: interactions make hot QCD a multi-scale system, where the relevant momentum scales  $|k| \sim \pi T$ ,  $|k| \sim gT$  and  $|k| \sim g^2 T$  are called hard, soft and ultrasoft, respectively. At large temperatures, the gauge coupling is small and hence these three scales are well separated, allowing for a clean effective theory setup that factorizes the partition function, such that the pressure can be re-written as the sum of contributions from these three scales [13],

$$p_{\text{QCD}}(T) = p_{\text{hard}}(T) + p_{\text{soft}}(T) + p_{\text{ultrasoft}}(T). \quad (2)$$

Physics at those three different energy scales is governed by 4-dimensional (4d) thermal QCD, by a 3d adjoint Higgs model, and by 3d pure Yang-Mills theory, respectively. Soft and ultrasoft effects hence originate from 3-dimensional theories, a phenomenon known as dimensional reduction, as can be easily understood from the fact that long-range (small momentum) physics does not see the compact timelike dimension. As is common practice in effective field theory, couplings of the effective theory are related to the parameters of the original one (in this case  $T$  and  $g$ ) through matching computations. It turns out that, after the matching has been done in the weak-coupling sense, the three parts of the pressure parametrically start at different orders in the dimensionless 4d gauge coupling  $g$ ,

$$p_{\text{hard}}(T) \sim T^4(1 + g^2 + g^4 + g^6 + \dots), \quad (3)$$

$$p_{\text{soft}}(T) \sim T^4(g^3 + g^4 + g^5 + g^6 + \dots), \quad (4)$$

$$p_{\text{ultrasoft}}(T) \sim T^4(g^6 + \dots). \quad (5)$$

Now, as is well-known from effective theories, whenever a new physical scale enters a problem, one typically meets large logarithms (of the ratio of matching scales). Therefore, one might call the sum of the expansions shown above the ‘physical leading order’ of the pressure. Indeed, it is

known that whenever one truncates the series before the order  $g^6$ , the series does not show signs of convergence at intermediate temperatures [?], a telltale sign that these large logarithms are missing. Only at parametric order  $g^6$ , for the first time contributions from all physical energy scales have entered, giving a true leading-order approximation.

### 3. Current status

Of all the coefficients shown in Eqs. (3-5), only the  $g^6$  term of Eq. (3) remains unknown to date. The longest-distance input,  $p_{\text{ultrasoft}}$ , is a non-perturbative contribution that has been determined with a clean lattice measurement [15] and matched from the lattice to the continuum ( $\overline{\text{MS}}$ ) scheme via numerical stochastic perturbation theory [16]. This alleviates the infrared problem mentioned above.

Contributions to  $p_{\text{soft}}$  and  $p_{\text{hard}}$  are purely perturbative, the  $g^6$  coefficients entailing four-loop diagrams that are relatively straightforward to evaluate in the 3d theory [17], but much more involved in the 4d one [18]. Figure 2 shows typical diagrams at this level, all being of vacuum type. The open problem to be tackled is to develop a method that allows to systematically solve the large number of four-loop sum-integrals that appear in the short-distance piece  $p_{\text{hard}}$ . Once this difficult but conceptionally clear perturbative contribution has been fixed and hence a result for the 'physical leading order' of the pressure is available, it would of course be valuable to check convergence of the series by evaluating the next-to-leading (NLO,  $g^7$ ) order. Somewhat surprisingly, this step would not represent a huge technical complication, as it originates from the 3d theory only, allowing for an application of the full power of automated Feynman integral reduction algorithms (albeit at the five-loop level) that have been and are currently being developed for zero-temperature applications such as collider physics.

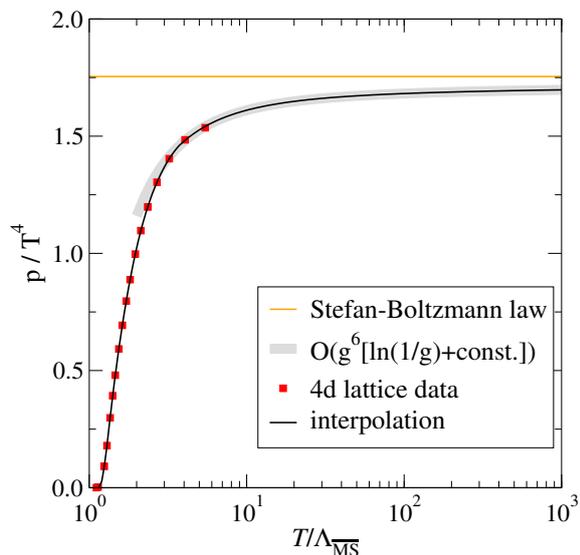
$$\begin{aligned} \Phi_4 = & \frac{1}{72} \text{[diagram 1]} - \frac{1}{4} \text{[diagram 2]} - \frac{1}{6} \text{[diagram 3]} + \frac{1}{12} \text{[diagram 4]} - \frac{1}{2} \text{[diagram 5]} - \frac{1}{2} \text{[diagram 6]} \\ & - 1 \text{[diagram 7]} - \frac{1}{3} \text{[diagram 8]} + \frac{1}{6} \text{[diagram 9]} + \frac{1}{6} \text{[diagram 10]} + \frac{1}{8} \text{[diagram 11]} - \frac{1}{4} \text{[diagram 12]} \\ & + \frac{1}{4} \text{[diagram 13]} - \frac{1}{2} \text{[diagram 14]} + \frac{1}{8} \text{[diagram 15]} + \frac{1}{8} \text{[diagram 16]} + \frac{1}{16} \text{[diagram 17]} + \frac{1}{48} \text{[diagram 18]} \end{aligned}$$

**Figure 2.** Some Feynman diagrams contributing to the four-loop pressure. Wiggly/dotted lines denote gluons/ghosts, respectively.

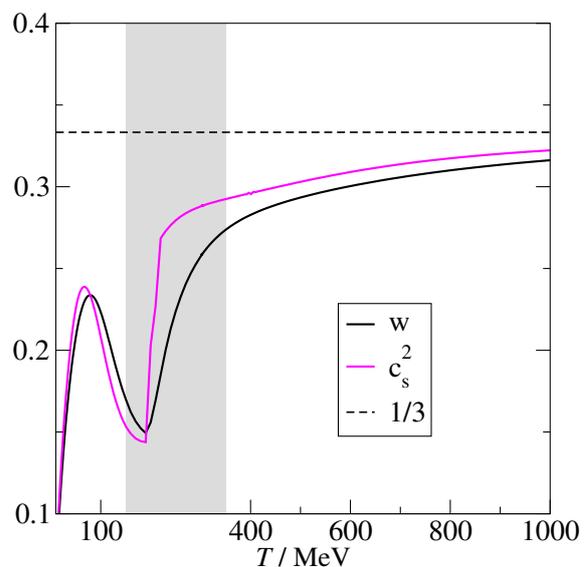
While working on the hard open problem of evaluating the required four-loop sum-integrals, one might be tempted to already show some results, in order to connect to phenomenology. To this end, as an attempt to strive for the currently best possible description of the pure-gluon sector of hot QCD, the single unknown  $g^6$  coefficient can be fixed by matching to available lattice data at intermediate temperatures  $T \sim 3 - 5T_c$  [7]. The result is shown in Fig. 3, and can be argued to show a friendly functional behavior, smoothly connecting low-temperature data with the weak-coupling expansion at high temperatures.

Once the gluonic sector is under control, the effective theory framework allows for straightforward inclusion of fermions, since their effects are contained in the matching coefficients. To obtain an estimate of the equation of state (EoS) with physical quark masses, one can follow the strategy of unquenching: starting from the  $N_f = 0$  results of Fig. 3, which correspond to  $m_q = \infty$ , one lowers  $N_f$  of them to their physical masses  $m_{q,\text{phys}}$  [7]. Schematically, the NLO result is  $p(N_f = 0) \times \frac{[c_0 + g^2 c_2](N_f, m_i, \mu_i)}{[c_0 + g^2 c_2](N_f = 0)}$  which, after going to physical units of energy by expressing  $\Lambda_{\overline{\text{MS}}}$  in units of MeV with the help of matching to a hadron resonance gas at low

temperature, allows to plot the dimensionless ratios of Fig. 4. Let us mention that the deviation of the (dimensionless) EoS from a third,  $(\frac{1}{3} - w(T))$ , is often called the 'trace anomaly' or 'interaction measure'. One observes significant structure at temperatures  $T \sim 100 - 400$  MeV, a feature that does not come as a surprise since close to a second-order phase transition, one would expect e.g. the sound speed to scale with a critical exponent,  $c(T) \sim (T - T_c)^{-\gamma}$ .



**Figure 3.** Interpolating curve (solid line) for the QCD pressure at  $N_f = 0$  [7]. In the perturbative curve (grey band) the unknown  $g^6$  constant has been adjusted to match lattice data [4] at  $T \sim 4T_c$ .



**Figure 4.** The equation-of-state parameter  $w = \frac{p(T)}{e(T)}$  and the speed of sound squared  $c_s^2 = \frac{p'(T)}{e'(T)}$  for QCD at  $N_f = 4$  [7].

#### 4. Conclusions

In summary, we have seen that phenomenologically relevant thermodynamic quantities of QCD, such as the pressure, can be determined to high accuracy with a mix of two main approaches: numerically at temperatures around 200 MeV, in the neighborhood of the critical one, and analytically at  $T \gg T_c$ . The latter approach involves multi-loop weak-coupling expansions and is efficiently organized in the framework of effective field theories, allowing for systematic improvements.

The relevant three-dimensional effective theories are interesting in their own right, opening up tremendous opportunities: fermions can be treated completely analytically, in contrast to the large problems they pose in the lattice approach. Furthermore, universality and superrenormalizability make these 3d theories an ideal playground for the development multi-loop methods.

Regarding the QCD pressure, the current status is that its 'physical leading order', in the sense defined above, remains unknown. A mild open problem being a matching of lattice- and continuum results for general  $N_c$ , the more difficult task to be completed is the evaluation of a large number of four-loop sum-integrals. However, the pressure shows a friendly functional behavior when the single yet-unknown coefficient is fitted.

#### Acknowledgments

This work was supported in parts by FONDECYT project number 1151281.

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