# Inverse magnetic catalysis in the linear sigma model

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**Abstract.**
We compute the critical temperature for the chiral transition in the background of a magnetic field in the linear sigma model, including the quark contribution and the thermo-magnetic effects induced on the coupling constants at one loop level. For the analysis, we go beyond mean field aproximation, by taking one loop thermo-magnetic corrections to the couplings as well as plasma screening effects for the boson's masses, expressed through the ring diagrams. We found inverse magnetic catalysis, i.e. a decreasing of the critical chiral temperature as function of the intensity of the magnetic field, which seems to be in agreement with recent results from the lattice community.

## 1. Introduction

In recent years there has been an increasing interest in studying the QCD phase diagram in the presence of a magnetic field. Particularly, the effect of the magnetic field on the critical temperature for chiral phase transition, has been studied in different effective models [1, 2, 3, 4, 5, 6, 7] as well as in lattice QCD where initially it has been found that magnetic catalysis takes place, i.e. that the critical temperature for chiral phase transition becomes higher in the presence of a magnetic field [8, 9]. However, recent improved lattice calculations have found the opposite behavior [10, 11, 12].

In this work, we studied the sigma model with quarks, finding inverse magnetic catalysis in agreement with the latest lattice results. The main idea was to go beyond mean field approximation, by taking thermo-magnetic corrections to the couplings. In this way we found that the boson's coupling decreases with the magnetic field which is a substantial ingredient in order to find inverse magnetic catalysis. The idea of correcting the vertices has also been implemented in QCD [13], where the thermal and magnetic corrections to the quark-gluon vertex have been calculated in the Hard Thermal Loop(HTL) approximation. From the quark-gluon vertex we can obtain the dependence of the QCD coupling with the magnetic field. The result shows that the coupling decreases as function of the intensity of the magnetic field. The reader should go to the original reference [14] for a complete description of the technical details.

## 2. Efective Potential

The model is given by the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(D_\mu\vec{\pi})^2 + \frac{\mu^2}{2}(\sigma^2+\vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 \\ &+ i\bar{\psi}\gamma^\mu D_\mu\psi - g\bar{\psi}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi, \end{aligned} \tag{1}$$

where $\psi$ is an SU(2) isospin doublet, $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ is an isospin triplet and $\sigma$ is an isospin singlet, with

$$D_\mu = \partial_\mu + iqA_\mu, \tag{2}$$

is the covariant derivative. $A^\mu$ is the vector potential corresponding to an external magnetic field directed along the $\hat{z}$ axis,

$$A^\mu = \frac{B}{2}(0, -y, x, 0), \tag{3}$$

and $q$ is the particle's electric charge. $A^\mu$ satisfies the gauge condition $\partial_\mu A^\mu = 0$. Since $A^3 = 0$, the gauge field only couples to the charged pion combinations, namely

$$\pi_\pm = \frac{1}{\sqrt{2}}(\pi_1 \mp i\pi_2). \tag{4}$$

The neutral pion is taken as the third component of the pion isovector, $\pi^0 = \pi_3$. The gauge field is taken as classical and thus we do not consider loops involving the propagator of the gauge field in internal lines. The squared mass parameter $\mu^2$ and the self-coupling $\lambda$ and $g$ are taken to be positive.

To allow for an spontaneous breaking of symmetry, we let the $\sigma$ field to develop a vacuum expectation value $v$

$$\sigma \to \sigma + v, \tag{5}$$

which can later be taken as the order parameter of the theory. After this shift, the Lagrangian can be rewritten as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}[\sigma(\partial_\mu + iqA_\mu)^2\sigma] - \frac{1}{2}\left(3\lambda v^2 - \mu^2\right)\sigma^2 \\ &- \frac{1}{2}[\vec{\pi}(\partial_\mu + iqA_\mu)^2\vec{\pi}] - \frac{1}{2}\left(\lambda v^2 - \mu^2\right)\vec{\pi}^2 + \frac{\mu^2}{2}v^2 \\ &- \frac{\lambda}{4}v^4 + i\bar{\psi}\gamma^\mu D_\mu\psi - gv\bar{\psi}\psi + \mathcal{L}_I^b + \mathcal{L}_I^f, \end{aligned} \tag{6}$$

where $\mathcal{L}_I^b$ and $\mathcal{L}_I^f$ are given by

$$\begin{aligned}\mathcal{L}_I^b &= -\frac{\lambda}{4}\Big[(\sigma^2 + (\pi^0)^2)^2 \\ &+ 4\pi^+\pi^-(\sigma^2 + (\pi^0)^2 + \pi^+\pi^-)\Big], \\ \mathcal{L}_I^f &= -g\bar{\psi}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi, \end{aligned} \tag{7}$$

and represent the Lagrangian describing the interactions among the fields $\sigma$, $\vec{\pi}$ and $\psi$, after symmetry breaking. From Eq. (6) we see that the $\sigma$, the three pions and the quarks have masses given by

$$\begin{aligned} m_\sigma^2 &= 3\lambda v^2 - \mu^2, \\ m_\pi^2 &= \lambda v^2 - \mu^2, \\ m_f &= gv, \end{aligned} \tag{8}$$

respectively.

Using Schwinger's proper-time method, the expression for the one-loop effective potential for one boson field with squared mass $m_b^2$ and absolute value of its charge $q_b$ at finite temperature $T$ in the presence of a constant magnetic field can be written as

$$\begin{aligned} V_b^{(1)} &= \frac{T}{2}\sum_n \int dm_b^2 \int \frac{d^3k}{(2\pi)^3}\int_0^\infty \frac{ds}{\cosh(q_b Bs)} \\ &\times\ e^{-s(\omega_n^2+k_3^2+k_\perp^2 \frac{\tanh(q_bBs)}{q_bBs}+m_b^2)}, \end{aligned} \tag{9}$$

where $\omega_n = 2n\pi T$ are boson Matsubara frequencies. Similarly, the expression for the one-loop effective potential for one fermion field with mass $m_f$ and absolute value of its charge $q_f$ at finite temperature $T$ in the presence of a constant magnetic field can be written as

$$\begin{aligned} V_f^{(1)} &= -\sum_{r=\pm1} T\sum_n \int dm_f^2 \int \frac{d^3k}{(2\pi)^3}\int_0^\infty \frac{ds}{\cosh(q_f Bs)} \\ &\times\ e^{-s(\tilde{\omega}_n^2+k_3^2+k_\perp^2 \frac{\tanh(q_fBs)}{q_fBs}+m_f^2+rq_fB)}, \end{aligned} \tag{10}$$

where $\tilde{\omega}_n = (2n+1)\pi T$ are fermion Matsubara frequencies. The sum over the index $r$ corresponds to the two possible spin orientations along the magnetic field direction.

Including the $v$-independent terms, choosing the renormalization scale as $\tilde{\mu} = e^{-1/2}\mu$ and after mass and charge renormalization, it has been shown in Ref. [15] that the thermo-magnetic effective potential in the small to intermediate field regime, in a high temperature expansion can be written as

$$\begin{aligned} V^{(\mathrm{eff})} &= -\frac{\mu^2}{2}v^2 + \frac{\lambda}{4}v^4 + \sum_{i=\sigma,\pi^0}\left\{\frac{m_i^4}{64\pi^2}\left[\ln\left(\frac{(4\pi T)^2}{2\mu^2}\right) - 2\gamma_E + 1\right]\right. \\ &- \left.\frac{\pi^2T^4}{90} + \frac{m_i^2T^2}{24} - \frac{T}{12\pi}(m_i^2+\Pi)^{3/2}\right\} + \sum_{i=\pi_+,\pi_-}\left\{\frac{m_i^4}{64\pi^2}\left[\ln\left(\frac{(4\pi T)^2}{2\mu^2}\right) - 2\gamma_E + 1\right]\right. \\ &- \frac{\pi^2T^4}{90} + \frac{m_i^2T^2}{24} + \frac{T(2qB)^{3/2}}{8\pi}\zeta\left(-\frac{1}{2}, \frac{1}{2} + \frac{m_i^2+\Pi}{2qB}\right) \\ &- \left.\frac{(qB)^2}{192\pi^2}\left[\ln\left(\frac{(4\pi T)^2}{2\mu^2}\right) - 2\gamma_E + 1 + \zeta(3)\left(\frac{m_i}{2\pi T}\right)^2 - \frac{3}{4}\zeta(5)\left(\frac{m_i}{2\pi T}\right)^4\right]\right\} \\ &- \sum_{f=u,d}\left\{\frac{m_f^4}{16\pi^2}\left[\ln\left(\frac{(\pi T)^2}{2\mu^2}\right) - 2\gamma_E + 1\right] + \frac{7\pi^2T^4}{180} - \frac{m_f^2T^2}{12}\right. \\ &+ \left.\frac{(q_fB)^2}{24\pi^2}\left[\ln\left(\frac{(\pi T)^2}{2\mu^2}\right) - 2\gamma_E + 1\right]\right\}, \end{aligned} \tag{11}$$

where $q$ is the absolute value of the charged pions' charge ($q = 1$), $q_u = 2/3$, $q_d = 1/3$ are the absolute values of the $u$ and $d$ quarks, respectively and $\gamma_E$ is Euler's gamma. Though we take the quark masses as equal, the notation emphasizes that the effective potential is evaluated accounting for the different quark charges. We have introduced the leading temperature plasma screening effects for the boson's mass squared, encoded in the boson's self-energy $\Pi$. For the Hurwitz zeta function $\zeta(-1/2, z)$ in Eq. (11) to be real, we need that

$$-\mu^2 + \Pi > qB, \tag{12}$$

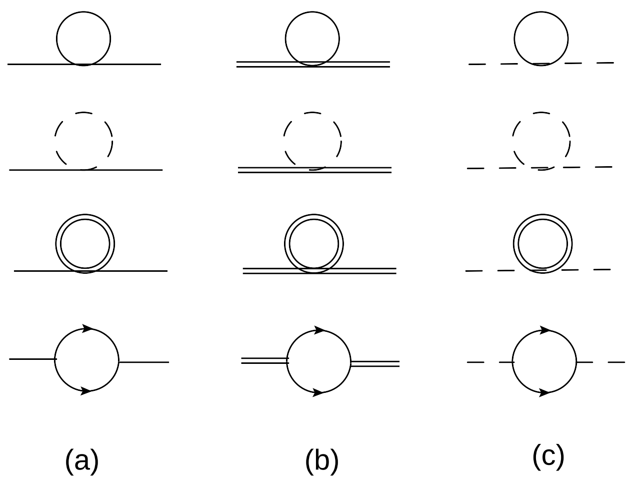


**Figure 1.** Feynman diagrams contributing to the one loop bosons' self-energies. The dashed line denotes the charged pion, the continuous line is the sigma, the double line represents the neutral pion and the continuous line with arrows represents the fermions.

condition that comes from requiring that the second argument of the Hurwitz zeta function satisfies $z > 0$, even for the lowest value of $m_b^2$ which is obtained for $v = 0$. Furthermore, for the large $T$ expansion to be valid, we also require that

$$qB/T^2 < 1. \tag{13}$$

The diagrams representing the bosons' self-energies are depicted in Fig. 1. Each column corresponds to the diagrams contributing to the self-energy of a given boson. The total self-energy for any boson is identical to the other's and thus we concentrate on computing the diagrams in column (a). The contribution from the individual diagrams require of the expressions [hereby capital letters are used to denote four-momenta in Eucledian space, e.g. $K \equiv (\omega_n, \mathbf{k})$]

$$\begin{aligned}
\Pi_{a1}(m_\sigma^2) &= \lambda T \sum_n \int \frac{d^3k}{(2\pi)^3} D(K; m_\sigma^2), \\
\Pi_{a2}(m_{\pi^0}^2) &= \lambda T \sum_n \int \frac{d^3k}{(2\pi)^3} D(K; m_{\pi^0}^2) \\
\Pi_{a3}(m_{\pi^\pm}^2) &= \lambda T \sum_n \int \frac{d^3k}{(2\pi)^3} D_B(K; m_{\pi^\pm}^2), \\
\Pi_{a4}(P; m_f) &= -N_f g^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \\
&\times \; \mathrm{Tr}\, S_B(K; m_f) S_B(P-K; m_f),
\end{aligned} \tag{14}$$

where $N_f$ is the number of fermions and the corresponding propagators are given by

$$\begin{aligned}
D(K; m_i^2) &= \frac{1}{K^2 + m_i^2}, \\
D_B(K; m_i^2) &= \int_0^\infty ds \frac{e^{-s(\omega_n^2 + k_3^2 + k_\perp^2 \frac{\tanh(qBs)}{qBs} + m_i^2)}}{\cosh(qBs)},
\end{aligned}$$

$$
\begin{aligned}
S_B(K;m_f) &= \int_0^\infty ds \frac{e^{-s(\widetilde{\omega}_n^2+k_3^2+k_\perp^2\frac{\tanh(q_f Bs)}{q_f Bs}+m_f^2)}}{\cosh(q_f Bs)} \\
&\times \left[(\cosh(q_f Bs) - i\gamma_1\gamma_2 \sinh(q_f Bs))(m_f - \not{k}_\parallel) - \frac{\not{k}_\perp}{\cosh(q_f Bs)}\right], \qquad (15)
\end{aligned}
$$

and for the charged particle propagators we have used Schwinger's proper-time representation. For the computation of $\Pi_{a3}$ we work in the *infrared limit*, namely, $P = (0, \mathbf{p} \to 0)$ and with the hierarchy of scales $qB, m_i^2 < T^2$. It has been shown [16] that this limit can be formally implemented by straightforward setting $P = 0$ in the third of Eqs. (14). The leading contribution at high temperature from each of these diagram is

$$
\begin{aligned}
\Pi_{a1}(m_\sigma^2) &= \lambda \frac{T^2}{12} \\
\Pi_{a2}(m_{\pi^0}^2) &= \lambda \frac{T^2}{12} \\
\Pi_{a3}(m_{\pi^\pm}^2) &= \lambda \frac{T^2}{12} \\
\Pi_{a4}(0;m_f) &= N_f g^2 \frac{T^2}{6}, \qquad (16)
\end{aligned}
$$

and therefore, considering the permutation factors, the total self-energy is given by

$$
\begin{aligned}
\Pi &= 3\Pi_{a1}(m_\sigma^2) + \Pi_{a2}(m_{\pi^0}^2) + 2\Pi_{a3}(m_{\pi^\pm}^2) + \Pi_{a4}(0;m_f) \\
&= \lambda \frac{T^2}{2} + N_f g^2 \frac{T^2}{6}. \qquad (17)
\end{aligned}
$$

## 3. One loop thermo-magnetic couplings and critical temperature

Let us now compute the one-loop correction to the coupling $\lambda$, including thermal and magnetic effects. Figure 2 shows the Feynman diagrams that contribute to this correction. Columns (a), (b), (c), (d), (e) and (f) contribute to the correction to the $\sigma^4$, $(\pi^0)^4$, $(\pi^+)^2(\pi^-)^2$, $\sigma^2\pi^+\pi^-$, $(\pi^0)^2\pi^+\pi^-$ and $\sigma^2(\pi^0)^2$ terms of the interaction Lagrangian in Eq. (7), respectively

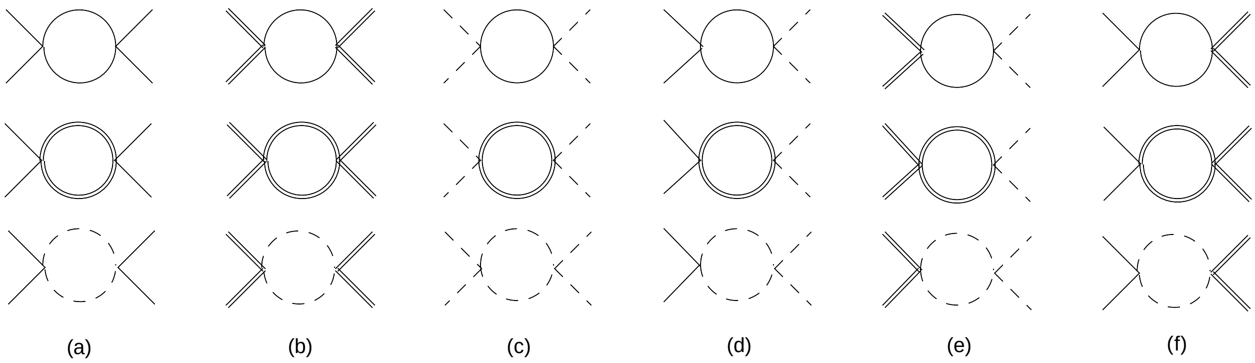


**Figure 2.** One-loop Feynman diagrams that contribute to the thermal and magnetic correction to the coupling $\lambda$. The dashed line denotes the charged pion, the continuous line is the sigma and the double line represents the neutral pion.

Once again we work in the *infrared limit*, namely, $P_i = (0, \mathbf{p} \to 0)$ and with the hierarchy of scales where $qB, m_i^2 < T^2$. Considering the permutation factors and the contribution from the

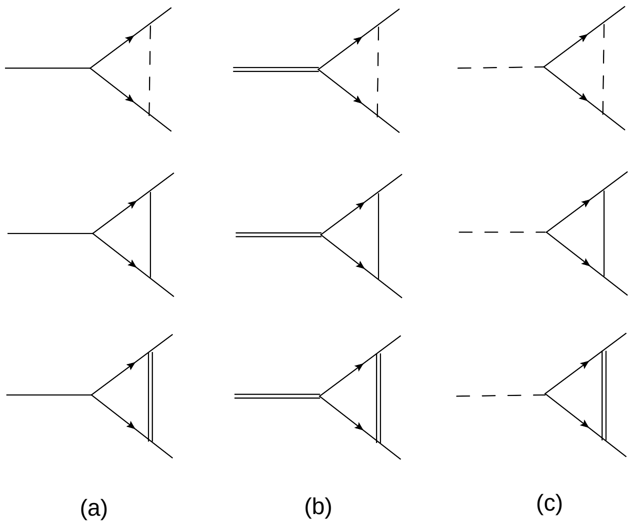


**Figure 3.** One-loop Feynman diagrams that contribute to the thermal and magnetic correction to the coupling $g$. The dashed line denotes the charged pion, the continuous line is the sigma, the double line represents the neutral pion and the continuous line with arrows represents the quarks.

$s$, $t$ and $u$-channels, the correction to the self-coupling $\lambda$ to one-loop order is given by

$$\lambda_{\text{eff}} = \lambda \left[1 + 24\lambda \left(9I(0; m_\sigma^2) + I(0; m_\pi^2) + 4J(0; m_\pi^2)\right)\right], \tag{18}$$

where

$$I(0; m_i^2) = \frac{T}{8\pi} \frac{1}{(m_i^2 + \Pi)^{1/2}} - \frac{1}{16\pi^2} \Big[ \ln \left( \frac{(4\pi T)^2}{2\mu^2} \right) + 1 - 2\gamma_E + \zeta(3) \left( \frac{\sqrt{m_i^2 + \Pi}}{2\pi T} \right)^2 \Big], \tag{19}$$

$$\begin{aligned} J_{n\neq 0}(0; m_i^2) &= -\frac{1}{16\pi^2} \Big[ \ln \left( \frac{(4\pi T)^2}{2\mu^2} \right) + 1 - 2\gamma_E + \zeta(3) \left( \frac{\sqrt{m_i^2 + \Pi}}{2\pi T} \right)^2 \Big] \\ &- \frac{(qB)^2}{1024\pi^6 T^4} \zeta(5). \end{aligned} \tag{20}$$

Note that $\lambda_{\text{eff}}$ depends on $v$ through the dependence on the boson masses. Let us furthermore take the approximation where we evaluate $\lambda_{\text{eff}}$ at $v = 0$. The rationale is that we are pursuing the effect on the critical temperature which is the temperature where the curvature of the effective potential at $v = 0$ vanishes.

We now turn to the calculation of the thermo-magnetic correction of the coupling $g$. Figure 3 shows the Feynman diagrams that contribute to this correction. We are interested in computing an effective value for this coupling, $g_{\text{eff}}$, also for $v = 0$, in the same manner we did for $\lambda_{\text{eff}}$. Columns (a), (b) and (c) contribute to the correction to the quark-$\sigma$, quark-$\pi^0$ and quark-$\pi^\pm$ terms of the interaction Lagrangian of Eq. (7), respectively. Therefore the correction to $g$ at one-loop order is given by

$$g_{\text{eff}} = g[1 + g^2 L(0, 0)], \tag{21}$$

where

$$L(0,0) = \frac{g^3}{8\pi^2}\left[-2\gamma_E + 1 - \ln\left(\frac{T^2\pi^2}{2\mu^2}\right) - 7\left(\frac{m_\pi^2}{8\pi^2 T^2}\right)\zeta(3) + 31\left(\frac{m_\pi^2}{8\pi^2 T^2}\right)^2\zeta(5)\right.$$
$$\left. + \quad \frac{3410}{9}\frac{(qB)^2}{(4\pi T)^4}\zeta(5)\right]. \tag{22}$$

In order to use a set of values for the couplings $\lambda$ and $g$ appropriate for the description of the phase transition note that the curvature of the effective potential vanishes for $v = 0$. Since the boson thermal masses are proportional to this curvature, they also vanish at $v = 0$. This observation provides a condition to obtain a relation between the model parameters at $T_c$ that can be supplemented with information from the physical masses of the pion and sigma in vacuum [22].

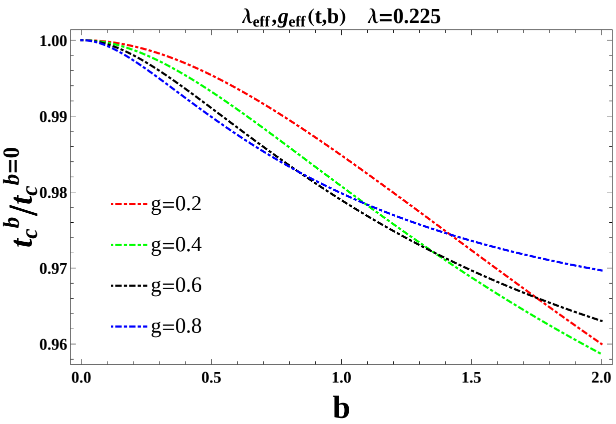


**Figure 4.** Color on-line. Effect of the full thermo-magnetic dependence of couplings on the critical temperature for a fixed value of the tree level $\lambda = 0.225$ and different values of the tree level $g$ as a function of $b = qB/\mu^2$. In all cases the critical temperature is a decreasing function of $b$.

Let us now study the effect that the thermo-magnetic corrections to the couplings have on the critical temperature. We first look at the cases where we set the couplings to their tree level values and where only thermal effects are included. Figure 4 shows the critical temperature for the case where we consider the full thermo-magnetic dependence of the couplings, obtained from setting the second derivative of Eq. (11) equal to zero at $v = 0$, normalized to the critical temperature for vanishing magnetic field. Figure 4 shows the case when we set the tree-level coupling $\lambda$ to a fixed value and vary the tree-level coupling $g$.

## 4. Conclusions

In summary, we have shown that when including the one-loop thermo-magnetic effects for the couplings in the linear sigma model with fermions interacting with an external magnetic field, the critical temperature for the chiral transition is a decreasing function of the field strength. This behavior is a direct consequence of the decrease of the boson self-coupling with the field strength. The effect of the fermions is marginal and the main contribution comes from the charged pions. We emphasize that the thermo-magnetic dependence of the couplings has been computed –as opposed to assumed– within the model itself.

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