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Integrability properties of a coupled KdV system and its supersymmetric extension

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Abstract. We discuss several integrability properties of a coupled KdV system. We obtain a new generalization of the already known static solutions for the system. We then consider the supersymmetric extension of the coupled KdV system, it is a new integrable system. We show that for particular Grassmann algebras the system is the limit of a Clifford algebra valued system with nice stability properties. We briefly discuss the hamiltonian structures of this supersymmetric integrable system.

1. Introduction

The Korteweg-de Vries (KdV) equation has relevant applications in low energy physics as well as in high energy physics. Several extensions of KdV have been proposed. Some of them are the coupled KdV systems, the supersymmetric KdV equation and operatorial extensions.

Recently [1] we introduced a parametric coupled KdV system which includes the complex version of KdV equation for a particular value of the parameter as well as one of the Hirota-Satsuma coupled systems. We found a Bäcklund transformation, the Gardner transformation and we showed the permutability theorem for the system. In particular we found an infinite family of multisolitonic and periodic solutions for the system. On the other side a supersymmetric extension of KdV equation has been considered in the literature. Although it has infinite conserved quantities as the KdV equation, only one local hamiltonian structure has been found for it in distinction to the KdV case which is bihamiltonian.

In this work we present some new interesting properties about the solutions to the parametric coupled KdV system and we show also that for particular Grassmann algebras, which are the basic algebraic structures to define the supersymmetric extensions of the KdV equation, the supersymmetric system is bihamiltonian, in the sense we will explain in this work. In particular, one of the Hirota-Satsuma coupled systems is contained in a supersymmetric system. Consequently, this supersymmetric system has all the integrability properties shown in [1].

2. The parametric coupled KdV system and some of its relevant properties

The parametric coupled KdV system is given by

$$u_t + uu_x + u_{xxx} + \lambda vv_x = 0 \tag{1}$$

$$v_t + u_xv + v_xu + v_{xxx} = 0, \tag{2}$$

where λ is a real parameter and u, v are rapidly decreasing real valued functions depending on the temporal and spatial variables t and x respectively. The system contains, for $\lambda = -1$, the complex version of KdV and for $\lambda = 0$ includes the ninth Hirota-Satsuma system given in [2] (after a suitable change of the variables). For the case $\lambda = 1$ the system decouples in to two KdV equations, in terms of $u \pm v$ respectively; for the other values of the parameter λ ($\lambda \neq 0$) a simple change in the variable v reduces it to $\lambda = \pm 1$.

The system defined by (1) and (2) has applications as a physics model in a two-layer liquid, see [3], and also has solitonic solutions, obtained using the Hirota approach [4].

In [1] it was proven the following

Lemma 1 *Let $r, s \in C^{\infty}_{\downarrow}$ be a solution of the following ϵ-parameter partial differential equations (called the Gardner system)*

$$r_t + r_{xxx} + r r_x + \lambda s s_x - \frac{1}{6}\varepsilon^2 \left[\left(r^2 + \lambda s^2\right) r_x + 2\lambda r s s_x\right] = 0$$
$$s_t + s_{xxx} + r s_x + s r_x - \frac{1}{6}\varepsilon^2 \left[\left(r^2 + \lambda s^2\right) s_x + 2 r s r_x\right] = 0.$$

Then $u, v \in C^{\infty}_{\downarrow}$ defined through the relations (called the Gardner transformation)

$$u = r + \varepsilon r_x - \frac{1}{6}\varepsilon^2 \left(r^2 + \lambda s^2\right)$$
$$v = s + \varepsilon s_x - \frac{1}{3}\varepsilon^2 r s$$

are solutions of the system (1),(2).

As a direct consequence of lemma 1 we have that the system (1),(2) has infinite polynomial conserved quantities in the fields u, v and its spatial derivatives (making the system integrable in this sense for every value of λ) and they are explicitly obtained from the first two conserved quantities of the Gardner system. The first few of them are

$$\int_{-\infty}^{+\infty} u dx \quad , \qquad \int_{-\infty}^{+\infty} v dx,$$
$$\int_{-\infty}^{+\infty} \left(u^2 + \lambda v^2\right) dx \quad , \qquad \int_{-\infty}^{+\infty} u v dx,$$
$$\int_{-\infty}^{+\infty} \left(\frac{1}{3}u^3 + \lambda u v^2 - \lambda (v_x)^2 - (u_x)^2\right) dx \quad , \qquad \int_{-\infty}^{+\infty} \left(\frac{1}{2}u^2 v - u_x v_x + \frac{1}{6}\lambda v^3\right) dx.$$

We proposed in [1] for the system (1), (2) the following Bäcklund transformation. If (w, y) and (w', y'), where $w_x = u, w'_x = u'$ and $y_x = v, y'_x = v'$ satisfy the following equations (Bäcklund transformation)

$$w_x + w'_x = 2\eta - \frac{1}{12}(w - w')^2 - \frac{\lambda}{12}(y - y')^2, \tag{3}$$

$$\begin{aligned} w_t + w'_t = {} & \frac{1}{6}(w - w')(w_{xx} - w'_{xx}) + \frac{\lambda}{6}(y - y')(y_{xx} - y'_{xx}) - \frac{1}{3}w_x^2 - \\ & - \frac{1}{3}{w'_x}^2 - \frac{1}{3}w_x w'_x - \frac{\lambda}{3}y_x^2 - \frac{\lambda}{3}{y'_x}^2 - \frac{\lambda}{3}y_x y'_x, \end{aligned} \tag{4}$$

$$y_x + y'_x = 2\mu - \frac{1}{6}(w - w')(y - y'), \tag{5}$$

$$\begin{aligned} y_t + y'_t = {} & \frac{1}{6}(w - w')(y - y')_{xx} + \frac{1}{6}(w - w')_{xx}(y - y') - \\ & - \left(\frac{2}{3}w_x y_x + \frac{2}{3}w'_x y'_x + \frac{1}{3}w_x y'_x + \frac{1}{3}w'_x y_x\right) \end{aligned} \tag{6}$$

on an open set $\Omega \subset \mathbb{R}^2$ and

$$(w - w')^2 - \lambda (y - y')^2 \neq 0$$

on Ω, then (w, y) and (w', y') are two (different) solutions of the system

$$Q_1(w, y) \equiv w_t + \frac{1}{2}(w_x)^2 + w_{xxx} + \frac{1}{2}\lambda (y_x)^2 = 0 \tag{7}$$

$$Q_2(w, y) \equiv y_t + w_x y_x + y_{xxx} = 0, \tag{8}$$

and consequently $w_x = u, y_x = v, w'_x = u', y'_x = v'$ are solutions of the original system (1), (2) over the corresponding set in the plane. We observe that (7), (8) is essentially our original system.

It was also proven in [1] the following

Theorem 1 *(Permutability theorem) Let w_{12}, y_{12} be the solution of (7),(8) obtained from the Bäcklund transformation following the sequence*

$$(w_0, y_0) \to_{\eta_1} (w_1, y_1) \to_{\eta_2} (w_{12}, y_{12})$$

and (w_{21}, y_{21}) the solution following the sequence

$$(w_0, y_0) \to_{\eta_2} (w_2, y_2) \to_{\eta_1} (w_{21}, y_{21}).$$

Then $w_{12} = w_{21}, y_{12} = y_{21}$ and

$$w_{12} - w_0 = \frac{24(\eta_1 - \eta_2)(w_1 - w_2)}{(w_1 - w_2)^2 - \lambda (y_1 - y_2)^2}$$

$$y_{12} - y_0 = \frac{-24(\eta_1 - \eta_2)(y_1 - y_2)}{(w_1 - w_2)^2 - \lambda (y_1 - y_2)^2}.$$

We notice that the denominator in the above formulas is the same expression which appears in the assumptions given for the Bäcklund transformation. This condition is necessary in order to have regular solutions. In the case $\lambda = -1$, the denominator becomes

$$(w_1 - w_2)^2 + (y_1 - y_2)^2. \tag{9}$$

3. On the explicit solutions from the Bäcklund transformation

In [1] the following set of solutions for (1), (2) were found, using the Bäcklund transformation : for any $\eta > 0$ and $\lambda = -1$

$$u(x,t) = 4\eta \frac{\left[1 - \frac{3\mathcal{C}}{\eta A} \cosh(ax + b)\right]}{\left[\cosh(ax + b) - \frac{3}{\eta}\frac{\mathcal{C}}{A}\right]^2}, \quad v(x,t) = -\frac{\rho}{A} a \frac{\sinh(ax + b)}{\left[\cosh(ax + b) - \frac{3}{\eta}\frac{\mathcal{C}}{A}\right]^2}, \tag{10}$$

for any $\eta < 0$ and $\lambda = -1$

$$u = \frac{-4|\eta| \left(1 + \frac{3\mathcal{C}\epsilon}{|\eta|\hat{A}} \cos(ax + b)\right)}{\left(\epsilon \cos(ax + b) + \frac{3\mathcal{C}}{|\eta|\hat{A}}\right)^2}, \quad v = \frac{\rho a}{\hat{A}} \frac{\epsilon \sin(ax + b)}{\left(\epsilon \cos(ax + b) + \frac{3\mathcal{C}}{|\eta|\hat{A}}\right)^2} \tag{11}$$

and finally for $\eta = 0$ and $\lambda = -1$ (and for any value of the parameters $\mathcal{C}, H$ and $\rho \neq 0$)

$$u = w_x = \frac{C^2 \frac{\rho^2}{12} - 12C^4(x+H)^2}{\left[\left(\frac{\rho}{12}\right)^2 + C^2(x+H)^2\right]^2}, \quad v = y_x = \frac{-2C^3\rho\,(x+H)}{\left[\left(\frac{\rho}{12}\right)^2 + C^2(x+H)^2\right]^2}. \tag{12}$$

The solutions are solitonic, periodic and static (independent of t) respectively. For the details about the parameters involved in the solutions see [1].

We can also use the expressions given for the permutability theorem and the explicit formulas for w_{12}, y_{12} to generate new solutions from old ones with respect our original system (1), (2), in particular new multisolitonic solutions.

In distinction with what occurs with the Bäcklund transformation for the KdV equation, we can use directly w_{12} and y_{12} in terms of the regular solutions presented before to find new regular solutions of the coupled KdV system. In fact, the denominator in the expressions for w_{12}, y_{12} is manifestly positive and $\mathcal{C}^\infty$. Moreover, we are going to show that if we choose adequately the parameters of the given solitonic solutions the denominator in the expressions for w_{12}, y_{12} is strictly positive and the new solutions are regular solutions.

An interesting result about the regularity of the solutions, in the case of solitonic ones, is the following (see also [1])

Theorem 2 *For any value of the parameters $\eta_1 > 0, \rho_1 \neq 0, \mathcal{C}_1$ and $\eta_2 > 0, \rho_2 \neq 0, \mathcal{C}_2$ such that $\eta_1 \neq \eta_2$ and*

$$\frac{\mathcal{C}_1}{\eta_1 \rho_1} = \frac{\mathcal{C}_2}{\eta_2 \rho_2}$$

the solutions for the coupled KdV system with $\lambda = -1$ obtained from the permutability formulas are regular.

4. About the static solutions for the parametric coupled KdV system

In the most simple case of (3),(4),(5) and (6) the Bäcklund transformation reduces to

$$w_x = 2\eta - \frac{1}{12}w^2 \tag{13}$$

$$w_t = \frac{1}{6}w w_{xx} - \frac{1}{3}{w_x}^2. \tag{14}$$

Clearly, we have started with the trivial solution and we have assumed that the field y is identically zero. We have thus the Bäcklund transformation for the scalar KdV equation given by

$$u_t + u u_x + u_{xxx} = 0. \tag{15}$$

For $\eta = 0$ and assuming $w_t = 0$, that is, w is only a function of the spatial variable x we get from (13) $w_{xx} = -\frac{1}{6} w w_x$ and substituting this in (14) we get $-\frac{w_x}{3}\left(\frac{w^2}{12} + w_x\right) = 0$. The conclusion is that for this special type of solutions equation (14) gives only known information about equation (13).

We also have that if w is a solution of (13) (for $\eta = 0$) then $w_x = u$ satisfies $\frac{u^2}{2} + u_{xx} = 0$ and this automatically implies that $u_t = 0$, because we have

$$u_t + \left(\frac{u^2}{2} + u_{xx}\right)_x = 0.$$

For example, a static solution for the equation (15) in the case of $\eta = 0$ can be obtained directly integrating the equation $w_x = -\frac{1}{12}w^2$, giving

$$w = -\frac{12}{x+K},$$

where K is a constant of integration. We note however that this solution is a singular solution, making a substantial difference with the solutions given by (12) (for the case $\lambda = -1$ and $\eta = 0$), which are regular solutions.

This argument can be generalized to the Bäcklund transformation (3),(4),(5),(6). In fact, the result about the static solution can be extended as follows

Lemma 2 *Let w, y a solution of the parametric equations*

$$w_x = -\frac{1}{12}\left(w^2 + \lambda y^2\right) \tag{16}$$

$$y_x = -\frac{1}{6}wy. \tag{17}$$

Then $u = w_x, v = y_x$ is a static solution of equations (1),(2).

Proof of Lemma 1 *From (16),*

$$\begin{aligned} u_x &= w_{xx} = -\frac{1}{12}\left(2ww_x + 2\lambda yy_x\right) = \frac{w^3}{72} + \frac{3}{72}\lambda y^2 w \\ u_{xx} &= \frac{3}{72}w^2 w_x + \frac{3}{72}\lambda\left(2yy_x w + y^2 w_x\right) = -\frac{1}{288}\left(w^2+\lambda y^2\right)^2 - \frac{\lambda}{72}y^2w^2. \end{aligned}$$

Then (1) becomes

$$u_t + \left(\frac{u^2}{2} + u_{xx} + \lambda\frac{v^2}{2}\right)_x = 0, \tag{18}$$

and when replacing the previous results we obtain

$$\frac{u^2}{2} + u_{xx} + \lambda\frac{v^2}{2} = 0.$$

Consequently any solution of the system (16),(17) is a static solution of (1). In the same way we can check that equation (2) is satisfied.

Remark: This result generalizes the formulas given by (12) for static solutions of the system (1),(2).

5. The supersymmetric extension

The parametric coupled KdV system (1),(2) has a supersymmetic extension. It can be constructed from the $N = 1$ supersymmetric KdV equation [5] by considering the fields valued on a Z^λ algebra [6] generated by e_1, e_2:

$$e_1 \circ e_1 = e_1, e_1 \circ e_2 = e_2, e_2 \circ e_2 = \lambda e_1. \tag{19}$$

If we express the super KdV fields $U = ue_1 + ve_2, \xi = \mu e_1 + \nu e_2$ then the new supersymmetric transformations becomes

$$\begin{aligned} \delta u &= \epsilon_1\mu' + \lambda\epsilon_2\nu', \ \delta v = \epsilon_1\nu' + \epsilon_2\mu' \\ \delta\mu &= \epsilon_1 u + \lambda\epsilon_2 v, \ \delta\nu = \epsilon_1 v + \epsilon_2 u, \end{aligned} \tag{20}$$

where ϵ_1 and ϵ_1 are two independent odd Grassmann supersymmetric parameters.

The supersymmetric extension is

$$\begin{aligned} u_t &= -u''' + 6uu' + 6\lambda v'v - 3\mu\mu'' - 3\lambda\nu\nu'' \\ v_t &= -v''' + 6(vu)' - 3\mu\nu'' - 3\nu\mu'' \\ \mu_t &= -\mu''' + 3(u\mu + \lambda v\nu)' \\ \nu_t &= -\nu''' + 3(u\nu + v\mu)'. \end{aligned} \tag{21}$$

This system is invariant under the supersymmetric transformation (20).

In particular, if $\lambda = 0$ the system (1),(2) corresponds to one of the Hirota-Satsuma integrable equations [2, 7]. The system (21) for $\lambda = 0$ becomes its supersymmetric extension.

The system (21), for any value of λ, has an infinite sequence of local conserved quantities. It is a new integrable system. In particular if we consider the Grassmann algebra to have only one generator e, then the system (21), for $\lambda = 0$, reduces to

$$\begin{aligned} u_t &= -u''' + 6uu' \\ \mu_t &= -\mu''' + 3(u\mu)' \\ v_t &= -v''' + 6(vu)' \\ \nu_t &= -\nu''' + 3(u\nu + v\mu)'. \end{aligned} \tag{22}$$

If in addition, we take $v = \nu = 0$, the system becomes

$$\begin{aligned} u_t &= -u''' + 6uu' \\ \mu_t &= -\mu''' + 3(u\mu)'. \end{aligned} \tag{23}$$

In [8] we introduced the system

$$\begin{aligned} u_t &= -u''' - uu' - \tfrac{1}{4}\left(\mathcal{P}\left(\xi\bar{\xi}\right)\right)' \\ \xi_t &= -\xi''' - \tfrac{1}{2}(\xi u)'. \end{aligned} \tag{24}$$

It arises from a supersymmetric breaking of the $N = 1$ super KdV system. If we perform a scale transformation $\xi \rightarrow \alpha\hat{\xi}$, assuming that the Clifford algebra has only one generator, and take the limit $\alpha \rightarrow 0$ leaving $\hat{\xi}$ and μ finite we obtain the system (23). The interesting point is that (23) is a supersymmetric system, it has a hamiltonian structure and an infinite sequence of conserved quantities, consequently system (24) in the above limit has those properties. For $\alpha \neq 0$ (24) has only a finite number of conserved quantities. Hence in the $\alpha \rightarrow 0$ limit we obtain an enhancement of the symmetries of the system.

Consequently the system (23) has two hamiltonian structures, although one of it arises after a limit process. The hamiltonian structures are not compatible. This is a new aspect to be investigated since it is well known that supersymmetric KdV equations only have one local hamiltonian structure.

6. Conclusions

We discussed the integrability properties of a coupled KdV system [1]. We obtained a new generalization of the static solutions to the system. We also obtained new properties of the supersymmetric extension of the coupled KdV system, which is a new integrable system. For the particular case of a Grassmann algebra with only one odd generator the supersymmetric system is the limit of a Clifford valued system with nice stability properties. Hence we expect that the same stability properties will be valid for the supersymmetric system. It has two non-compatible hamiltonian structures one arising from its supersymmetric structure and the other one arising from the Clifford system in the limit procedure. This interesting aspect will be analyzed elsewhere.

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