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Truncated γ-exponential models for tidal stellar systems

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Abstract. We introduce a parametric family of models to characterize the properties of astrophysical systems in a quasi-stationary evolution under the incidence evaporation. We start from an one-particle distribution $f_\gamma\left(\mathbf{q},\mathbf{p}|\beta,\varepsilon_s\right)$ that considers an appropriate deformation of Maxwell-Boltzmann form with inverse temperature β, in particular, a power-law truncation at the scape energy ε_s with exponent $\gamma > 0$. This deformation is implemented using a generalized γ-exponential function obtained from the *fractional integration* of ordinary exponential. As shown in this work, this proposal generalizes models of tidal stellar systems that predict particles distributions with *isothermal cores and polytropic haloes*, e.g.: Michie-King models. We perform the analysis of thermodynamic features of these models and their associated distribution profiles. A nontrivial consequence of this study is that profiles with isothermal cores and polytropic haloes are only obtained for low energies whenever deformation parameter $\gamma < \gamma_c \simeq 2.13$. This study is a first approximation to characterize a self-gravitating system, so we consider equal to all the particles that constitute the system.

1. Introduction

Gravitation is an example of attractive interaction that is unable to confine particles. It is always possible that some particles acquire sufficiently energy and escape from gravitational influence of the system. Evaporation is an unavoidable process that drives dynamical evolution of astrophysical systems in Nature. Needless to say that it is a very important ingredient to explain structure and evolution of the so-called tidal stellar systems, such as globular clusters [1, 2]. In general, the effects of evaporation crucially depend on the relaxation mechanisms that are present in the microscopic dynamics, which depend on concrete conditions of a given particular system [3, 4, 5].

We propose in this work a parametric family of astrophysical models that accounts for the influence of evaporation under different relaxation regimes: the *truncated γ-exponential models* [6]. These phenomenological models constitutes a suitable generalization of some models of tidal stellar systems available in the literature, such as


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Wooley & Dickens truncated isothermal model [7], as well as King's models of globular clusters [8, 9] and Wilson models [10]. This proposal enable us to analyze the influence of the evaporation on the thermodynamical behavior and distribution profiles. Moreover, it provides a unification framework of the known isothermal and polytropic profiles considered in hydrodynamic models of stellar systems [11]. Remarkably, these models are sufficiently amenable to allow an analytical derivation of most of their associated hydrodynamic quantities.

The paper is organized into sections as follows. Second section is devoted to introduce the parametric family of quasi-stationary distributions, the derivation of their associated hydrodynamic quantities, as well as some details about the integration of the associated nonlinear Poisson equation. Afterwards, third section is devoted to discuss numerical results obtained from this model, both information concerning to thermodynamical description as well as distribution profiles. Some conclusions are drawn in forth section.

2. A parametric family of quasi-stationary one-body distributions

Observational evidences and theoretical analysis suggest that distribution profiles of many stellar systems can be explained considering a quasi-stationary one-particle distribution with a mathematical form close to Maxwell-Boltzmann distribution [12]:

$$f_{MB}\left(\mathbf{q},\mathbf{p}\right) \sim \exp\left[-\beta\varepsilon(\mathbf{q},\mathbf{p})\right], \tag{1}$$

with $\varepsilon(\mathbf{q},\mathbf{p}) = \mathbf{p}^2/2m + m\phi\left(\mathbf{q}\right)$ being the mechanical energy of a given particle with momentum $\mathbf{p}$ located at the position $\mathbf{q}$ with a mean field gravitational potential $\phi\left(\mathbf{q}\right)$. The incidence of evaporation introduces a deformation of the above distribution, specifically, a truncation for energies above a certain *escape energy* ε_s. Some examples of distributions that introduce this kind of deformation are truncated isothermal models of Wooley and Dickens [7], Michie-King models [8, 9] and Wilson models [10]. Quasi-stationary one-body distributions of all these models can be expressed using certain truncated exponential function $E_t(x)$ as follows:

$$f_{qs}(\mathbf{q},\mathbf{p}|\beta,\varepsilon_s) = AE_t(x), \tag{2}$$

where $x = \beta\left[\varepsilon_s - \varepsilon(\mathbf{q},\mathbf{p})\right]$. The function $E_t(x)$ adopts the following forms:

$$\begin{gathered} E_{WD}(x) = \exp(x)H(x),\ E_{MK}(x) = \left[\exp(x)-1\right]H(x) \\ E_{W}(x) = \left[\exp(x)-1-x\right]H(x) \end{gathered} \tag{3}$$

for each ones of these models, respectively, with $H(x)$ being the Heaviside step function.

Our interest is to propose a generalization of evaporation truncation scheme considered in the above astrophysical models. Firstly, let us express them into a unifying form as follows:

$$f_{\gamma}(\mathbf{q},\mathbf{p}|\beta,\varepsilon_s) = AE_{\gamma}(x), \tag{4}$$

where $E_\gamma(x)$ is a certain function obtained from the *truncation of exponential function* $\exp(x)$, with γ being a deformation parameter. At first glance, the truncation schemes

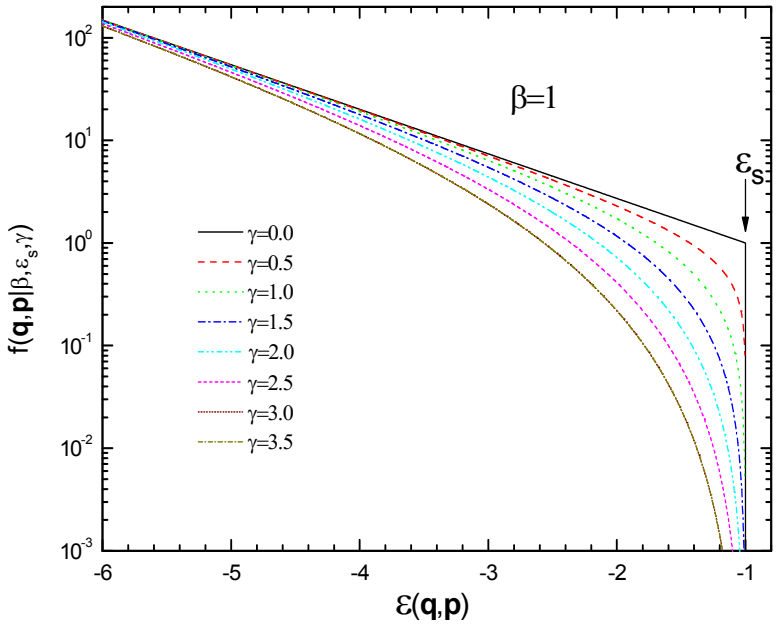


Figure 1. One-particle distributions *versus* mechanical energy $\varepsilon(\mathbf{q},\mathbf{p})$ corresponding to truncated γ-exponential model (4) for some values of deformation parameter γ. We have employed prefixed values $\varepsilon_s = -1$ and $\beta = 1$ for the scape energy and the inverse temperature, respectively.

(3) could be generalized for any nonnegative integer $\gamma = n$ subtracting the first n terms of power-expansion of exponential function:

$$E_n(x) = \begin{cases} \sum_{k=0}^{+\infty} x^{n+k}/(n+k)!, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Precisely, definitions (3) correspond to the particular cases $\gamma = 0$, $\gamma = 1$ and $\gamma = 2$. A more general definition for any nonnegative real number γ is the following:

$$E_\gamma(x) \equiv E\,(x;\gamma) = \sum_{k=0}^{\infty} \frac{x^{\gamma+k}}{\Gamma\,(\gamma+1+k)} \equiv \frac{1}{\Gamma(\gamma)} \int_0^x \eta^{\gamma-1} \exp(x-\eta) d\eta, \tag{6}$$

which is obtained replacing factorial function $(n+k)!$ by the Gamma function $\Gamma(\gamma+1+k)$. Hereinafter, this function will be referred to as γ-*exponential function*, and the same one will be equivalently denoted as $E_\gamma(x)$ or $E\,(x;\gamma)$. Strictly speaking, this function represents fractional integration of ordinary exponential function [13].

The proposed family of quasi-stationary distributions describe the influence of evaporation with two independent parameters: the escape energy ε_s and the deformation parameter γ. This family of distributions is schematically shown in figure 1. The same ones exhibit a Maxwell-Boltzmann profile (1) for energies $\varepsilon(\mathbf{q},\mathbf{p})$ that are far enough from below the escape energy ε_s. Besides, they show a *power-law truncation* at ε_s

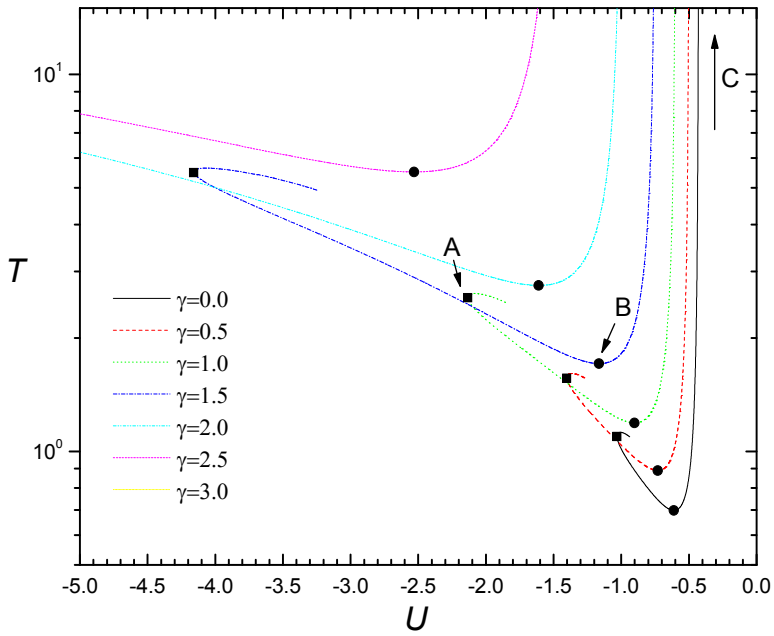


Figure 2. Behavior of the caloric curves T *versus* U. Additionally, we have included the following notable points: (squares) critical point of gravothermal collapse U_A; (circles) critical point of isothermal collapse U_B; (triangles) critical point of evaporation disruption U_C.

with exponent γ. The increasing of deformation parameter γ characterizes a larger deviation from Maxwell-Boltzmann profile (1). For a phenomenological viewpoint, a larger value of the deformation parameter γ describes a stronger influence of evaporation, or equivalently, the incidence of a weaker relaxation mechanism.

3. Structure equation and observables

Truncation due to evaporation leads to spherical-symmetric distributions that are located inside a region of finite radius R_t, the so-called *tidal radius* [1]. Rewriting the escape energy $\varepsilon_s = m\phi_s$ using the surface gravitational potential ϕ_s, one can express R_t as follows:

$$R_t = -GM/\phi_s, \tag{7}$$

where M is the system total mass. Introducing the dimensionless potential $\Phi = \Phi(\mathbf{q})$:

$$\Phi(\mathbf{q}) = \beta m \left[\phi_s - \phi(\mathbf{q})\right], \tag{8}$$

one can express the mass density ρ in characteristic units $\rho_c = M/R_t^3$ as follows:

$$\rho = \frac{\kappa}{\eta} E\left(\Phi; \gamma + \frac{3}{2}\right), \tag{9}$$

where η is the dimensionless inverse temperature:

$$\eta = \beta \frac{GMm}{R_t} \tag{10}$$

and κ a certain integration parameters defined below. Mathematical form of dimensionless potential is obtained from integration of the following nonlinear Poisson problem:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2 \frac{d}{d\xi}\Phi(\xi)\right] = -4\pi E\left[\Phi(\xi);\gamma+\frac{3}{2}\right]. \tag{11}$$

We have considered here the dimensionless radius:

$$\xi = \sqrt{\kappa} r/R_t, \tag{12}$$

where r is the physical radius, $\kappa = Gm^2\beta CR^2$ is an auxiliary constant, R_t is the tidal radius. Problem (11) can be solved in a numerical way by demanding the following conditions at the origin:

$$\Phi(0) = \Phi_0,\ \Phi'(0) = 0, \tag{13}$$

which is integrated till the vanishing of dimensionless potential at the surface of the system with dimensionless radius $\xi_c = \sqrt{\kappa}$:

$$\Phi(\xi_c) = 0, \Phi'(\xi_c) = -\frac{\eta}{\xi_c}. \tag{14}$$

The total energy $U = K + V$ can be obtained from the total kinetic K and total potential energy V:

$$\begin{aligned} V &= -\frac{1}{2} - \frac{1}{2\eta^2\xi_c}\int_0^{\xi_c} E\left[\Phi(\xi);\gamma+\frac{3}{2}\right]\Phi(\xi)\ 4\pi\xi^2 d\xi, \\ K &= \frac{3}{2\eta^2\xi_c}\int_0^{\xi_c} E\left[\Phi(\xi);\gamma+\frac{5}{2}\right] 4\pi\xi^2 d\xi, \end{aligned} \tag{15}$$

which are expressed in characteristic units $U_c = GM^2/R_t$.

4. Results and discussions

According the asymptotic behavior of truncated γ-exponential function, the particles density (9) describes a power-law dependence for Φ small with exponent $n = \gamma + 3/2$, while an exponential dependence for Φ large enough:

$$\rho(\Phi;\gamma) = \begin{cases} \propto \Phi^n, & \Phi << 1, \\ \propto \exp(\Phi), & \Phi >> 1. \end{cases} \tag{16}$$

Consequently, this family of models unifies the known isothermal and polytropic profiles of hydrodynamic astrophysical models [1, 11]. Since the deformation parameter γ is nonnegative and the polytropic profiles are unstable for $n \geq 5$ [11], the admissible values of the polytropic exponent n are restricted to the interval $3/2 \leq n < 5$, which correspond

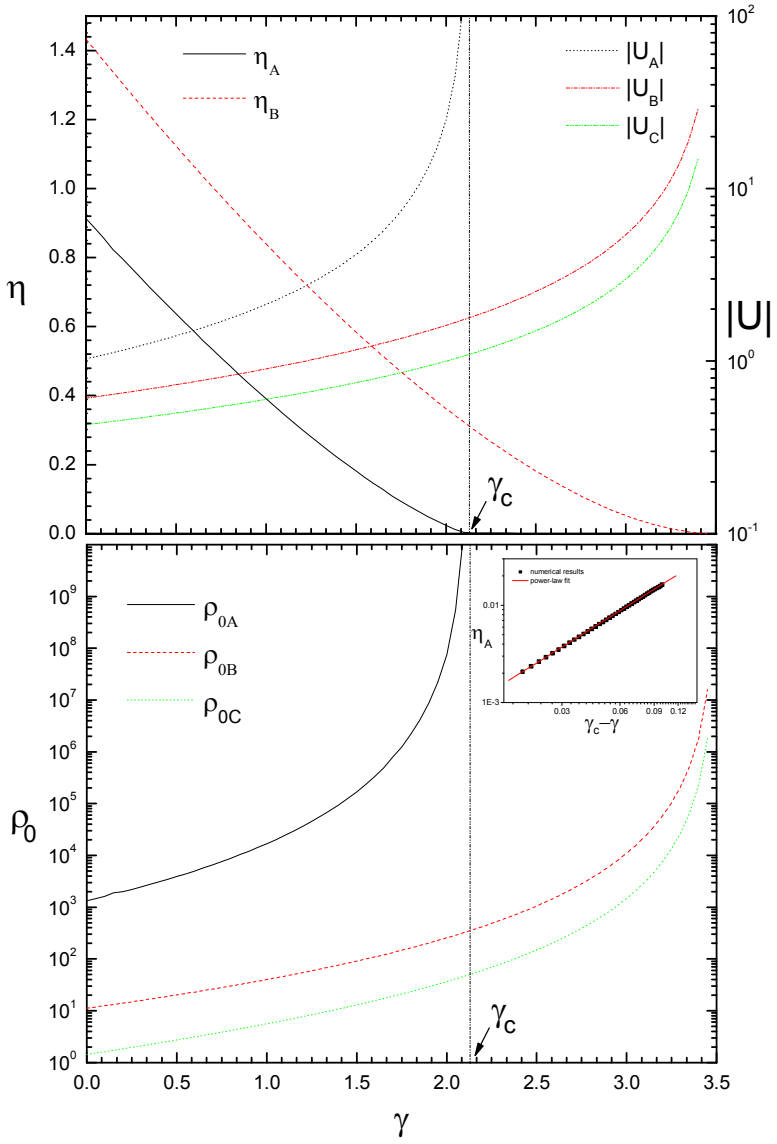


Figure 3. Thermodynamic quantities associated with the notable points *versus* deformation parameter γ: Panel a) the inverse temperatures (η_A, η_B) and the modulus of energies (U_A, U_B, U_C); Panel b) the central particles densities $(\rho_{0A}, \rho_{0B}, \rho_{0C})$. Inset: A power-law fit $\eta_A(\gamma) = A\,|\gamma_c - \gamma|^{\alpha}$ that provides a more precise estimation for the notable value $\gamma_c = 2.1310.003$

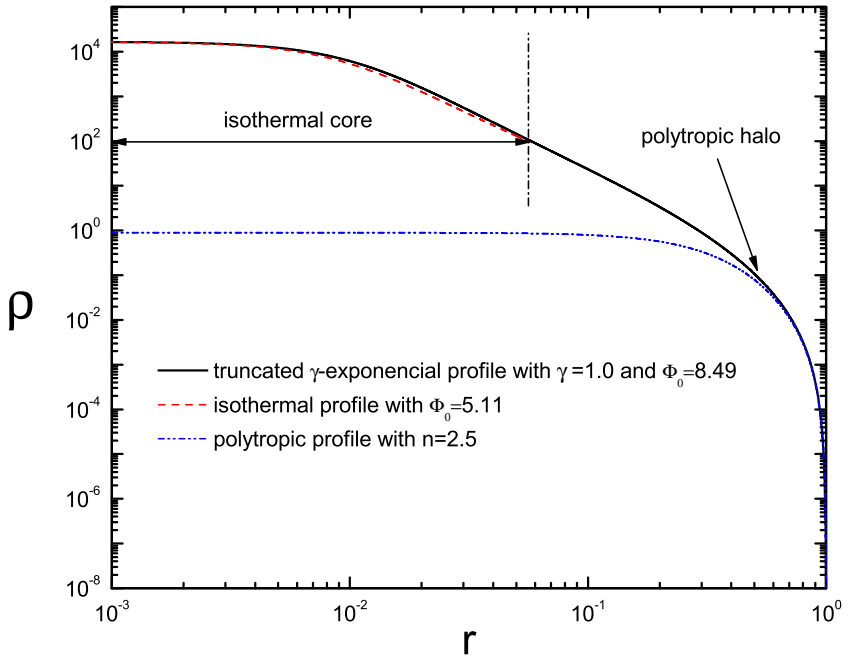


Figure 4. Truncated γ-exponential profile with $\gamma = 1$ (Michie-King profile) corresponding to the point of gravothermal collapse and its comparison with isothermal and polytropic profiles using *log-log scales*. Accordingly, the proposed family of models can describe distribution profiles that exhibit isothermal cores and polytropic haloes.

to the interval $0 \leq \gamma < 7/2$ for the deformation parameter γ. This restriction will be manifested in the thermodynamic quantities and distribution profiles.

Dependencies of the temperature T *versus* the dimensionless energy U are shown in figure 2 for different values of deformation parameter γ. All quasi-stationary configurations obtained from these models have always negative values for the energy U. Moreover, one can recognize the existence of three notable points: the critical point of gravothermal collapse U_A, the critical point of isothermal collapse U_B and the critical point of evaporation disruption U_C [4, 5]. All quasi-stationary configurations are located inside the energy region $U_A \leq U \leq U_C$. Stable quasi-stationary configurations inside the energy range $U_A \leq U \leq U_B$ exhibit *negative heat capacities* $C < 0$, which is a distinguished feature of thermodynamics of astrophysical systems [14, 15]. For the

sake of a better understanding about the influence of deformation parameter γ, we have calculated dependence of some thermodynamic quantities at the notable points. Such results are shown in figure 3. The increasing of the deformation parameter γ provokes a systematic decreasing of inverse temperatures (η_A, η_B), and the increasing of absolute values of energies (U_A, U_B, U_C) and their associated central particles densities $(\rho_{0A}, \rho_{0B}, \rho_{0C})$. Interestingly, the inverse temperature η_A at the notable point of gravothermal collapse vanishes when $\gamma \geq \gamma_c \simeq 2.13$. This means that both the total energy U_A and the temperature T_A diverge at this point as a consequence of divergence of the central density ρ_A. When γ is below the point γ_c the system develops a gravothermal collapse at the finite energy U_A and when the deformation parameter γ is above the point γ_c, the system should release an infinite amount of energy. Additionally, one observes a divergence in the other thermodynamic variables when the deformation parameter γ approaches its maximum admissible value $\gamma_m = 7/2$, which is related to instability of polytropic profiles for $n \geq 5$.

We show in figure 4 a distribution profile with deformation parameter $\gamma = 1$ at the point of gravothermal collapse, which corresponds to a Michie-King profile with lowest energy. As clearly shown in this figure, the inner region, the core, can be well-fitted with an isothermal profile [14], while the outer one, the halo, is well-fitted with a polytropic profile [11]. Isothermal cores disappears for energies sufficiently large, so that, distribution profiles are practically polytropic. For a better illustration, dependence of distribution profiles on the deformation parameter γ and the internal energy U is shown in figure 5, specifically, twelve profiles corresponding to three notable points and forth different values of deformation parameter γ. Particles concentration in the inner regions decreases with the increasing of the internal energy U. The increasing of the deformation parameter γ produces distribution profiles with more diluted haloes, and consequently, more dense cores.

We have shown in figure 6 what is considered as the phase diagram of truncated γ-exponential models in the plane $\gamma - \Phi_0$. For each value of deformation parameter γ, the admissible values of the central dimensionless potential Φ_0 are located inside the interval $0 \leq \Phi_0 \leq \Phi_A(\gamma)$, where $\Phi_A(\gamma)$ corresponds to the critical point of gravothermal collapse U_A. Central values of dimensionless potential Φ_0 above dependence $\Phi_A(\gamma)$ correspond to nonphysical on unstable configurations (white region). Additionally, we have included dependence $\Phi_B(\gamma)$ associated with the point of isothermal collapse U_B. Configuration between dependencies $\Phi_B(\gamma) \leq \Phi_0 \leq \Phi_A(\gamma)$ exhibit negative heat capacities (dark gray region), while those one with central dimensionless potential Φ_0 belonging to interval $0 < \Phi_0 < \Phi_B(\gamma)$ exhibit positive heat capacities (light gray region). It is remarkable that dependence $\Phi_A(\gamma)$ is weakly modified by a change in the deformation parameter γ for values below the notable point γ_c. However, this function experiences an sudden change above this notable value. As expected, this behavior accounts for a sudden change in behavior of distribution profiles: the proposed models can exhibit profiles with isothermal cores for $\gamma < \gamma_c$, while they only exhibit profiles without isothermal cores for $\gamma \geq \gamma_c$.

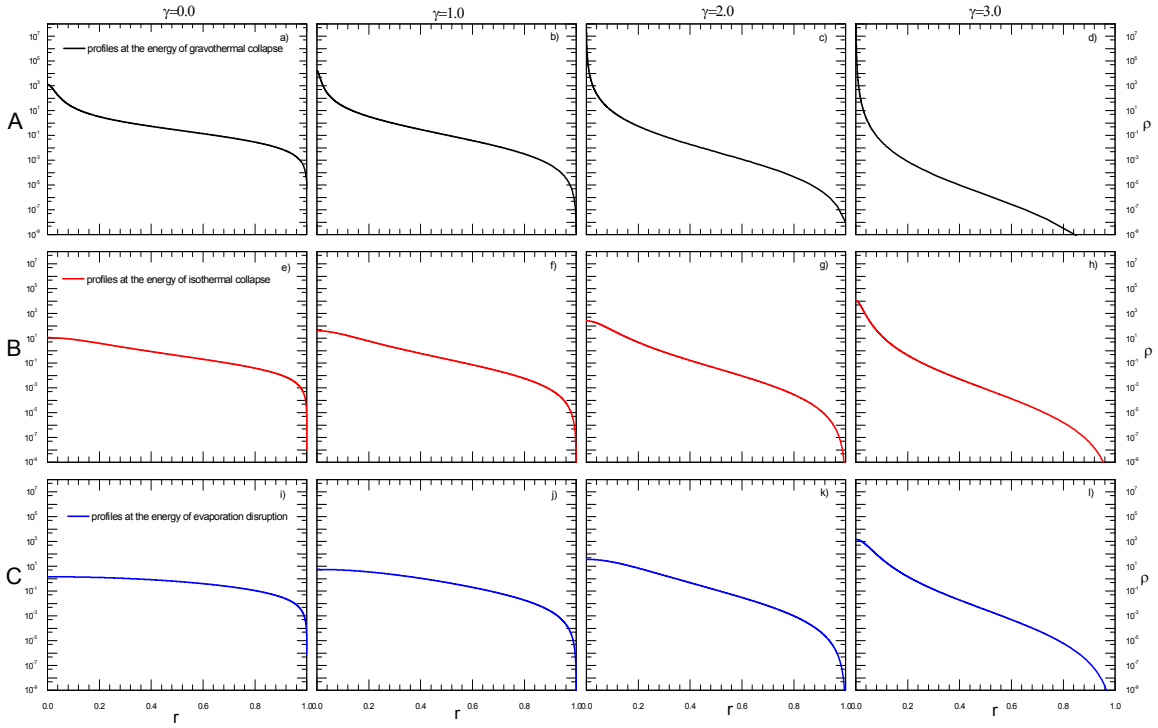


Figure 5. Dependence of distribution profiles on the deformation parameter γ for three notable values of the internal energy U. Panels a)-c) Distribution profiles at the energy of gravothermal collapse U_A for three values of deformation parameter with $\gamma < \gamma_c$. All them exhibit isothermal cores and polytropic haloes. Panel d) A distribution profile at gravothermal collapse with deformation parameter $\gamma > \gamma_c$. Note that this profile does not exhibit an isothermal core. Panels i)-l) Distribution profiles very near the energy of evaporation disruption U_C are everywhere polytropic. Panels e)-h) Transitional profiles at the energy of isothermal collapse U_B. These profiles hardly differ from polytropic profiles i)-l) because of they exhibit more dense cores.

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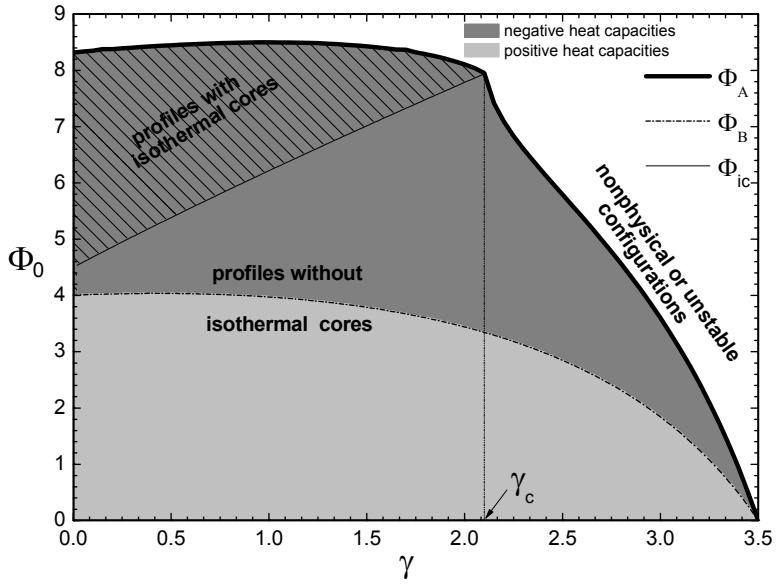


Figure 6. Phase diagram of truncated γ-exponential models in the plane of integration parameters $\gamma - \Phi_0$.

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