

Solution of conharmonic curvature tensor in General Relativity

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Abstract. Conharmonically flat space time is studied in the frame work of FRW cosmological model. Einstein's field equations with variable cosmological constant are solved by using the law of variation of cosmological constant that scales as a^{-m} (where a is scale factor and m is a positive constant). A new class of exact solutions of the field equations has been obtained in which cosmological constant decreases with cosmic time. Physical and geometrical properties are also discussed in detail.

1. Introduction

The importance of conharmonic curvature tensor is very well known [1, 2] which is given as

$$L_{ijk}^h = R_{ijk}^h - \frac{1}{(n-2)}(g_{ij}R_k^h - g_{ik}R_j^h + \delta_k^h R_{ij} - \delta_j^h R_{ik}) \quad (1)$$

where R_{ijk}^h is the Riemannian curvature tensor and R_{ij} is the Ricci tensor. A space time whose conharmonic curvature tensor vanishes at each point is called conharmonically flat space time. Conharmonic curvature tensor represents the deviation of the space time from conharmonic flatness. Ahsan and Siddiqui [3], Siddiqui and Ahsan [4] have studied the significance of conharmonic curvature tensor in general theory of relativity. The significance of the space of constant curvature is very well known in cosmology. Due to cosmological principle, we have in the rest system of matter, there is no preferred point and no preferred direction, the three dimensional space being constituted in the same way everywhere. Narlikar [5] defines that the cosmological solutions of Einstein's field equations which contain a three dimensional space-time surface of a constant curvature are the Robertson-Walker metrics, while a four dimensional space of constant curvature is the deSitter model of the universe. deSitter model possesses a three dimensional space of constant curvature and this belongs to Robertson-Walker metrics. Recently Kumar and Shrivastava [6] studied FRW cosmological model for conharmonically flat space time. Einstein's field equations with variable cosmological term are solved by using the law of variation of Hubble Parameter. In this paper, we consider FRW space time filled with perfect fluid in presence of time dependent cosmological constant. To solve the Einstein's field equations we assume that cosmological constant scales with a^{-m} (where a is scale factor and m is a positive constant) [7]. The paper is organized as follows : In Sec. 2, the model and field equations have been presented. The field equations have been solved in Sec. 3 by using the decaying law of Λ . Exact R-W models have been obtained in Sec. 3.1, 3.2 and 3.3. The physical behavior of the models have been discussed in detail. In the last section i.e. in Sec. 4 concluding remark have been expressed.



2. Field equations

The Friedman-Robertson-Walker (FRW) universe in four dimensional space-time is described by the following metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad (2)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $a(t)$ represents the scale factor. The θ and ϕ parameters are the usual azimuthal and polar angles of spherical co-ordinates, with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The coordinates (t, r, θ, ϕ) are called comoving coordinates. The constant k denotes the curvature of the space. When $k = 1$, the universe is closed and is open for $k = -1$ whereas $k = 0$ represents flat universe.

The Einstein's field equations are given by

$$R_{\mu\nu} - \frac{1}{2}R_k^k g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \quad (3)$$

We assume that the energy momentum tensor of a perfect field is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \quad (4)$$

We take the equation of state as

$$p = \omega\rho, \quad 0 \leq \omega \leq 1 \quad (5)$$

where p and ρ are respectively pressure and energy density of the cosmic fluid and U_μ is unit flow vector such that $U_\mu = (1, 0, 0, 0)$. The conharmonic curvature tensor for four dimensional space time is

$$L_{ijk}^h = R_{ijk}^h - \frac{1}{2}(g_{ij}R_k^h - g_{ik}R_j^h + \delta_k^h R_{ij} - \delta_j^h R_{ik}) \quad (6)$$

For conharmonically flat space time, we have

$$g_{ij}R_k^h - g_{ik}R_j^h + \delta_k^h R_{ij} - \delta_j^h R_{ik} = 2R_{ijk}^h \quad (7)$$

Contracting this with $h = j$ and summing over j , we have

$$R_{ik} = -\frac{1}{4}R_l^l g_{ik} \quad (8)$$

Thus for conharmonically flat space time, the Einstein's field equations (3) reduce to

$$R_{\mu\nu} = -\frac{8\pi}{3}T_{\mu\nu} + \frac{\Lambda}{3}g_{\mu\nu} \quad (9)$$

For the metric (2) and energy momentum tensor (4) the Einstein's field equations (9) reduce to

$$8\pi\rho + \Lambda = -9(\dot{H} + H^2) \quad (10)$$

$$8\pi p + \Lambda = -3(\dot{H} + 3H^2 + \frac{2k}{a^2}) \quad (11)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Here and elsewhere an over-head dot denotes ordinary differentiation with respect to time t .

3. Solution of field equations

The system of equations (5), (10) and (11) supply only three equations in four unknowns (a, p, ρ, Λ). One extra equation is needed to solve the system completely. The phenomenological Λ decay scenarios have been considered by a number of researches Chen and Wu [8] considered $\Lambda \propto a^{-2}$ (a is scale factor of the Robertson-Walker model), Hoyle et. al [9] considered $\Lambda \propto a^{-2}$ while $\Lambda \propto a^{-m}$ was considered by Olsan and Jordan [10]; Pavon [11], Maia and Silva [12], Silveira and Waga [13, 14] . Thus we take the decaying vacuum energy density

$$\Lambda = \frac{\sigma}{a^m} \quad (12)$$

where σ and m are positive constants, For $k = 0$, from equations (5), (10), (11) and (12) one finds that

$$\frac{\ddot{a}}{a} + \frac{2}{1+3\omega} \left(\frac{\dot{a}}{a}\right)^2 + \frac{(1+\omega)}{3(1+3\omega)} \sigma a^{-m} = 0 \quad (13)$$

On integration of (13), we have the scale factor

$$a = \left[\frac{m}{2} \sqrt{\frac{2\sigma(1+\omega)}{3(3m\omega - 6\omega + m - 6)}} t + t_0 \right]^{\frac{2}{m}} \quad (14)$$

where t_0 is a constant of integration. The integration constant is related to the choice of origin of time.

Now we analyze scenarios for different values of ω .

3.1. Matter dominated solution (Cosmology for $\omega = 0$)

For $\omega = 0$, from equation (14), we obtain the scale factor

$$a(t) = \left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]^{\frac{2}{m}} \quad (15)$$

In this case, the spatial volume V , matter density ρ pressure p and cosmological constant Λ are given by

$$V = \left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]^{\frac{6}{m}} \quad (16)$$

$$\rho = \frac{m\sigma}{4\pi(m-6)} \left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]^{-2} \quad (17)$$

$$p = 0 \quad (18)$$

$$\Lambda = \frac{\sigma}{\left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]^2} \quad (19)$$

The expansion scalar θ and density parameter Ω are given by

$$\theta = 3 \sqrt{\frac{2\sigma}{3(m-6)}} \frac{1}{\left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]} \quad (20)$$

$$\Omega = \frac{8\pi\rho}{3H^2} = m \quad (21)$$

The deceleration parameter q for the model is

$$q = \frac{m}{2} - 1 \quad (22)$$

The vacuum energy density ρ_v and critical density ρ_c are given by

$$\rho_v = \frac{\Lambda}{8\pi} = \frac{\sigma}{8\pi \left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]^2} \quad (23)$$

$$\rho_c = \frac{3H^2}{8\pi} = \frac{\sigma}{4\pi(m-6) \left[\frac{m}{2} \sqrt{\frac{2\sigma}{3(m-6)}} t + t_0 \right]^2} \quad (24)$$

Here we observe that the spatial volume V is zero at $t = t'$, where $t' = -\frac{2t_0}{m\sqrt{\frac{2\sigma}{3(m-6)}}}$ and expansion scalar θ is infinite, which shows that the universe starts evolving with zero volume at $t = t'$ with an infinite rate of expansion. At $t = t'$, the scale factor vanishes therefore the space time exhibits point type singularity at initial epoch. The energy density and cosmological constant diverge at the initial singularity. As t increases, the scale factor and spatial volume increase but the expansion scalar decreases. Thus the rate of expansion slows down with increasing time. Also the cosmological parameters (ρ, ρ_v, ρ_c) and Λ decreases as t increases. As $t = \infty$, the scale factor and volume become infinite whereas (ρ, ρ_v, ρ_c) and Λ tend to zero. Therefore the model would essentially give an empty universe for large time t .

Furthermore, we observe that $\Lambda \propto \frac{1}{t^2}$ which follows from the model of Kalligas et al [15]; Berman and Som [16]; Berman [17]; Berman et al [18] and Bertolami [19,20]. This form of Λ is physically reasonable as observations suggests that Λ is very small in the present universe.

3.2. Zeldovich fluid distribution (Cosmology for $\omega = 1$)

In this case, from equation (14), the scale factor $a(t)$ becomes

$$a(t) = \left[\frac{m}{2} \sqrt{\frac{\sigma}{3(m-3)}} t + t_0 \right]^{\frac{2}{m}} \quad (25)$$

Here spatial volume V , matter density ρ , pressure p and cosmological constant Λ are

$$V = \left[\frac{m}{2} \sqrt{\frac{\sigma}{3(m-3)}} t + t_0 \right]^{\frac{6}{m}} \quad (26)$$

$$\rho = p = \frac{m\sigma}{6\pi(m-3)} \left[\frac{m}{2} \sqrt{\frac{\sigma}{3(m-3)}} t + t_0 \right]^{-2} \quad (27)$$

$$\Lambda = \frac{\sigma}{\left[\frac{m}{2} \sqrt{\frac{\sigma}{3(m-3)}} t + t_0 \right]^2} \quad (28)$$

The expansion scalar θ and deceleration parameter q are given by

$$\theta = 3 \sqrt{\frac{\sigma}{3(m-3)}} \frac{1}{\left[\frac{m}{2} \sqrt{\frac{\sigma}{3(m-3)}} t + t_0 \right]} \quad (29)$$

$$q = \frac{m}{2} - 1 \quad (30)$$

In this case, we observe that the spatial volume V is zero at $t = \frac{-2t_0}{m\sqrt{\frac{2\sigma}{3(m-3)}}}$ and expansion scalar θ is infinite at $t = t''$, which shows that the universe starts evolving with zero volume and infinite rate of expansion initially. At $t = t''$, the energy density ρ , pressure p , cosmological constant Λ are infinite. As t increases the spatial volume increases but the expansion scalar decreases. Thus the expansion rate decreases as time increases. As t tends to ∞ the spatial volume V becomes infinitely large. As t increases, all the parameters ρ, p, Λ, θ decrease and tend to zero asymptotically. Therefore the model essentially gives an empty universe for large values of t .

3.3. Radiation dominated solution (Cosmology for $\omega = \frac{1}{3}$)

In this case, behavior of the cosmological model is same as in Sec. 3.2 which we have already discussed.

4. Conclusion

We have investigated Conharmonically flat space time in the frame work of FRW cosmological model. Einstein's field equations with variable cosmological constant are solved by using the law of variation of cosmological constant that scales as a^{-m} (where a is scale factor and m is a positive constant) suggested by Olsan and Jordan [10], Pavon [11], Maia and Silva [12], Silveira and Waga [13, 14]. In all the cases, we have found that the cosmological term being very large at initial time decreases as time increases which is in accordance with the observations.

References

- [1] Mishra R 1984 *Structure on a Differential Manifold and their Applications* (Chandrama Prakashan Allahabad)
- [2] Yano K, Kon M 1984 *Structure on Manifolds* (World Scientific publishing co. Singapore)
- [3] Ahsan Z, Siddiqui S 2009 *International J. Theor. Phys.* **48** 3202-3212
- [4] Siddiqui S, Ahsan Z 2012 *Differ. Geom. Dyn. Syst.* **12** 213-220 (2012)
- [5] Narlikar J 1978 *General Relativity and Gravitation* (The Macmillan co. of India)
- [6] Kumar R, Shrivastava S 2012 *Int. J. Theor. Phys.* doi:10.1007/5/10773-012-1364-7
- [7] Tiwari R 2010 *Research in Astrophysics and Astronomy* **10** 4
- [8] Chen W and Wu Y 1990 *Phys. Rev. D* **41** 695
- [9] Hoyle F, Burbidge G and Narlikar J 1997 *MNRAS* **286** 173
- [10] Olsan T and Jordan T 1987 *Phys. Rev. D* **35** 3258
- [11] Pavon D 1991 *Phys. Rev. D* **43** 375
- [12] Maia M and Silva G 1994 *Phys. Rev. D.* **50**, 7233
- [13] Silveira V and Waga I 1994 *Phys. Rev. D* **50** 4890
- [14] Silveira V and Waga I 1997 *Phys. Rev. D* **56** 4625
- [15] Kalligas D, Wesson P and Everitt C 1989 *General Relativity and Gravitation* **24** 351 (1992).
- [16] Berman M, Som M and Gomide F 1989 *Gen. Rel. Gravit.* **21** 287
- [17] Berman M 1990 *Int. J. Theor. Phys.* **29** 567
- [18] Berman M and Som M 1990 *Int. J. Theor. Phys.* **19** 1411
- [19] Bertolami O 1986 *Nuovo Cimento* **93** 36
- [20] Bertolami O 1986 *Fortschr. Phys.* **34** 829