

# Retardation effect in the recoil corrections to the fine shift of $S$ energy levels of hydrogen-like atoms

S Churochkina and A Udalova

Saratov State University, 410012, Astrakhanskaya St. 83, Saratov, Russia

E-mail: klechshevskaya@mail.ru

**Abstract.** Contributions of recoil effects to the fine shift of  $S$  energy levels of hydrogen-like atoms have been considered. The detailed analysis of the influence of retardation effect on the value of the logarithmic in respect to the mass ratio correction from the two photon diagram of sequential exchange by Coulomb and transverse photons has been realized.

## 1. Introduction

The problem of refinement of calculations of recoil corrections for the energy levels of hydrogen-like atoms (HLA) has attracted a lot of attention during last few years [1–9]. Being simplest possible bound state in quantum electrodynamics (QED) the HLA allows to verify experimentally all fundamental QED predictions. On the other hand, recently, owing to the breakthrough in the Doppler-free two-photon spectroscopy [10] the value of the frequency of key for the theory of  $1S$ - $2S$  transition in the hydrogen atom has been evaluated with the precision up to 10th Hz [11]

$$f_{1S-2S} = 2\,466\,061\,413\,187\,018\,(11)\,\text{Hz}. \quad (1)$$

And, it certainly challenges all theoretical QED predictions [1,8,9]. That is why, a few extended reviews (see for example [9]) devoted to the theoretical description of the HLA have been published. In this regards, the problem of the achievement of the consistency between the modern experimental measurements of the fine shift of energy levels of HLA and the corresponding theoretical results has been marked as one out of the most stimulating issues for the theory.

Indeed, it has been established that neither the Dirac equation nor the Schrödinger one suffice for the full description of the HLA, because of its bound state nature. In this case, the QED field-theoretical methods [12–17] developed for bound states required to describe such phenomena as: Lamb shift, both fine shift and hyperfine splitting corrections due to the radiation and recoil effects. It is worth mentioning that, historically, the treatment of the two particle bound state problem had been done firstly on the ground of the Bethe-Salpeter equation [12]. Afterwards, different methods based on the field-theoretical quasipotential approach (QA) emerged [18,19].

As it is known, all electrodynamic corrections to the energy levels of HLA may be written in the form of the series expansion in powers of three small parameters  $\alpha$ ,  $Z\alpha$  and  $m_1/m_2$  (where  $\alpha$  is the fine structure constant,  $Z$  is the charge of nucleus being considered,  $m_1$  and  $m_2$  are the masses of the light and the heavy particle, respectively) which determine the properties of the bound state. Corrections which depend on  $Z\alpha$  and  $m_1/m_2$  are called recoil corrections.



In particular, in this work we focus on the calculation of recoil corrections to the fine shift of  $S$  energy levels of HLA on the ground of QA. For the first time this problem was treated in [20]. The result was,

$$\Delta E = \frac{1}{\pi} \frac{(Z\alpha)^5 \mu^3}{m_1 m_2} \frac{1}{n^3} \left[ A + \frac{2}{3} \delta_{l0} \ln(Z\alpha)^{-1} - \frac{8}{3} \ln[k_0(n)] + \right. \\ \left. + \frac{2}{m_2^2 - m_1^2} \delta_{l0} \left( m_1^2 \ln \frac{m_2}{\mu} - m_2^2 \ln \frac{m_1}{\mu} \right) \right], \quad (2)$$

where  $n$  is the principal quantum number,  $l$  is the orbital quantum number,  $A$  is constant,  $\ln[k_0(n)]$  is the Bethe logarithm,  $\mu = \frac{m_1 m_2}{m_1 + m_2}$  is the reduced mass.

This recoil correction  $\Delta E$  is generated by the kern of interactions between two photons. In expression (2) there is a contribution linear in the logarithm of the mass ratio of particles

$$\Delta E^{\ln \beta} = \frac{2}{\pi} \frac{(Z\alpha)^5 \mu^3}{m_1 m_2} \frac{\beta^2}{n^3(1 - \beta^2)} \ln \beta^{-1}, \quad \beta = \frac{m_1}{m_2}. \quad (3)$$

It should be emphasized that, the evaluation of other logarithmic on the mass ratio contributions to the fine shift of energy levels of the HLA has not been counted entirely yet [21–29]. In our opinion, the improvement achieved in the precision of the measurement of the  $1S$ - $2S$  transition frequency increases the value of such estimation. And, even going deeply, in this article, we calculate the influence of the retardation effect on the value of the logarithmic in respect to the mass ratio correction from the two photon diagram of sequential exchange by Coulomb and transverse photons.

## 2. Retardation effect

The relativistic modification of the Coulomb interaction has to be required under the investigation of recoil effects in the HLA up to the order  $\frac{(Z\alpha)^6 \mu^3}{m_1 m_2} \beta \ln \beta^{-1}$ , i.e. integrals of the following type should be considered

$$i = \int \frac{d^3 q}{(q^2 + \alpha^2 \mu^2)^2 \mathbf{k}_q (\mathbf{k}_q + s)}, \quad \mathbf{k}_q = |\mathbf{k} - \mathbf{q}|. \quad (4)$$

The integral from (4) with  $s = 0$  gives the integral with Coulomb interaction

$$I = \int \frac{d^3 q}{(q^2 + \alpha^2 \mu^2)^2 \mathbf{k}_q^2}. \quad (5)$$

After integrating with respect to angles in (4) and (5) they take the form:

$$i = \frac{2\pi}{k} \int_0^\infty \frac{q dq}{(q^2 + \alpha^2 \mu^2)^2} \ln \frac{k + q + s}{|k - q| + s}, \quad I = \frac{2\pi}{k} \int_0^\infty \frac{q dq}{(q^2 + \alpha^2 \mu^2)^2} \ln \frac{k + q}{|k - q|}. \quad (6)$$

Using the well-established properties of integrals one can conclude that for any  $k$  and  $s$  the condition  $i < I$  should be fulfilled.

In turn, corrections to the Coulomb energy levels can be found by calculating integrals containing expressions from (6) as a part of its integrands along with the parameter  $k$  as the variable of integration.

For a start, let's analyze the result of the integration of expressions from (6) with respect to momentum  $q$

$$i = \frac{2\pi}{k} \left[ \frac{1}{4} \ln \frac{k^2 + \alpha^2 \mu^2}{s^2} \left( \frac{1}{(k+s)^2 + \alpha^2 \mu^2} - \frac{1}{(k-s)^2 + \alpha^2 \mu^2} \right) + \right. \\ \left. + \frac{\pi}{4\alpha\mu} \left( \frac{k+s}{(k+s)^2 + \alpha^2 \mu^2} + \frac{k-s}{(k-s)^2 + \alpha^2 \mu^2} \right) + \right. \\ \left. + \left( \frac{k+s}{(k+s)^2 + \alpha^2 \mu^2} - \frac{k-s}{(k-s)^2 + \alpha^2 \mu^2} \right) \frac{1}{2\alpha\mu} \arctan \frac{k}{\alpha\mu} \right], \quad (7)$$

$$I = \frac{\pi^2}{\alpha\mu} \frac{1}{k^2 + \alpha^2 \mu^2}. \quad (8)$$

Now, in accordance with general rules of estimation of integrals, we initially consider everywhere the leading contribution with  $I$  ( $s = 0$ ) in integrands. If it leads to the contribution proportional to  $\frac{(Z\alpha)^6 \mu^3}{m_1 m_2} \beta \ln \beta^{-1}$  then the corresponding integral with  $i$  ( $s \neq 0$ ) should be counted as well.

As an example, we treat the Feynman diagram with parallel photon lines corresponding to the exchange by one Coulomb and one transverse photons.

The contribution to the fine shift from sequential exchange by Coulomb and transversal photons is defined by the expression

$$\Delta E_{\text{par}} = \langle \Psi'_{nS} | (K_C G_0 K_T + K_T G_0 K_C)_{0F}^+ | \Psi'_{nS} \rangle, \quad (9)$$

where  $G_0$  is a Green function of free fermions,  $K_C$  and  $K_T$  are kernels, describing the exchange by one Coulomb and one transversal photons respectively, symbol  $(\dots)_{0F}^+ = F^{-1} G_0 (\dots) G_0^+ F^{-1}$ ,  $F = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})(E - \varepsilon_{1p} - \varepsilon_{2p})^{-1}$ ,  $\varepsilon_{ip} = \sqrt{p^2 + m_i^2}$ ,  $i = 1, 2$ ,  $E$  is the total energy of a system of two relativistic particles,  $\Psi'_{nS}(\mathbf{p}) = \Omega_p \Psi_{nS}(\mathbf{p})$ ,  $\Omega_p$  is a factor, emerged because of the relativistic character of the quasipotential equation,  $\Psi_{nS}$  is a wave function which defines the state of a two-fermion system with a total energy  $E$ .

In particular, the logarithmic contribution to the fine shift of the  $1S$  energy level of the HLA (with  $Z = 1$ ) appeared from the following integral

$$\Delta E_{\text{par}}(\ln^2 \beta^{-1}) = -\frac{\alpha^7 \mu^5}{2\pi^6} \int \frac{d^3 p N_p \Omega_p}{(p^2 + \alpha^2 \mu^2)^2} \int \frac{d^3 q N_q \Omega_q}{(q^2 + \alpha^2 \mu^2)^2} \times \\ \times \int \frac{d^3 k k^2}{\varepsilon_{1k} \varepsilon_{2k} (\mathbf{p} - \mathbf{k})^2 \mathbf{k}_q (\mathbf{k}_q + \varepsilon_{1k} + \varepsilon_{1q}) (\varepsilon_{1k} + \varepsilon_{2k} + E_1 + E_2)}, \quad (10)$$

where

$$N_p = \sqrt{\frac{(\varepsilon_{1p} + m_1)(\varepsilon_{2p} + m_2)}{2\varepsilon_{1p}\varepsilon_{2p}}}, \quad \Omega_p = \frac{(\varepsilon_{1p} + m_1)(\varepsilon_{2p} + m_2)}{2\mu(\varepsilon_{1p} + \varepsilon_{2p} + E_1 + E_2)}.$$

Leaving in (10) only terms proportional to  $\frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}$ , we got

$$I_1 = -\frac{\alpha^7 \mu^3}{8\pi^6} m_1 \left\{ \int \frac{d^3 p}{(p^2 + \alpha^2 \mu^2)(\varepsilon_{1p} + m_1)} \int \frac{d^3 q}{(q^2 + \alpha^2 \mu^2)^2} \int \frac{d^3 k k^2}{\mathbf{k}_p^2 \mathbf{k}_q} \times \right. \\ \times \frac{1}{\varepsilon_{1k} \varepsilon_{2k} (\mathbf{k}_q + \varepsilon_{1k} + E_1)(\varepsilon_{1k} + \varepsilon_{2k} + E_1 + E_2)} + \int \frac{d^3 p}{(p^2 + \alpha^2 \mu^2)^2} \int \frac{d^3 q}{(q^2 + \alpha^2 \mu^2)(\varepsilon_{1q} + m_1)} \times \\ \left. \times \int \frac{d^3 k k^2}{\varepsilon_{1k} \varepsilon_{2k} \mathbf{k}_p^2 \mathbf{k}_q (\mathbf{k}_q + \varepsilon_{1k} + E_1)(\varepsilon_{1k} + \varepsilon_{2k} + E_1 + E_2)} \right\} = I'_1 + I''_1. \quad (11)$$

Let's focus on the first term from (11). By counting integrations with respect to momenta  $\mathbf{p}$  and  $\mathbf{q}$  it can be rewritten as

$$I'_1 = -\frac{\alpha^7 \mu^3}{2\pi^5} m_1 \int_0^\infty \frac{dk k^4}{\varepsilon_{1k} \varepsilon_{2k} (\varepsilon_{1k} + \varepsilon_{2k} + E_1 + E_2)} j_p j_q, \quad (12)$$

where

$$j_p = \int \frac{d^3 p}{(p^2 + \alpha^2 \mu^2)(\varepsilon_{1p} + m_1) \mathbf{k}_p^2} = \frac{2\pi}{k^2 + \alpha^2 \mu^2} \left( -1 + \frac{\varepsilon_{1k}}{k} \ln \frac{\varepsilon_{1k} + k}{m_1} \right) + \frac{\pi^2}{m_1 k} \arctan \frac{k}{\alpha \mu} + \frac{2\pi}{k} \left( \frac{k}{3\varepsilon_{1k}^2} + \frac{1}{\varepsilon_{1k}} \left( 1 + \frac{m_1^2}{3\varepsilon_{1k}^2} \right) \ln \frac{\varepsilon_{1k} + k}{m_1} \right),$$

$$j_q = \int \frac{d^3 q}{(q^2 + \alpha^2 \mu^2)^2 \mathbf{k}_q (\mathbf{k}_q + \varepsilon_{1k} + E_1)} = \frac{\pi}{k m_1} \left( \frac{1}{\alpha \mu} \arctan \frac{k}{\alpha \mu} - \frac{\pi}{2\alpha \mu} \frac{k}{\varepsilon_{1k} + m_1} + \frac{k}{2m_1(\varepsilon_{1k} + m_1)} \ln \frac{\varepsilon_{1k} + E_1}{\varepsilon_{1k} - E_1} \right).$$

It is worth mentioning here, that  $j_p$  from (12) contains the Coulomb factor  $\frac{1}{\mathbf{k}_p^2}$  whereas  $j_q$  holds its relativistic modification  $\frac{1}{\mathbf{k}_q(\mathbf{k}_q + \varepsilon_{1k} + E_1)}$ . And, if we make the replacement  $\mathbf{k}_q + \varepsilon_{1k} + E_1 \rightarrow \mathbf{k}_q$  in  $j_q$ , then, one can easily verify that,  $j_q$  reduces to  $I$  from (8). It leads to the condition  $j_q < I$ . Following the above-mentioned criterium we must evaluate the integral  $I'_1$  because its leading contribution with  $I$  instead of  $j_q$  is nonzero.

Evaluating the integral  $I'_1$  up to the order of  $\frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}$  we come to the integral

$$I'_1 = -\frac{1}{\pi^2} \frac{\alpha^6 \mu^3}{m_1 m_2} m_1 (m_1 + m_2) \int_0^\infty \frac{dk}{\varepsilon_{2k} (\varepsilon_{1k} + m_1) (\varepsilon_{2k} + m_2)} \ln \frac{\varepsilon_{1k} + k}{m_1}. \quad (13)$$

After calculating it we have

$$I'_1(j_q) = -\frac{1}{4\pi^2} \frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}. \quad (14)$$

If we use in (12)  $I$  instead of  $j_q$  then, after passing through the same steps we obtain

$$I'_1(I) = -\frac{2}{4\pi^2} \frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}. \quad (15)$$

Therefore, the relativistic modification of the Coulomb interaction reduces its effect twice in comparison with the non-relativistic case.

The second term in (11) can be evaluated following the same procedure as it was used for the estimation of the first one. It yields, in the relativistic case

$$I''_1 = -\frac{1}{2\pi^2} \frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}. \quad (16)$$

Thus, the total correction to the fine shift of  $1S$  energy level of the HLA from the contribution presented by (10) takes the form

$$\Delta E_{\text{par}}(\ln^2 \beta^{-1}) = I_1 = -\frac{3}{4\pi^2} \frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}. \quad (17)$$

For comparison, the non-relativistic version of the same correction can be written as

$$\Delta E_{\text{par}}(J_q) = I_1 = -\frac{1}{\pi^2} \frac{\alpha^6 \mu^3}{m_1 m_2} \beta \ln^2 \beta^{-1}. \quad (18)$$

### 3. Conclusion

Therefore, we can conclude, that the relativistic modification of the Coulomb interaction reduces the value of  $\Delta E$ . It should be counted in further calculations of corrections to energy levels of the HLA.

### References

- [1] Mohr P J, Taylor B N and Newell D B 2012 *Rev. Mod. Phys.* **84** 1527
- [2] Karshenboim S G, Pavone F S, Bassani F, Inguscio M and Hänsch T W 2001 *The Hydrogen Atom: Precision Physics of Simple Atomic Systems* vol 570 (Berlin, Heidelberg, New York, Barcelona, Hong Kong, London, Milan, Paris, Singapore, Tokyo: Springer)
- [3] Karshenboim S G 2008 *Precision Physics of Simple Atoms and Molecules* vol 745 (Berlin, Heidelberg: Springer)
- [4] Dvoeglasov V V, Tyukhtyaev Yu N and Faustov R N 1994 *Fiz. Elem. Chastits At. Yadra* **25** 144 [*Phys. Part. Nucl.* **25** 58]
- [5] Kinoshita T 1996 *Rep. Prog. Phys.* **59** 1459
- [6] Sapirstein J 2006 *Handbook of Atomic, Molecular and Optical Physics*, ed Drake G W F (Heidelberg: Springer) p 413
- [7] Mohr P J and Taylor B N 2006 *Handbook of Atomic, Molecular and Optical Physics*, ed Drake G W F (Heidelberg: Springer) p 429
- [8] Eides M I, Grotch H and Shelyuto V A 2001 *Phys. Rep.* **342** 63
- [9] Eides M I, Grotch H and Shelyuto V A 2007 *Theory of light hydrogenic bound states* Springer Tracts Mod. Phys. **222** (Berlin, Heidelberg, New York: Springer)
- [10] Hänsch T W 2006 *Rev. Mod. Phys.* **78** 1297
- [11] Matveev A, Parthey Ch G, Predehl K, Alnis J, Beyer A, Holzwarth R, Udem Th, Wilken T, Kolachevsky N, Abgrall M *et al* 2013 *Phys. Rev. Lett.* **110** 230801
- [12] Bethe H A and Salpeter E E 1951 *Phys. Rev.* **84** 1232
- [13] Logunov A A and Tavkhelidse A N 1963 *Nuovo Cimento* **29** 380
- [14] Kadyshevsky V G 1968 *Nucl. Phys. B* **6** 125
- [15] Grotch H and Yennie D R 1969 *Rev. Mod. Phys.* **41** 350
- [16] Gross F 1969 *Phys. Rev.* **186** 1448
- [17] Dulyan L S and Faustov R N 1975 *Teor. Mat. Fiz.* **22** 314 [*Theor. Math. Phys.* **22** 220]
- [18] Faustov R N 1964 *JINR preprint* P-1572 (Dubna: JINR)
- [19] Tyukhtyaev Yu N 1982 *Teor. Mat. Fiz.* **53** 419 [*Theor. Math. Phys.* **53** 1217]
- [20] Fulton T and Martin P C 1954 *Phys. Rev.* **95** 811
- [21] Boikova N A, Tyukhtyaev Yu N and Faustov R N 1998 *Yad. Fiz.* **61** 866 [*Phys. At. Nucl.* **61** 781]
- [22] Boikova N A, Tyukhtyaev Yu N and Faustov R N 2001 *Yad. Fiz.* **64** 986 [*Phys. At. Nucl.* **64** 917]
- [23] Boikova N A, Kleshchevskaya S V, Tyukhtyaev Yu N and Faustov R N 2003 *Yad. Fiz.* **66** 925 [*Phys. At. Nucl.* **66** 893]
- [24] Boikova N A, Nunko N E, Tyukhtyaev Yu N and Faustov R N 2002 *Teor. Mat. Fiz.* **132** 339 [*Theor. Math. Phys.* **132** 1179]
- [25] Boikova N A, Kleshchevskaya S V, Tyukhtyaev Yu N and Faustov R N 2006 *Teor. Mat. Fiz.* **149** 325 [*Theor. Math. Phys.* **149** 1577]
- [26] Boikova N A, Boikova O A, Kleshchevskaya S V and Tyukhtyaev Yu N 2008 *Izv. Saratov. un-ta. Novaya seriya. Fizika* **8** 42
- [27] Boikova N A, Kleshchevskaya S V, Tyukhtyaev Yu N and Faustov R N 2009 *Yad. Fiz.* **72** 300 [*Phys. At. Nucl.* **72** 272]
- [28] Boikova O A, Kleshchevskaya S V, Tyukhtyaev Yu N and Faustov R N 2010 *Yad. Fiz.* **73** 1024 [*Phys. At. Nucl.* **73** 988]
- [29] Churochkina S V 2014 *Geteromagnitnaya mikroelektronika* **17** 23