Interaction of neutrons with a birefringent medium moving with an acceleration

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Abstract. Recent experiments demonstrated that the energy of a neutron traversing an accelerated sample of a refractive medium changes. Later, it was realized that such an accelerated-medium effect (AME) is quite a general phenomenon. This paper discusses some special features of the effect for a birefringent medium. In this case, AME shows quite new features. In neutron optics, where birefringence is due to the spin dependence of the refractive index, AME results in a nonstationary state with a precessing spin.

1. Introduction

The first theoretical study devoted to the neutron wave interaction with moving matter appeared in the late 1970s [1]. It triggered the appearance of a number of experimental [3–5] and theoretical [6,7] studies. A review of those studies and a comparison of the results obtained in conventional and neutron optics are given in [8]. By analogy with light optics, neutron experiments with moving media became known as neutron Fizeau experiments.

While having a number of similar features, phenomena accompanying the propagation of electromagnetic and neutron waves through moving matter are markedly different [9]. Fresnel's drag of light and the phase shift of electromagnetic waves in the Fizeau experiment are due to the relativistic nature of the velocity transformation. In contrast, neutron optics deals with nonrelativistic particles, so that in order to go over to a different reference frame, one has to perform a Galilean transformation of the respective wave function. Further, neutron interaction with matter can be described in terms of an effective potential,

$$U = \frac{2\pi\hbar^2}{m}\rho b\,, \tag{1}$$

where ρ is the spatial density of nuclei, and b is the volume-averaged coherent scattering length. The motion of matter itself leaving the effective potential unchanged does not have any effects on the neutron and the phase shift can only be obtained by the motion of sample boundaries. If, however, a layer of matter moves together with its boundaries as a coherent whole, the wave frequency in matter as measured in the laboratory frame is different from that in a vacuum. This will be discussed in more detail in the next section.

In the case of the motion of a matter layer at a constant velocity, Doppler frequency shifts resulting from the passage of a wave through two boundaries of the sample are equal in magnitude but opposite in sign. The net effect is zero, and the frequency of the wave that passed through the moving sample does not change. In the case of an arbitrary motion, such a cancellation does not occur, so that the frequency of the wave transmitted through the sample differs from that of the incident wave. For the first time, this was indicated by Tanaka in [10].

Apparently, the problem of neutrons passing through a layer of matter moving with a linear acceleration was first considered by Kowalski [11]. His theoretical approach was based on the notion of the neutron group and phase velocities and on the calculation of the time of wave propagation between relevant points in stationary and moving reference frames. As an intermediate result, he arrived at the conclusion that the energy of a neutron changes after its transmission through a sample moving at a not-too-high acceleration. Later on, the same issue was considered by Nosov and Frank [12]. Their analysis, in which a consistent calculation of the velocity of a neutron upon entering the


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sample and exiting it through another surface was performed, was in fact based on the classical approach.

The first brief report on the experimental observation of the change in the energy of neutrons propagating through an accelerated layer of matter appeared in [13]. The results of a more detailed study of this effect, along with basic theoretical conclusions, were presented in [14]. In the same paper, the authors put forth the idea that the observed effect was a manifestation of quite a general phenomenon – the accelerated medium effect (AME) – inherent in waves and particles of different nature. Recently, the acceleration and deceleration of neutrons were observed by applying a specific time-of-flight method [15]. The acceleration of the samples in those experiments was several tens of *g* units, the value of the energy transferred to a neutron fell within the range of $(2\div6)\times10^{-10}$ eV, and the experimental results obtained by two different methods were in agreement with the theory to a precision of about 10%. Recently the energy change of neutrons passing through the accelerated crystal close to the Bragg angle was observed in [16]. The first stage of AME investigations with UCN was summarized in [17], where some features of neutron interaction with an accelerated sample of birefringent matter were also briefly discussed. The similar phenomena in neutron and neutrino physics were also discussed in [18].

In the next section, we discuss the problem of neutron wave propagation through a moving medium and show how the propagation of a neutron wave through a layer of conventional refracting matter moving with an acceleration leads to a change in energy. In the following two sections we discuss some features of the AME manifestation in the case of a birefringent medium and some possibilities of their experimental observation.

2. Doppler effect upon the refraction of a neutron wave at the boundary of a moving medium and accelerated matter effect

We now consider the problem of neutron-wave refraction at the boundary of a half-infinite layer of matter moving at a constant velocity V along the x axis. Representing, as usual, the initial state of the neutron as a plane wave, $\psi(x, t) = \exp[i\,(k_0 x - \omega_0 t)]$, we find the neutron wave function in the reference frame co-moving with matter. According to [19] it has the form

$$\psi'\left(x',t\right)=\psi\left(x'+Vt,t\right)exp\left[-i\left(k_V x'+\omega_V t\right)\right], \tag{2}$$

where

$$k_V=\frac{mV}{\hbar},\quad \omega_V=\frac{mV^2}{2\hbar}. \tag{3}$$

In this reference frame, the wave number and frequency of the incident wave are

$$k'=k_0-k_V,\quad \omega'=\omega_0+\omega_v-kV\ . \tag{4}$$

Upon refraction at the matter boundary, which is stationary in this reference frame, only the wave number changes, so that the wave function inside matter assumes the form

$$\psi_i'(x',t)=exp\left[i\left(k'n'x'-\omega't\right)\right]. \tag{5}$$

where $n' = n(k')$ is the wave-number-dependent refractive index,

$$n\left(k\right)=\sqrt{1-k_b^2/k^2}\ , \tag{6}$$

and $k_b^2=4\pi\rho b$ is the boundary value of the wave number.

After substituting (4) into (5) and returning to the laboratory frame we find for the wave number in the moving matter that

$$k_i = n'k_0\left(1+\frac{1-n'}{n'}\frac{V}{v_0}\right)\ , \tag{7}$$

where v_0 is the neutron velocity in a vacuum. One can easily see that this formula coincides with the well-known expression for the Fresnel drag in a moving medium (see, for example [20]).

For the frequency of a neutron wave in a medium, as measured in the laboratory frame, we obtain

$$\omega_i = \omega_0 + (n'-1)(k_0 - k_V)V\ .$$

Assuming that velocity of matter is small in relation to v_0, we obtain the following expression to the first order terms in V/v_0 :

$$\omega_i = \omega_0 + (n'-1)k_0V \qquad \left(V << v_0\right). \tag{8}$$

Equation (8) describes the Doppler effect occurring upon neutron-wave refraction at the boundary of a moving medium. It coincides with the similar expression obtained in [21] to the first order in *V/c* for light refraction at the boundary of a moving medium.

We now consider the propagation of a wave through a moving sample and find the phase shift caused by the presence of matter. For a stationary sample, it is $\chi = \left(k_i - k_0\right)d$, where *d* is the sample thickness. Substituting expression (9) into it and setting $k_i = n'k_0 - (n'-1)k_V$, we obtain

$$\chi = \left(n'-1\right)\left(k_0 - k_V\right)d\ . \tag{9}$$

Substituting into (9) the definition $n' = \sqrt{\left(k_0 - k_V\right)^2 - k_b^2} \times \left(k_0 - k_{\mathrm{v}}\right)^{-1}$ we arrive at the relation

$$\chi = \left[\sqrt{\left(k_0 - k_V\right)^2 - k_b^2} - \left(k_0 - k_V\right)\right]d\ , \tag{10}$$

which coincides with formula (5) from [6].

As concerns the frequency of a wave passing through a sample, it was indicated in the Introduction that in the case where a layer of matter moves at a constant velocity, the frequency shifts resulting from the passage of the wave through two boundaries are equal in magnitude but opposite in sign. The net effect is then zero. In the case of an accelerated motion, the frequency shifts at the entrance and exit surfaces of the sample are unequal and do not cancel each other, since, during the time of wave propagation through the sample, the velocity of the boundaries changes by $\Delta V = a\tau$, where *a* is the acceleration of the sample and τ is the propagation time. Setting the latter to $\tau = d/\left(nv_0\right)$ and neglecting, for the time being, the dispersion of the medium, we find that the difference between the frequencies of the incident and transmitted waves is $\Delta\omega = ad\left(1-n\right)k_0/nv_0$ The change in the energy is then given by

$$\Delta E = mad\left(\frac{1-n}{n}\right) \qquad \left(a\tau << v_0\right), \tag{11}$$

in perfect agreement with the results obtained in [11, 12]. Relation (11) was obtained to the first order in V/v_0 and $\Delta V/v_0$.

It should be noted that the above way of deriving (11) for AME is only valid under the adopted assumptions, since it completely ignores the fact of the accelerated motion of the medium itself. In neutron optics, this does not result in an error, at least at moderate accelerations, since (and as long as) the description of the medium with the aid of an effective potential (1) is true here and the motion of the region within which the potential is operative by no means affects the state of the particle inside it.

At the same time some arguments were presented recently that the concept of effective potential may be wrong in the case of a very large acceleration of matter [22].

It is worth mentioning that for derivation of Eq. (11) we have only used the fact that the wave number in accelerated matter is not equal to that in a vacuum. Thus, the effect should occur at an accelerated motion of refractive matter as well as of a finite spatial region containing a force field. Below we do not draw a distinction between these two cases but define the refractive index as the ratio of the wave number in a medium (or in the area of the action of the potential) and in a vacuum, $n = k/k_0$. It allows us to use formula (11) in both cases. The refractive index for an area with the potential U is given by

$$n = \sqrt{1 - \frac{U}{E}}, \tag{12}$$

where E is the neutron energy in free space $(E > U)$.

3. Interaction of neutrons with moving matter in the case of double refraction

3.1 The case of permanent velocity

Birefrigent matter is characterized by two refractive indices in accordance with the two values of polarization of the incident wave. In neutron optics, two values of the refractive index, $n_{\pm}$, correspond to two different projections of the neutron spin onto the physical axis. For a sample moving with a constant speed, relations (7) and (9) hold. In the case of arbitrary polarization of the initial wave function

$$\Psi_0(x,t) = \begin{pmatrix} A_+^0 \\ A_-^0 \end{pmatrix} exp\left[i(kx - \omega t)\right],$$

the state inside the matter is

$$\Psi_i(x,t) = \begin{pmatrix} A_+^0\, exp(ik_{i+}x) \\ A_-^0\, exp(ik_{i-}x) \end{pmatrix} exp\left(-i\omega_i t\right), \tag{13}$$

and the spin is rotating during the time of neutron propagation inside the sample. After passing the moving sample the state is

$$\Psi_f(x,t) = \begin{pmatrix} A_+^0\, exp\, i\chi_+ \\ A_-^0\, exp\, i\chi_- \end{pmatrix} exp\left[i(kx - \omega t)\right], \tag{14}$$

where the phase shifts $\chi_{\pm}$ caused by the presence of the sample are:

$$\chi_{\pm} = \left(n'_{\pm} - 1\right)\left(k_0 - k_V\right)d\,. \tag{15}$$

The phase difference $\Phi(V) = \chi_+(V) - \chi_-(V)$ determines the spin rotation angle [23].

3.2 The case of accelerated matter

As it has been mentioned above, the neutron wave number changes only at the matter boundaries. So, if we talk about the effects arising prior to refraction at the exit surface, the motion of the sample affects only the effective path length travelled by the neutron inside the sample. In the case of an accelerated motion, the effective thickness of the sample is apparently equal to $d_{eff} = d + a\tau^2/2$.

Let us now address the issue concerning the frequencies of the two waves transmitted through accelerated matter. It is obvious that in the case of double refraction we have to modify Eq. (11) as

$$\Delta E_{\pm} = mad\left(\frac{1-n_{\pm}}{n_{\pm}}\right), \qquad \Delta\omega_{\pm} = \frac{\Delta E_{\pm}}{\hbar}. \tag{16}$$

After the passage of an accelerated doubly refracting sample, the two spin components of the neutron wave function differ in frequency, so that the wave function is a time-dependent superposition and the final state has now the form

$$\Psi(x,t) = \begin{pmatrix} A_+ \\ A_- \end{pmatrix} exp\left[i(kx-\omega t)\right], \quad A_{\pm} = A_{\pm}^0\, exp\left[-i\left(\Delta k_{\pm}x + \Delta\omega_{\pm}t + \chi_{\pm}\right)\right]. \tag{17}$$

The wave function $\Psi(x,t)$ describes a nonstationary state with a precessing spin. The precession angle is obtained as the time-dependent difference of the phase angles of the two spin components; that is,

$$\phi(x,t) = \left(k_+ - k_-\right)x + \Omega t + \Phi(t), \tag{18}$$

where $k_{\pm} = k_0 + \Delta k_{\pm}$, $\Omega = \omega_+ - \omega_-$ and $k_+^2 - k_-^2 = 2m\Omega/\hbar$. Introducing the average velocity $\hat{v}$ for the two components, we arrive at an expression for the spin precession angle in the form

$$\phi(x,t) = \Omega\left(t - \frac{x}{\hat{v}}\right) + \Phi(t), \quad \hat{v} = \frac{\hbar}{m}\frac{k_+ + k_-}{2}. \tag{19}$$

A time dependence arises in the phase shift Φ since, in the accelerated-motion case, which is considered here, the velocity V and the wave number kV are not constants; as a result, the phases $\chi_{\pm}$ given by Eqs. (9) and (10) change with time. In the approximation of a small change in energy, $\Delta E_{\pm}/E << 1$, the effective velocity $\hat{v}$ coincides with the classical velocity. As follows from Eq. (19), the direction of the neutron spin is invariable in the reference frame moving at the neutron velocity $\hat{v}$ with respect to the laboratory frame. At a fixed observation point, however, the spin direction changes with time periodically at the frequency Ω. The situation is similar to that which was first observed in the experiment reported in [24], where the energy of the two spin components was changed by means of two resonant spin flippers operating at slightly different frequencies. It is also quite typical of neutron resonant spin-echo instruments and related facilities [25-27]. A similar situation must occur upon the dynamic reflection and refraction of neutrons at the boundary of a magnet having a variable induction [28, 29].

In all of the aforementioned cases [24–29], a nonstationary state arises owing to the interaction of the neutron magnetic moment with an alternating magnetic field – that is, via a strictly quantum process. A feature distinguishing the AME case for birefringent matter under discussion is that a nonstationary state arises as a difference effect in a classical phenomenon such as the Doppler frequency shift.

From the practical point of view, it is very important that the periodic change under discussion in the polarization direction can be measured even if the oscillation frequency Ω and the respective energy transfer $\hbar\Omega$ are quite small. Under the assumption that $\Delta n = n_+ - n_- \ll n = \left(n_+ + n_-\right)/2$ it follows from (16) that

$$\left|\Omega\right| = \left|\Delta\omega_+ - \Delta\omega_-\right| \simeq \frac{mad}{\hbar}\frac{\Delta n}{n}. \tag{20}$$

3.3 Birefringence in neutron optics and AME

In neutron optics, several physical reasons may be responsible for the origin of birefringence. First, we will dwell on the simplest case of interaction between the neutron magnetic moment μ and the magnetic field B. In accordance with (12), the spatial domain where a nonzero magnetic field is present is characterized by two values of the refractive index:

$$n_{\pm} = \sqrt{1 \mp \frac{\mu B}{E}}\,. \tag{21}$$

In line with the aforesaid, neutron propagation through such a domain moving with an acceleration results in the formation of a nonstationary state. From Eqs. (16) and (20), one can readily find that in a conventional weak-field approximation $\mu B << E$, the beat frequency Ω is

$$\Omega \simeq \frac{\mu B}{\hbar}\frac{mad}{E} = \omega_L \frac{mad}{2E}, \tag{22}$$

where $\omega_L = 2\mu B/\hbar$ is the Larmor frequency.

It is highly probable that this simplest case is the most attractive for a first experimental demonstration of the neutron spin rotation resulting from the interaction with an accelerated birefringent medium. Below, we will discuss some details of such an experiment.

In addition to a moving domain where a magnetic field is present, it is useful to consider the case of a magnetic sample. If the sample material is characterized by an effective potential U and a magnetic induction B, the two values of the neutron refractive index are

$$n_{\pm} = \sqrt{1 - \frac{U \mp \mu B}{E}}\,. \tag{23}$$

If $U \pm \mu B << E$, relation (22) remains valid in this case inclusive.

A nonmagnetic sample placed in a magnetic field can also act as a birefringent material for a neutron wave [30]. In a sense, we might speak here about ordinary matter placed in a birefringent medium with the refractive index given by (21). The two spin components then have the wave numbers

$$k_{\pm} = k_0\sqrt{1 \mp \frac{\mu B}{E}}\,, \quad E = \frac{\hbar^2}{2m}k_0^2\,, \tag{24}$$

where k_0 is the wave number in the absence of a field. The effective refractive index of the sample is then

$$n_{\pm} = \sqrt{1 - \frac{U}{E \mp \mu B}}\,. \tag{25}$$

The accelerated motion of a sample in a constant magnetic field also leads to the formation of states with unequal frequencies for the two spin components of the wave function and to the precession frequency caused by AME and given by

$$\Omega \simeq \omega_L \frac{U}{2E} maL\,, \qquad \left(\frac{\mu B}{E} << 1\right). \tag{26}$$

In general, birefringence may also occur in the absence of a magnetic field. By way of example, we focus on the nuclear pseudomagnetism phenomenon occurring in the case where polarized neutrons propagate through matter containing polarized nuclei [31, 32]. Because of the spin dependence of nuclear forces, the coherence lengths for neutron–nucleus scattering, $b_{\pm}$, are different for the two values of the total spin of the neutron and nucleus, $J = I \pm 1/2$, where I is the spin of the nucleus. Employing the notation $b = \left(b_+ + b_-\right)/2$ and $\Delta b = b_+ - b_-$, we obtain an expression for the refractive indices in the form

$$n_{\pm} = \sqrt{n^2 \mp p_{nuc} \frac{2\pi\rho}{k_0^2} \Delta b}\,, \tag{27}$$

where p_{nuc} is the polarization of the sample and $n^2 = 1 - 4\pi\rho b / k_0^2$. Assuming that the effect of nuclear pseudomagnetism may be detected for small values of polarization, we consider the case of $p_{nuc} \ll 1$. Then for any realistic values of k_0, relation (29) can be written as

$$n_{\pm} = n \mp p_{nuc} \frac{\pi\rho}{nk_0^2} \Delta b\,. \tag{28}$$

Substituting (28) into (20), we obtain a relation for the frequency of precession arising due to AME:

$$\Omega = p_{nuc} \frac{mad}{\hbar} \frac{2\pi\rho}{n^2 k_0^2} \Delta b\,. \tag{29}$$

The inversely proportional dependence of the precession frequency on the square of the wave number gives sufficient grounds to believe that the spin-dependent scattering lengths Δb can be measured by using very slow neutrons even in the case of rather low nuclear polarization.

Finally, birefringence could be due to the parity violation in neutron-nucleus interactions. But AME in this case is very small [18] and we shall not discuss it here.

3.4 Possibility of experimental observation of AME in the case of birefringence

Spin rotation in the case of neutron propagation through an accelerated birefringent sample can be detected in experiments similar to those where one observes the Larmor spin precession in a refracting sample [33]. The principal difference is that in this case one has to measure the time dependence of the spin precession angle within the time that is commensurate with the spin-rotation period $T \approx 1/\Omega$. This means that the total rotation angle $\phi_{max} = \Omega t$ can be comparatively large even at a very small value of the ratio $\Delta n / n$. On the other hand, the measurement time is limited by the time interval within which the accelerated sample remains inside the measuring instrument used. It is obvious that in order to improve statistics, the measurement must be repeated many times.

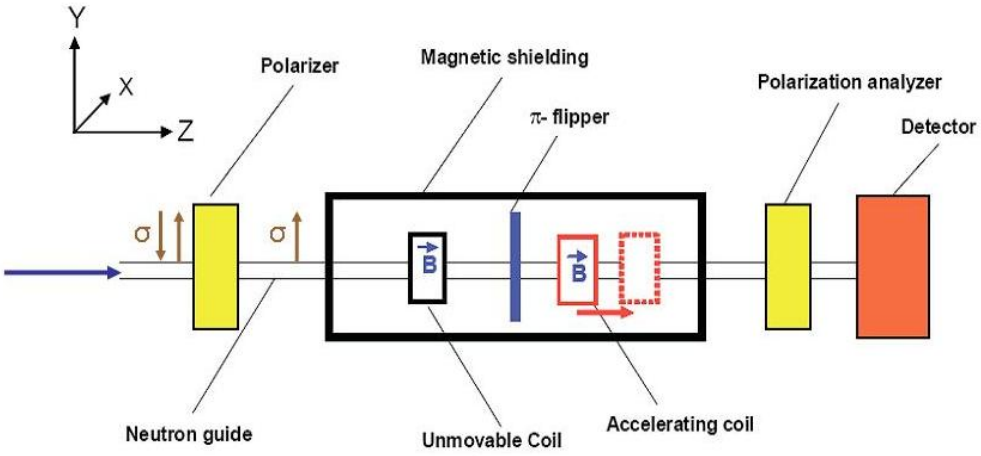


Figure 1. Layout of an experiment aimed at observing the rotation of the spin of a neutron propagating through a region with a magnetic field and moving with an acceleration.

As an example, we will consider a possible experiment aimed at observing AME in the case where a bounded region with a magnetic field moves with an acceleration (see Fig. 1). The respective experimental facility is a classic neutron spin-echo spectrometer [34]. Neutrons polarized along the y axis enter nonadiabatically the region where there exists a magnetic field B directed along the z axis and created by an immobile coil. Throughout the time within which a neutron remains inside the coil, its spin precesses in an ordinary fashion. Having passed through a π-flipper, neutrons enter the second coil identical to the first one. In the stationary case, the spin-precession angles in the two coils cancel each other exactly, so that the total precession angle is zero. According to (19), the accelerated motion of the second coil along the beam direction leads to the appearance of an additional time-dependent precession angle.

Since the transmission of an analyzer placed in front of the detector depends on the spin direction, the polarization-vector rotation associated with the motion of the second coil leads to a change in the detector counting rate. If the effect is small, an additional π/2-flipper can be placed in front of the analyzer. This corresponds to the well-known method of orthogonally crossed polarizer–analyzer pairs.

Figure 2, which illustrates the estimated results of such an experiment, gives the spin-precession angle φ and the quantity $1-cos\phi$ versus time. The latter is proportional to the detector counting rate under ideal conditions. By ideal conditions, we mean a hundred-percent efficiency of the polarizer and analyzer, the total monochromation of the beam, and the absence of any background. The instant at which the coil begins its motion is taken for a reference point. The measurement starts at a later instant, whereby one takes into account the neutron time of flight from the coil to the detector (0.4 s).

These calculations were performed for an experiment with ultracold neutrons of speed *4.5* m/s. We assumed that the magnetic field is 100 μT in both coils and that the path length inside the coil is 1 cm. Under these conditions, the neutron refractive indices (21) for the two spin components are estimated at $n_{\pm} = 1 \mp 2.8 \times 10^{-5}$.

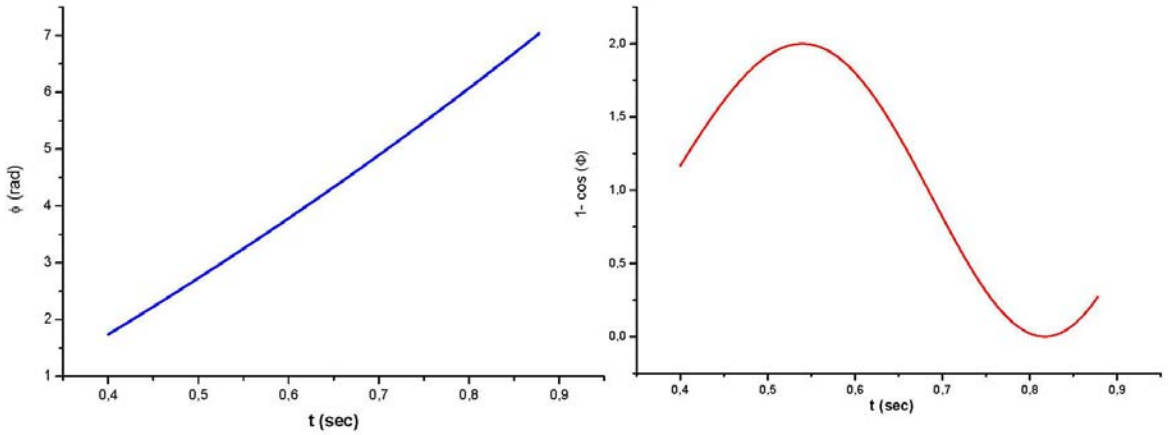


Figure 2. Results obtained by simulating the experiment illustrated in Fig. 1: (a) spin rotation angle as a function of time and (b) variation of the detector counting rate under ideal conditions as a function of time.

Relation (22) allows us to estimate the precession frequency at Ω ≈ 4.2 rad/s. The corresponding energy transfer to the neutron is $\Delta E = \pm\hbar\Omega/2 \approx \pm 1.5 \times 10^{-15}$ eV. In the calculations, we assume that the coil moves at an acceleration of about 50 cm/s^2 over the period of T = 0.5 s, traveling within this

period a distance of about 10 cm and reaching a speed of about 24.5 cm/s. The detector is placed at a distance of 180 cm from the initial position of the coil.

The results of the calculation are likely to favor the statement that for the above properties of a birefringent sample and the adopted parameters of its motion, the AME-induced effect of a nonstationary spin rotation can be reliably detected. Specifically, Fig. 2 shows that the angle of polarization-vector rotation is close to 2π, whereas the sensitivity of modern experiments permits measuring values that are many orders of magnitude smaller.

4. Conclusion

This article has been devoted to the discussion of a new aspect of the AME – namely, the case of an accelerated motion of birefringent matter. In this case, which was mentioned earlier in [17], the AME acquires quite new features. In neutron optics, birefringence is caused by the spin dependence of the refractive index; as a result, the AME gives rise to a nonstationary state with a rotating spin.

Physical reasons that lead to the double-refraction phenomenon in neutron optics have been considered, and the respective formulas for the resulting spin-precession frequency have been derived for this case. For the simplest case of double refraction in a magnetic field, we have considered a possible implementation of an experiment aimed at observing the effect under discussion. The results of a simulation of an experiment that would involve ultracold neutrons and in which a magnetic field with the refractive indices of $n_{\pm} = 1 \pm 3 \times 10^{-5}$ would move at an acceleration of 0.05 g for 0.5 s have been presented. Under such conditions, the energy transfer to the neutron would be about $\pm 1.5 \times 10^{-15}$ eV. Nevertheless, the AME induced by the precession of the neutron spin is quite measurable.

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