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On some Aspects of Gravitomagnetism and Correction for Perihelion Advance

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Abstract. In 1918 Joseph Lense and Hans Thirring, discovered the gravitomagnetic effect when studied solutions to the Einstein field equations using the weak field and slow motion approximation of rotating systems. They noted that when a body falls towards a massive object in rotation it feels a force perpendicular to its movement. The equations that they obtained were similar to Maxwell's equations of electromagnetism, now known as Maxwell's equations for gravitomagnetism. Some authors affirm that the gravitomagnetic effect can cause precession then in this paper we calculate the precession that gravitomagnetic effect cause in Mercury's perihelion advance. To make this we calculate the field between dipoles to measure the influence that the Sun has on Mercury, taking into account the gravitomagnetic field that the Sun and Mercury produces when they rotate around themselves. In addition, we calculate the ratio of the dipole force (of all solar system planet's) and the Newton's gravitational force to see how much is smaller.

1. Introduction

In 1918, Joseph Lense and Hans Thirring found gravitomagnetism (GM), while studying solutions to the Einstein field equations, using the weak field approximation for systems with rotation. They noted that when a body falls, towards a rotating massive object, it feels a perpendicular force to its movement. The equations that they obtained were similar to Maxwell's equations of electromagnetism, now known as Maxwell's equations for GM presented in (1), (2), (3) and (4)

$$\nabla \cdot \vec{g} = -4\pi G \rho_G \tag{1}$$

$$\nabla \cdot \vec{b} = 0 \tag{2}$$

$$\nabla \times \vec{g} = -\frac{1}{c}\frac{\partial \vec{b}}{\partial t} \tag{3}$$

$$\nabla \times \vec{b} = \frac{1}{c}\left(-4\pi G \vec{J}_G + \frac{\partial \vec{g}}{\partial t}\right), \tag{4}$$

where c is the speed of light in vacuum, G is the gravitational constant, $\vec{g}$ refers to gravito-electric field and $\vec{b}$ gravito-magnetic field.

The GM is expected by Einstein's theory of General Relativity. When a planet, star, black hole or something massive rotates, it drags space-time around it, this action is call "frame dragging".

Einstein theory shows that all gravitational forces correspond to the curvature of space-time, the twist here is GM.
However, when the GM can cause some effects? Some researchers[4],[5] and [8] say that it can cause precession orbiting satellites, and this would make a gyroscope (for example, placed in Earth orbit) swing. Some NASA researchers led by physicist Ignazio Ciufolini tried to detect equatorial bulge pulls satellites as well, which causes a precession billions of times greater than GM. Ciufolini and their researchers then had to subtract that effect, which then left the doubt whether the precision was enough to detect the GM. Many scientists accept these results, however others are skeptical.
In fact, there are some satellites in Earth's orbit to study about the gravitomagnetic precession[1] and [2], for this study they used the Laser Geodynamic Satellites, the LAGEOS and LAGEOS II. The researchers found a small precess (about 5 per cent) consistent with GM. Gravitomagnetic effects as can be seen in[1],[2] and [5] can be theoretically measured for some systems, such as black holes, super massive stars and even rotation curve of galaxies.
Bearing in mind the gravitomagnetic concepts and formalism, we calculate the field between two dipoles to measure the influence that the Sun has on Mercury, taking into account the gravitomagnetic field that the Sun and Mercury produces to rotate around themselves, and whether the addition of this force influence the perihelion advance of Mercury. Moreover, we calculate the ratio of the dipole force (of all solar system planet's) and the Newton's gravitational force to see how much is smaller.

2. Dipole moment of a Planet

In this section, we will calculate the dipole moment equation of a planet, to do this we will consider a sphere spinning in your own axis and we used the expression for dipole moment taken from[7] and doing appropriate modifications to the gravitomagnetic formalism, we have:

$$\vec{\mu} = \frac{1}{2c}\int \vec{r} \times \vec{J} d^3x,$$

And

$$J = \rho\omega r \sin\theta, \tag{5}$$

But,

$$v = \omega r \sin\theta,$$

This expression becomes:

$$\vec{\mu} = \int (xJ\cos\phi\hat{e}_z + yJ\sin\phi\hat{e}z - zJ\sin\phi\hat{e}_y - zJ\cos\phi\hat{e}_x)d^3x, \tag{6}$$

Applying coordinate transformation and, replacing (5) in (6), we have:

$$\vec{\mu} = \frac{\rho\omega}{2c}\left(\frac{r^5}{5}\frac{4\pi}{3}\hat{e}_z + \frac{r^5}{5}\frac{4\pi}{3}\hat{e}_z\right) = \frac{\rho\omega r^5 4\pi}{15c}\hat{e}_z.$$

But, into the case of homogeneous sphere

$$\rho = \frac{M}{\frac{4}{3}\pi r^3},$$

Replacing ρ and simplifying, the expression becomes:

$$\vec{\mu} = \frac{\sqrt{G}Mwr^2}{5c}\hat{e}_z. \tag{7}$$

Where G is the gravitational constant, c is the speed of light, M is the planet mass, r is the planet radius and w is the angular velocity of the planet.

3. Sun Gravitomagnetic field

The expression below is the Sun's gravitomagnetic dipole field produced by its mass in rotation, this expression was taken from Jackson[7] and gives the value of gravitomagnetic field outside of the Sun:

$$\vec{B}_{\odot} = \frac{3(\vec{\mu}_{\odot} \cdot \vec{R})\vec{R} - R^2 m}{R^5}.$$

In spherical coordinates, with $\vec{\mu}_{\odot} = \mu_{\odot}\vec{e}_z$, this equation can be rewritten as

$$\vec{B}_{\odot} = \frac{\mu_{\odot}}{R^3}(2\cos\theta\hat{e}_r + \sin\theta\hat{e}_{\theta}). \tag{8}$$

Where $\vec{\mu}_{\odot}$ is the Sun dipole moment and $\vec{R}$ is the position vector from the Sun to the planet.

4. Potential and interaction force between dipoles

Now in this section, we can calculate the gravitomagnetic potential to finally find the expression for the interaction force between the dipoles. For this we will use the expression for the potential energy of a dipole in a gravitomagnetic field $\vec{B}$

$$U = -(\vec{\mu}_p \cdot \vec{B}_{\odot}),$$

where $\vec{\mu}_p$ is the planet's dipole moment. Now working with

$$\vec{\mu}_p = \frac{\sqrt{G}M_p w_p r_p^2}{5c}\hat{e}_z = K_p\hat{e}_z,$$

and

$$\vec{\mu}_{\odot} = \frac{\sqrt{G}M_{\odot} W_{\odot} r_{\odot}^2}{5c}\hat{e}_z = K_{\odot}\hat{e}_z,$$

the potential reads

$$U = -\frac{K_p K_{\odot}}{R^3}(2\cos^2\theta - \sin^2\theta).$$

The force between dipoles is given by:

$$F_d = -\nabla U = \frac{3K_p K_{\odot}}{R^4}[(3\sin^2\theta - 6\cos^2\theta)\hat{e}_r + (2\sin\theta\cos\theta + 4\cos\theta\sin\theta)\hat{e}_{\theta}].$$

Considering that $\theta = \frac{\pi}{2}$, we have that the force of the interaction between the dipoles is:

$$F_d = \frac{3K_p K_{\odot}}{R^4}\hat{e}_r.$$

Therefore, the force have dependence in the radial component. Now, we are in position to calculate the ratio between the gravitomagnetic gravitational forces.

Table 1. Ratio between the dipole force and the gravitational for the solar system planets.

Planet	F_d/F_{grav}
Mercury	$3,7931 \times 10^{-10}$
Venus	$2,6149 \times 10^{-10}$
Earth	$9,3934 \times 10^{-11}$
Mars	$6,0768 \times 10^{-12}$
Jupiter	$3,6582 \times 10^{-11}$
Saturn	$3,1088 \times 10^{-12}$
Uranus	$4,8549 \times 10^{-14}$
Neptune	$9,4391 \times 10^{-15}$

As we can see in Tab.(1), the gravitomagnetic force is much smaller than gravitational force.

5. The orbit equation

In this section, we will calculate the orbit equation starting from the Lagrangian:

$$L = T - V. \tag{9}$$

In our case the potential energy will be an effective potential with one term coming from the gravitational potential and the other from the dipole's potential energy

$$V_{eff} = -\frac{k}{r} + \frac{d}{r^3}.$$

Here r is the distance from the Sun to the planet, $k = GM_{\odot}M_p$ and d is strength of the potential energy of the dipoles and m the mass of the planet. Replacing this potential in Eq.(9) results

$$L = \frac{1}{2}mv_r^2 + \frac{l^2}{2mr^2} - \left(\frac{k}{r} + \frac{d}{r^3}\right),$$

where l is the angular momentum of the planet. Doing some manipulations, using $r = 1/u$ we reach:

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{l^2} + \frac{3dmu^2}{l^2}. \tag{10}$$

Notice that unless the term $3dmu^2/l^2$ the equation corresponds to a Newtonian equation for the orbit of a particle test in a gravitational field produced by a point mass body. So, this term corresponds to a perturbation when compared to mk/l^2.

In the Tab.(1) we have the comparison between the last two terms of the Eq.(10), for all solar system planets. As we can see the Mercury is the planet subjected to the strongest field, according to the data

$$r_{\text{orbital}} = 5{,}79 \times 10^{10} \text{ m},$$
$$\text{period} = 7{,}60 \times 10^{6} \text{ s}.$$

Thus,

$$\frac{3dmu^2}{mk} = 3{,}79 \times 10^{-10}.$$

This shows clearly that $3dmu^2/l^2$ is a correction term to mk/l^2. The Eq.(10) can be put in the form:

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{l^2} + \epsilon u^2, \tag{11}$$

where $\epsilon = 3dm/l^2$ and solved perturbatively [12], choose:

$$u = \sum_{k=0}^{\infty} \epsilon^k u_k = \epsilon^k u_k. \qquad \text{Einstein summation convention implied,}$$

substitute this expansion in the Eq.(11), given

$$\epsilon^k \frac{d^2 u_k}{d\theta^2} + \epsilon^k u_k = \frac{mk}{l^2} + \epsilon(\epsilon^k u_k)^2.$$

and collect terms with the same power of ϵ. The zero order of perturbation is the equation:

$$\frac{d^2 u_0}{d\theta^2} + u_0 = \frac{mk}{l^2},$$

whose solution with the appropriate initial conditions and with the semi-major axis a pointing in the x direction is

$$u_0 = \frac{mk}{l^2}(1 + e\cos\theta),$$

where e is the eccentricity given by the expression $l^2 = mka(1 - e^2)$. The next order of perturbation is obtained taking the terms in ϵ^1 and using value of u_0 obtained above, it reads

$$\frac{d^2 u_1}{d\theta^2} + u_1 = u_0^2.$$

The solution of the Eq.(10) up to the first order is

$$u = \frac{mk}{l^2}(1 + e\cos(\theta - \delta\theta)),$$

where

$$\delta\theta = \frac{3dm^2k}{l^4}\theta.$$

After one revolution ($\theta = \delta\theta + 2\pi \approx 2\pi$), so that the precession is

$$\delta\theta = \frac{6\pi dm^2 k}{l^4}.$$

Using the experimental data for Mercury we have: $\delta\theta \approx 1.27 \times 10^{-12}$ arcsec/Cy.

6. Conclusions

In this work we have focused on the gravitomagnetic field produced by two rotating spheres, in particular we considered that the field produced by the Sun and its influence on Mercury. Our work is a first step to analyse the gravitomagnetic field influence in solar system's planets. We concluded that as was expected the value for the advance of perihelion obtained by GM is very small comparing with the one obtained by GR that is $\delta\theta \approx 43$ arcsec/Cy. However we still search if it's possible that the GM have a significant effect in some system.

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