Ward identities and the analogous Goldberger-Treiman relation in a three-flavor Spectral Quark Model

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Abstract. This work presents the first results of an extension of the spectral quark model which includes different flavors. The spectral quark model is an approach based on a generalization of the Lehmann representation for the quark propagator. Gauge and chiral invariance are ensured with the help of gauge technique which provides particular solutions to the Ward-Takahashi identities. General conditions on the quark spectral function follow from natural physical requirements. In particular, the function is normalized, its positive momenta must vanish, while the physical observables depend on negative moments and the so-called log moments. As a consequence, the model is made finite. To allow the description of mesons constituted by different flavors of quarks we introduce different spectral functions and obtain vertex functions constructed from Ward-Takahashi identities that includes two different spectral (constituent) quark masses, allowing the physical description of strange mesons, for example. We obtain some observables based on the current approach and, in particular, the spectral version of the Kaon analogous Goldberger-Treiman relation.

1. Introduction

The non-perturbative behavior of Quantum-Chromodynamics (QCD) at low energies is an ubiquitous feature of this theory. The running coupling constant of the strong interaction at this level assumes large values, not allowing the perturbative treatment that is successfully employed in the high energy limit or in other theories (Quantum Electrodynamics, for example). Thus, the study of the hadronic states that lie on this range of energy has to be carried out by using non-perturbative approaches such as lattice QCD or effective models. Belonging to this second class, chiral quark models, i.e., effective models that present the quarks as the relevant degrees of freedom and incorporate the relevant symmetries of QCD, are very successful on the description of the QCD low energy phenomenology. In this case, the non-perturbative behavior of QCD manifests itself by the presence of a constituent mass for the quarks, the consequence of a non-perturbative vacuum. The constituent mass is the effective mass that includes the effects of quantum corrections due the interaction of quarks with gluons and the self interaction of the gluons.

As a price to be paid, effective models are not fundamental, and need the introduction of parameters external to the theory. Some of these models include other degrees of freedom, such as mesonic [1, 2] or gluonic fields [3] that, even ensuring their renormalizability, introduce extra couplings and masses not present on the underlying theory. Other purely fermionic models, as the well established Nambu-Jona-Lasinio model [4, 5] are non renormalizable, and results are dependent of the regularization schemes and regularization parameters employed on their treatment.

The Spectral Quark Model (SQM) is a recently formulated effective chiral quark model that naturally incorporate some of the essential features discussed above. The non perturbative behavior is introduced by using the Lehmann representation for the quark propagators. These propagators include the constituent mass as a spectral running mass, reproducing the effect of the dependence of the constituent mass with the energy scale of the processes involved. SQM is also constructed in such a way to preserve important symmetries of QCD, by means of the gauge technique [6, 7].

A surprisingly and relevant feature of SQM is its finiteness, constructed via the correct selection of the so called spectral conditions and implying in the independence of the regularization scheme and regularization parameters employed on the intermediary steps of the calculations. Although the introduction of the Lehmann spectral distributions is *per se* an introduction of extra parameters, the SQM provides, by imposing physical conditions (as unitarity, finiteness and symmetries expressed by Ward identities [8, 9]) the necessary relations to determine the relevant parameters of the spectral distribution - the spectral conditions. The model was successfully applied to describe the low energy hadrons phenomenology [10, 11, 12] and other effects such as the Polyakov loop [13], Chiral anomaly [14] and so on.

As originally formulated, the SQM includes only one quark flavor (expressed by an unique spectral distribution), being adequate to describe the observables related to one flavored states, as the mesons Pion, Eta, Rho and so on. As a consequence, SQM still could not be used to describe the phenomenology involving two different quarks, like strange mesons for example.

In this contribution we take the first steps towards the construction of a three flavored version of the Spectral Quark Model by postulating two different spectral distributions for the up/down and for the strange quarks. We will show here that it is possible to construct vector and axial vectors vertex functions with one current that satisfy the Ward Identities and involving only the two spectral distributions mentioned above. Besides, by using the axial vertex function, we obtain the spectral version of the analogous Goldberger-Treiman relation for the Kaon.

2. The Model

In order to develop the three flavored version of SQM, we closely follow the results presented in [10], employing the Lehmann representation for the up/down and strange quarks propagators:

$$S_s(p) = \int_C d\omega \frac{\rho_s(\omega)}{\not{p} - \omega} \qquad S_u(p) = \int_C d\omega \frac{\rho_u(\omega)}{\not{p} - \omega}, \tag{1}$$

where ω is the spectral constituent mass, $\rho_u(\omega)$ e $\rho_s(\omega)$ are the spectral distributions for quarks up/down and strange respectively and C denotes a contour in the complex ω plane. As in the one flavored version, physical requirements imposed some relations (the spectral conditions) for the moments of the spectral distributions, such as the normalization condition

$$\rho_{q,0} \equiv \int d\omega \rho_q(\omega) = 1. \tag{2}$$

We also define the positive, negative and log spectral momenta respectively as

$$\rho_{q,n} \equiv \int d\omega \omega^n \rho_q(\omega) \qquad n = 1, 2, ..., \tag{3}$$

$$\rho_{q,-n} \equiv \int d\omega \omega^{-n} \rho_q(\omega), \tag{4}$$

and

$$\rho'_{q,n} \equiv \int d\omega \log(\omega^2)\omega^n \rho_q(\omega). \tag{5}$$

In equations (3)-(5) the distribution $\rho_q(\omega)$ denotes generically any of the distributions $\rho_u(\omega)$ and $\rho_s(\omega)$. Other relevant quantities to be considered on the current approach are the one current vector and axial unamputated vertex functions, defined as

$$\Lambda_V^{\mu,a}(p,p') = iS_u(p')\Gamma_V^{\mu,a}(p,p')iS_s(p) = \int d^4x d^4x' \langle 0|T\{J_V^{\mu,a}(0)q(x')\overline{q}(x)\}|0\rangle e^{i(p'x'-px)}, \tag{6}$$

and

$$\Lambda_A^{\mu,a}(p,p') = iS_u(p')\Gamma_A^{\mu,a}(p,p')iS_s(p) = \int d^4x d^4x' \langle 0|T\{J_A^{\mu,a}(0)q(x')\overline{q}(x)\}|0\rangle e^{i(p'x'-px)}, \tag{7}$$

respectively. Here

$$J_V^{\mu,a}(x) = \overline{q}(x)\gamma^\mu \frac{\lambda_a}{2} q(x), \tag{8}$$

and

$$J_A^{\mu,a}(x) = \overline{q}(x)\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x). \tag{9}$$

are the vector and axial currents and the Γ's represent the corresponding amputated vertex functions. Also, λ_a are the Gell-Mann matrices and γ^μ are the Dirac matrices. The Ward-Takahashi (WT) identity for the full vector vertex reads

$$(p'-p)_\mu \Lambda_V^{\mu,a}(p,p') = S_u(p')\frac{\lambda_a}{2} - \frac{\lambda_a}{2} S_s(p), \tag{10}$$

and, for the axial vertex,

$$(p'-p)_\mu \Lambda_A^{\mu,a}(p,p') = S_u(p')\frac{\lambda_a}{2}\gamma_5 + \gamma_5 \frac{\lambda_a}{2} S_s(p). \tag{11}$$

The SQM also makes use of the gauge technique [15, 16] in order to determine particular representations of the vertex functions Eq.(6) and Eq.(7) that satisfy the WT identities, Eq.(10) and (11). We will determine these quantities, in the context of the three flavor version of SQM, on section 5. In the next sections, we will obtain some single expressions for the quark condensates and the vacuum energy density.

3. $\langle \bar{u}u \rangle$ and $\langle \bar{s}s \rangle$ quark condensates

The quark condensate is the vacuum expected value of a chiral fermion/antifermion pair, taken as the order parameter for the chiral phase transition from the chiral symmetric phase($\langle \bar{q}q \rangle = 0$) to the phase where this symmetry is broken ($\langle \bar{q}q \rangle \neq 0$). The quark condensate for a single flavor is given by

$$\langle \bar{q}q \rangle = -iN_c \int d\omega \rho_q(\omega) \int \frac{d^4p}{(2\pi)^4} Tr \frac{1}{\not{p} - \omega}, \tag{12}$$

where the trace is done on the Dirac matrices and $N_c = 3$ is the number of colors. As we need to perform the integration on p before to proceed the spectral integration, we need to use an auxiliary regularization method - which will be removed at the end of calculation, once that

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the integral over the momentum p is quadratically divergent. Here, we employ the sharp-cutoff regularization, obtaining

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2}\int d\omega \rho_q(\omega)\omega \left[\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{\Lambda^2+\omega^2}\right)\right]. \tag{13}$$

As before, the subscript q denotes both u or s flavors of the quarks. As we can see from Eq. 13, finiteness of the result at $\Lambda \to \infty$ requires that

$$\rho_{q,1} = \int d\omega \omega \rho_q(\omega) = 0, \tag{14}$$

and

$$\rho_{q,3} = \int d\omega \omega^3 \rho_q(\omega) = 0. \tag{15}$$

and thus

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2}\int d\omega \log(\omega^2)\omega^3 \rho_q(\omega) = -\frac{N_c}{4\pi^2}\rho'_{q,3}, \tag{16}$$

or

$$\rho'_{q,3} = -\frac{4\pi^2}{N_c}\langle \bar{q}q \rangle. \tag{17}$$

allowing the determination of the spectral momenta $\rho'_{q,3}$ from the experimental estimates of the quark condensates. Except for the possibility of employing different spectral conditions, this result is the same previously obtained on [10].

4. Vacuum energy density

The vacuum energy density (denoted as B) is the vacuum energy related to the spontaneous creation of virtual particle/antiparticle pairs. Within SQM, it can be evaluated from the vacuum expected value of the energy-momentum tensor for a purely quark model, given by, after explicitly taken the trace on flavors

$$\begin{aligned} g^{\mu\nu}B &= \langle \theta^{\mu\nu}(x) \rangle - \langle \theta^{\mu\nu}(x) \rangle_0 \\ &= -iN_c \int d\omega (2\rho_u(\omega) + \rho_s(\omega)) \int \frac{d^4p}{(2\pi)^4} Tr\left\{ \frac{1}{\not{p}-\omega} \times \left[\frac{1}{2}(\gamma^\mu p^\nu + \gamma^\nu p^\mu) - g^{\mu\nu}(\not{p}-\omega) \right] \right\} \\ &= -4iN_c \int d\omega (2\rho_u(\omega) + \rho_s(\omega)) \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu p^\nu - g^{\mu\nu}(p^2-\omega^2)}{p^2-\omega^2}, \end{aligned} \tag{18}$$

where $\langle \theta^{\mu\nu} \rangle_0$ is the expected value of the energy-momentum tensor evaluated for the free theory (with $\rho(\omega) = \rho(\omega') = \delta(\omega)$).

Thus, we obtain, for the vacuum energy density,

$$B = -iN_c \int d\omega (2\rho_u(\omega) + \rho_s(\omega)) \int \frac{d^4p}{(2\pi)^4} \frac{\omega^2}{p^2-\omega^2}. \tag{19}$$

Employing again a sharp cut-off intermediate regularization procedure, we obtain

$$B = -\frac{N_c}{16\pi^2}\int d\omega (2\rho_u(\omega) + \rho_s(\omega))\omega^2 \left[\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{\Lambda^2+\omega^2}\right)\right]. \tag{20}$$

Hence, the conditions that must be fulfilled for finiteness of B are

$$\rho_{q,2} = \int d\omega \omega^2 \rho_q(\omega) = 0, \tag{21}$$

and

$$\rho_{q,4} = \int d\omega \omega^4 \rho_q(\omega) = 0. \tag{22}$$

Finally,

$$\begin{aligned} B &= -\frac{N_c}{16\pi^2}\left\{2\int d\omega \rho_u(\omega)\omega^4 \log(\omega^2) + \int d\omega \rho_s(\omega)\omega^4 \log(\omega^2)\right\} \\ &= -\frac{N_c}{16\pi^2}(2\rho'_{u,4} + \rho'_{s,4}). \end{aligned} \tag{23}$$

Written in terms of the expected value for the gluon condensate, $\langle \frac{\alpha_s}{\pi} G^2 \rangle$, we have [7]

$$B = -\frac{N_c}{16\pi^2}(2\rho'_{u,4} + \rho'_{s,4}) = -\frac{9}{32}\langle \frac{\alpha}{\pi} G^2 \rangle, \tag{24}$$

or

$$(2\rho'_{u,4} + \rho'_{s,4}) = -\frac{3\pi^2}{2}\langle \frac{\alpha}{\pi} G^2 \rangle. \tag{25}$$

where α_s is the perturbative strong interaction constant, $G^2 = G^a_{\mu\nu} G^{a\mu\nu}$ and $G^{\mu\nu}$ is the gluon field strength tensor.

By collecting the results for the momenta of the spectral distributions, such as Eq.(17) and (25), one can find a set of equations relating the unknown spectral momenta to physical observables that allow the determination of the relevant momenta of ρ_u and ρ_s and giving predictive power to the SQM. In this contribution, however, we will concentrate on the obtaining of vertex functions that satisfy the Ward identities in the context of quarks with different flavors.

5. Vertices with one current from the gauge technique

The gauge technique consists on finding solutions for the chiral and electromagnetic WT identities, Eqs. (10) and (11), determining a particular representation for the vector and axial-vector vertices. For two different flavors, we write a first attempt solution for the vector WT identity as

$$\Lambda^{\mu,a}_V(p,p') = iS_u(p')\Gamma^{\mu,a}_V(p,p')iS_s(p) = \int d\omega' d\omega \rho_{us}(\omega,\omega') \frac{i}{\not p' - \omega'}\gamma^\mu \frac{\lambda_a}{2}\frac{i}{\not p - \omega}. \tag{26}$$

One should expect that the spectral distribution for two different flavors, $\rho_{us}(\omega,\omega')$, could be written as a separable function of the two spectral masses expressed in terms of the quark distribution functions ρ_u and ρ_s. In fact, this is really the case we will treat here, and is a remarkable feature that this possibility is assured by the normalization spectral condition, Eq. (2), as we shall see. So, we will assume

$$\rho_{us}(\omega,\omega') = \rho_u(\omega)\rho_s(\omega'). \tag{27}$$

We should stress, however, that the separable solution is not the only possibility. For example, the results for the one flavor SQM can be recovered from our approach if we write ρ_{us} (in fact, ρ_{uu}, in this case) as

$$\rho_{uu}(\omega,\omega') = \rho_u(\omega)\delta(\omega - \omega'), \tag{28}$$

and this solution is non-separable.

Inserting the attempt solution (26) on the vector WT identity, Eq.(10), and using Eq.(27), we obtain

$$
\begin{aligned}
(p'-p)_\mu iS_u(p')\Gamma_V^{\mu,a}(p,p')iS_s(p) = &-\frac{\lambda_a}{2}\int d\omega'\rho_u(\omega')\int d\omega\frac{\rho_s(\omega)}{\not{p}-\omega} \\
&+\frac{\lambda_a}{2}\int d\omega\rho_s(\omega)\int d\omega'\frac{\rho_u(\omega')}{\not{p}'-\omega'} \\
&+\int d\omega' d\omega\rho_u(\omega')\rho_s(\omega)\frac{i}{\not{p}'-\omega'}(\omega'-\omega)\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega},
\end{aligned} \tag{29}
$$

where we can explicitly see that the normalization spectral condition, Eq.(2), ensures that

$$
\begin{aligned}
(p'-p)_\mu iS_u(p')\Gamma_V^{\mu,a}(p,p')iS_s(p) = S_u(p')\frac{\lambda_a}{2}-\frac{\lambda_a}{2}S_s(p) \\
+\int d\omega' d\omega\rho_u(\omega')\rho_s(\omega)\frac{i}{\not{p}'-\omega'}(\omega'-\omega)\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}.
\end{aligned} \tag{30}
$$

The first attempt solution does not satisfy the WT identity, but can be used as a guide for a second *ansatz* for the vector vertex function, written as

$$
\Lambda_V^{\mu,a}(p,p') = \int d\omega' d\omega\rho_u(\omega')\rho_s(\omega)\frac{i}{\not{p}'-\omega'}\left(\gamma^\mu - \frac{(\omega'-\omega)q^\mu}{q^2}\right)\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}, \tag{31}
$$

where $q^\mu = (p'-p)^\mu$. One can easily show that Eq.(31) is a solution of the vector WT identity, Eq.(10). Also, if we replace the spectral two quarks distribution ρ_{us} by the one flavor non separable solution, Eq.(28), we recover the result of the original SQM.

Following the same steps, we obtain, for the solution of the axial WT identity, Eq. (11), the axial unamputated vertex,

$$
\Lambda_A^{\mu,a}(p,p') = \int d\omega' d\omega\rho_u(\omega')\rho_s(\omega)\frac{i}{\not{p}'-\omega'}\left(\gamma^\mu - \frac{(\omega'+\omega)q^\mu}{q^2}\right)\gamma^5\frac{\lambda_a}{2}\frac{i}{\not{p}-\omega}. \tag{32}
$$

The results showed here are similar to those obtained in ref. [10], but present two different features that are significant to the development of a three flavor version of SQM: (a) the vector current term inside the spectral integral of the vertex function presents a dependence on the spectral masses ω and ω', implying on the rising of extra terms on the expressions of observables involving the vector function, in contrast with the one flavored version, where the vector current contains only the γ^μ term; (b) Both vertex functions involve two spectral distributions integrated over two different spectral masses. This fact allows the emergence of new spectral moments involving non separable integrals in ω and ω' (convolution integrals of the distribution functions ρ_u and ρ_s). Nevertheless, none of the results presented in this contribution depends on these new spectral moments.

6. Analogous Goldberger-Treiman Relation for the Kaon

In this section, we present a version of the Goldberger-Treiman Relation (GT) for the kaon, in the context of the three flavor version of SQM. As it is well known, any chiral model which intends to be a prototype of QCD in low energies should satisfy the GT Relation, once that such relation is not dependent of any model, but, instead of, it is a consequence of the chiral symmetry breaking and the partial conservation of axial current hypothesis (PCAC), two of the most important elements of QCD. GT relation establishes a connection between quark and mesonic properties, like masses, meson weak constant decays and quark-meson coupling constants in low energies.

We can obtain the analogous GT Relation in the SQM from the axial unamputaded vertex expression. In such vertex, we take the limit $q^2 \to 0$, where q^μ is the meson momentum. In this limit, we mean that there is a coupling between the Goldstone bosons (kaons or pions) and the axial current, and such coupling is proportional to the meson momentum. As we can see, the axial unamputaded vertex has a pole in $(q^2 \to 0)$ and, the kaon decay, characterized by the axial vertex (32), is dominated by the pseudoscalar coupling and the axial vertex is associated with the wave function of the kaon (corresponding to the vertex $k \to q\bar{q}$) by the relation

$$\Lambda_A^{\mu,a}(p,p')|_{q\longrightarrow 0} \to -2f_k \frac{q^\mu}{q^2}\Lambda_k^a(p,p'), \tag{33}$$

where f_k is the kaon constant decay. From (33), we get

$$\Lambda_k^a(p,p') = \int d\omega' d\omega \rho_u(\omega')\rho_{\bar{s}}(\omega)\frac{i}{\not{p}' - \omega'}\frac{(\omega' + \omega)}{2f_k}\gamma^5 \frac{\lambda_a}{2}\frac{i}{\not{p} - \omega}, \tag{34}$$

and we can identify the coupling between quarks and the kaon by

$$g_k(\omega', \omega) = \frac{\omega' + \omega}{2f_k}. \tag{35}$$

which is the analogous Goldberger-Treiman relation for the kaon [17], connecting the coupling constant between the kaon meson and the up and strange quarks, the spectral quark masses and the kaon constant decay. It is immediate to see that the analogous GT relation reproduces the expected result in the one flavor SQM [10], where the two spectral masses are equal.

7. Conclusion

In this work we obtained the vector and axial vector vertex functions in the context of the Spectral Quark Model by using an approach that includes different flavors of quarks. The approach is based on employing two different spectral functions and, by using the gauge technique, obtaining vertex functions that are related to these spectral propagators and satisfy the corresponding vector and axial vector Ward-Takahashi identities. From the propagators and vertex functions we obtained the expressions for the quark condensate, vacuum energy density and the analogous Goldberger-Treiman relation for the kaon. We discussed the relevant differences between the current approach and the one flavor version, and the spectral two quarks distribution that leads the current version to the original one. We also obtained the spectral version of the kaon analogous Goldbeger-Treiman relation. The approach opens the possibility to obtain other observables of mesons with different content of quarks, such as the weak decay and the electromagnetic form factor of the strange mesons (work in progress).

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