

Thermodynamics of an exactly solvable confining quark model

Bruno W. Mintz¹

Departamento de Física Teórica, Instituto de Física, UERJ - Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, 20550-013, Maracanã, Rio de Janeiro, Brasil

E-mail: brunomintz@gmail.com

Abstract. The grand partition function of a model of confined quarks is exactly calculated at arbitrary temperatures and quark chemical potentials. The model is inspired by a version of QCD where the usual (perturbative) BRST symmetry is broken in the infrared, while possessing a quark mass function compatible with nonperturbative analyses of lattice simulations and Dyson-Schwinger equations. Even though the model is defined at tree level, we show that it produces a non-trivial and stable thermodynamic behaviour at any temperature or chemical potential. Results for the pressure, the entropy and the trace anomaly as a function of the temperature are qualitatively compatible with the effect of non-perturbative interactions as observed in lattice simulations. The finite density thermodynamics is also shown to contain non-trivial features, being far away from an ideal gas picture.

1. Introduction

Given the difficulty to address the problem of confinement in strongly interacting systems directly from its fundamental theory, Quantum Chromodynamics (QCD), several alternative approaches have been proposed. One of the main such approaches is that of effective models of QCD, which are quantum field theories that possess one or more fundamental aspects of the original theory but being nevertheless easier to have some information extracted.

Regarding the quark sector of QCD, two quite successful models in the description of chiral symmetry breaking and its restoration at high temperature are the Linear Sigma Model with quarks (LSM) [1], and the Nambu-Jona-Lasinio (NJL) Model [2]. In their original formulations, these models do not address the issue of quark confinement. Indeed, in both models quarks are effectively treated as on-shell quasiparticles. Two possible directions that can be taken towards a simultaneous description of both chiral and confinement dynamics are represented either by the coupling of the Polyakov loop to quark degrees of freedom (the so-called PLSM and PNJL models) [3, 4, 5, 6, 7, 8], or by considering nonlocal interactions between quarks as a result of their nonperturbative coupling to gluons, as in the nonlocal versions of the NJL model [9, 10, 11, 12, 13]. Following [14, 15, 16, 17, 18], we consider a third possibility, which is inspired by the (refined) Gribov-Zwanziger effective theory for infrared QCD [19, 20, 21], although not equivalent to it.

Although it has been long clear that quarks and gluons are confined to hadrons, a definite theoretical criterion for confinement is not yet a settled issue. A sufficient condition for the

¹ In collaboration with Letícia F. Palhares and Marcelo S. Guimarães.



absence of isolated quarks or gluons from asymptotic states (i.e., confinement) is assumption that they violate reflection positivity [23]. We therefore take violation of reflection positivity as our criterion for confinement, following, e.g., [10, 22]. In this work, we explore the thermodynamics of a quark model in which confinement is encoded in the positivity violation of the quark propagator.

2. A nontrivial solvable quark model

As evidence from lattice QCD [24] and Dyson-Schwinger Equations [25] studies shows, the zero-temperature quark propagator can be quite well parametrized by a momentum-dependent mass function compatible with the functional form

$$M_{eff}(p) = \frac{\Lambda}{p^2 + m^2} + m_0, \quad (1)$$

where m_0 is the quark current mass. Indeed, the data of [24] can be well fitted by Eq. (1), with the values $\Lambda = 0.196 \text{ GeV}^3$, $m^2 = 0.639 \text{ GeV}^2$, and $m_0 = 0.014 \text{ MeV}$ [16]. It is interesting to notice that, with such parameters, the resulting euclidean quark propagator

$$S(p) = \frac{1}{\gamma \cdot p + M_{eff}(p)} \quad (2)$$

displays violation of reflection positivity, indicating the confinement of quarks [17].

The mass function (1) can be obtained from the lowest-level quark propagator of the theory given by sum of the QCD lagrangian,

$$S_{QCD} = \int d^4x \left[\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi}_\alpha^i [i(\gamma_\mu)_{\alpha\beta} D_\mu^{ij} - m_0 \delta_{\alpha\beta} \delta^{ij}] \psi_\beta^j + ib^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right], \quad (3)$$

with the BRST invariant action

$$\begin{aligned} S_{\xi\lambda} &= s \int d^4x \left[-\bar{\eta}_\alpha^i \partial^2 \xi_\alpha^i + \bar{\xi}_\alpha^i \partial^2 \eta_\alpha^i + m^2 (\bar{\eta}_\alpha^i \xi_\alpha^i - \bar{\xi}_\alpha^i \eta_\alpha^i) \right] \\ &= \int d^4x \left[-\bar{\lambda}_\alpha^i \partial^2 \xi_\alpha^i - \bar{\xi}_\alpha^i \partial^2 \lambda_\alpha^i - \bar{\eta}_\alpha^i \partial^2 \theta_\alpha^i + \bar{\theta}_\alpha^i \partial^2 \eta_\alpha^i + m^2 (\bar{\lambda}_\alpha^i \xi_\alpha^i + \bar{\xi}_\alpha^i \lambda_\alpha^i + \bar{\eta}_\alpha^i \theta_\alpha^i - \bar{\theta}_\alpha^i \eta_\alpha^i) \right] \end{aligned} \quad (4)$$

and the coupling term

$$S_M = \int d^4x \left[M_1^2 (\bar{\xi}_\alpha^i \psi_\alpha^i + \bar{\psi}_\alpha^i \xi_\alpha^i) - M_2 (\bar{\lambda}_\alpha^i \psi_\alpha^i + \bar{\psi}_\alpha^i \lambda_\alpha^i) \right] \quad (5)$$

between the quark fields ψ and the auxiliary fields ξ and λ . The resulting action

$$S = S_{QCD} + S_{\xi\lambda} + S_M \quad (6)$$

has been shown to be renormalizable [15], being equivalent to QCD in the high-energy limit but radically changing the infrared sector of the theory (with respect to the perturbative picture). We interpret the extra fields and their interaction with the quark fields ψ as a local way (in the sense of QFT) to take into account an effective dressing of quarks by gluons. The auxiliary fields of the quark sector of (6), can be straightforwardly integrated out, providing an effective theory for the quarks, whose (nonlocal) action reads, in the quadratic approximation,

$$S_{nl} = \int d^4x \bar{\psi}_\alpha^i \left[i(\gamma_\mu)_{\alpha\beta} \delta^{ij} \partial_\mu - \delta^{ij} \delta_{\alpha\beta} \left(\frac{2M_1^2 M_2}{-\partial^2 + m^2} + m_0 \right) \right] \psi_\beta^j. \quad (7)$$

Notice that, in this model, the quark mass function (1) is directly derived from the effective action (7), with $\Lambda \equiv 2M_1 M_2$.

Although for now we only shall investigate the model in the quadratic level, it is perfectly possible to consider a loop expansion of the free energy [26]. It is also reassuring that the full local model (6) is a renormalizable QFT, as shown in [15].

3. The partition function

At lowest order, the theory defined by (6) has a quadratic action. Therefore, its grand partition function can be exactly calculated using standard techniques of Finite-Temperature Field Theory [26]. Besides having a quadratic action, the theory does not correspond to a free theory, a fact that can be clearly seen, e.g., from the nonlocal form of the equivalent effective action (7).

One can easily introduce temperature by compactifying one euclidean direction (conventionally the 4-direction), whereas the introduction of the chemical potential is more subtle. Starting from the local action (6), one must first calculate the hamiltonian \mathcal{H} . By identifying the quark number with the charge associated with the $U(1)$ global symmetry transformation

$$\psi(x) \longrightarrow e^{-i\alpha}\psi(x), \quad (8)$$

one can use Noether's theorem to calculate the corresponding conserved current density j^μ and let $\mathcal{N} = \int d^3x j^0$ be the quark number operator. The resulting grand partition function

$$Z(T, \mu) = \text{Tr} \exp \left[-\frac{\mathcal{H} - \mu\mathcal{N}}{T} \right], \quad (9)$$

can be cast in a functional integral form and straightforwardly calculated [18]. The result is conveniently split into a μ -independent term plus a (T, μ) -dependent term as

$$\begin{aligned} \frac{\log Z(T, \mu)}{2\beta V N_c N_f} &= \oint \log \left\{ \beta^2 \left[\mathbf{p}^2 + M_{n,\mathbf{p}}^2(0) + \omega_n^2 \right] \right\} + \oint \log \left\{ \frac{\mathbf{p}^2 + M_{n,\mathbf{p}}^2(\mu) - (i\omega_n + \mu)^2}{\mathbf{p}^2 + M_{n,\mathbf{p}}^2(0) + \omega_n^2} \right\} \\ &\equiv \frac{\log Z(T, 0)}{2\beta V N_c N_f} + \frac{\log Z^{(\mu)}(T, \mu)}{2\beta V N_c N_f}, \end{aligned} \quad (10)$$

where we used the standard sum-integral notation,

$$\oint(\dots) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}(\dots). \quad (11)$$

The $\mu = 0$ term can be split in a sum of four terms, two of which corresponding to positive pressures of particles with complex conjugate masses, one of which of a particle with real mass, and one of which with a negative contribution or the pressure. Each of this term can be calculated straightforwardly as the pressure of a free gas [26]. After the usual vacuum energy subtraction [26], one finds

$$\log Z(T, 0) = \log Z_0 + 4N_c N_f V \int \frac{d^3p}{(2\pi)^3} \log \left[\frac{(1 + e^{-\beta\varphi_1})(1 + e^{-\beta\varphi_2})(1 + e^{-\beta\varphi_3})}{(1 + e^{-\beta\varphi_0})^2} \right], \quad (12)$$

for the $\mu = 0$ contribution, where

$$\log Z_0 = 2N_c N_f \beta V \int \frac{d^3p}{(2\pi)^3} (\varphi_1 + \varphi_2 + \varphi_3 - 2\varphi_0) \quad (13)$$

is the pure vacuum contribution. The quantities φ_i ($i = 0, 1, 2, 3$) are known functions of the internal momentum p^2 [18].

The μ -dependent term is given by

$$\log Z^{(\mu)}(T, \mu) = 2\beta V N_c N_f \oint \log \left\{ \frac{\mathbf{p}^2 + M_{n,\mathbf{p}}^2(\mu) - (i\omega_n + \mu)^2}{\mathbf{p}^2 + M_{n,\mathbf{p}}^2(0) + \omega_n^2} \right\}. \quad (14)$$

A closed expression for (14) can be found at the zero-temperature limit. Using Cauchy theorem, one may write

$$\begin{aligned}\log Z(0, \mu) &= \log Z^{(\mu)}(0, \mu) = 2\beta V N_c N_f \int \frac{d^3 p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} f(i\theta + \mu) \\ &= 2\beta V N_c N_f \int \frac{d^3 p}{(2\pi)^3} \int_0^{\infty} \frac{d\theta}{2\pi} [f(i\theta + \mu) + f(-i\theta + \mu)],\end{aligned}\quad (15)$$

where

$$f(\xi) := \log \left\{ \frac{\Omega_{\mathbf{p}}^2(\xi^2) - \xi^2}{\Omega_{\mathbf{p}}^2[(\xi - \mu)^2] - (\xi - \mu)^2} \right\} \quad (16)$$

and

$$\Omega_{\mathbf{p}}^2(\zeta) := \mathbf{p}^2 + \left[\frac{M_3}{-\zeta + \mathbf{p}^2 + m^2} + m_0 \right]^2. \quad (17)$$

The expressions above allowed us to evaluate the partition function (10) exactly in order to compute several thermodynamical quantities. We present our results in the next section.

4. Results

From the partition function, one can derive any equilibrium thermodynamical quantities. Let us first show our results for the pressure

$$P(T, \mu) = \frac{T}{V} \log Z(T, \mu). \quad (18)$$

In Figure 1, we show our results for zero chemical potential, with a comparison between our model, a massless bag model and a gas of free massive particles with mass M_{thr} . Notice that, in the low temperature regime, the model result is quite close to that of a free, massive gas of mass $M_{thr} = 0.467$ GeV. As the temperature rises, the pressure of the model rises above that of the free gas. We interpret this as a consequence of the decreasing mass function, Eq. (1), as one roughly expects that the average momentum of field excitations should increase with temperature. A difference between two kinds of modelling should be expected, since our model has positivity-violating quarks as its elementary constituents.

Figure 2 displays our results for the normalized trace anomaly at zero chemical potential. Notice that the model approaches the massless (conformally invariant) behavior at high temperature faster than the massive free gas.

$$\Delta(T) = \frac{E - 3P}{T^4} = T \frac{\partial}{\partial T} \left(\frac{P}{T^4} \right). \quad (19)$$

The zero-temperature, finite chemical potential pressure (15) can be seen in Figure 3. Notice that the first excitations appear at a chemical potential $\mu \simeq M_{thr} = 0.467$ GeV, a mass scale which is not explicitly present in the model action (6). We thus interpret the threshold mass M_{thr} as a dynamically generated mass scale. Notice also that M_{thr} was used for the comparison between the model and a free gas in Figures 1 and 2, where a low-temperature agreement between the curves appears to be satisfactory. This indicates that a description purely in terms of a free gas of mass M_{thr} can only be reasonable just above the threshold, but not at higher densities. This is clearly a consequence of the momentum dependence of the mass function (1).

For completeness, we show in Figure 4 our results for the pressure at both finite temperature and finite chemical potential. As one could expect from general thermodynamical arguments, for a fixed value of chemical potential, increasing temperatures lead to higher pressures. Notice that, as soon as the temperature is nonzero, thermal activation allows field excitations for arbitrarily

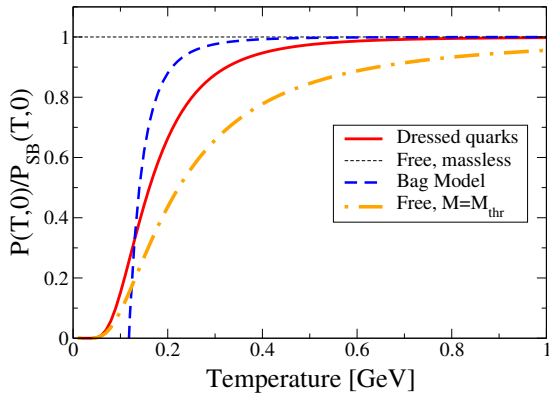


Figure 1. Pressure for the model quarks (solid, red line), for bag model quarks (dashed, blue line, with $B = (0.145 \text{ GeV})^4$), and for a gas of free but massive quarks (dashed-dotted yellow line, with mass $M_{thr} = 0.467 \text{ GeV}$). All lines are normalized by the Stefan-Boltzmann limit $P_{SB} = 7\pi^2 N_c N_f T^4 / 180$.

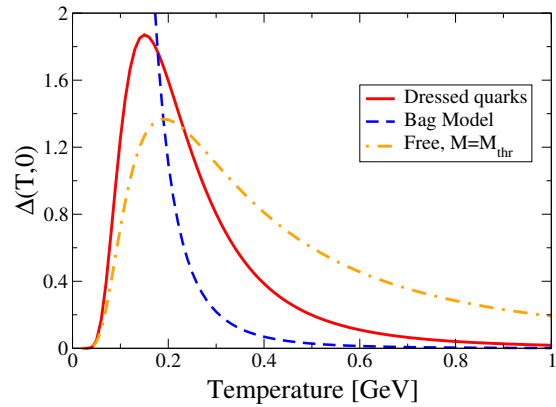


Figure 2. Trace anomaly at $\mu = 0$ for dressed quarks (solid, red line), for massless quarks under a bag pressure (dashed, blue line, with $B = (0.145 \text{ GeV})^4$), and for massive free quarks (dashed-dotted line, with mass $M_{thr} = 0.467 \text{ GeV}$).

low chemical potentials. This can be seen from the nonzero values as well as a smoothening of the pressure curve already below $\mu = M_{thr}$.

Let us finally notice that all our results are compatible with thermodynamical stability, a feature that is not always present in quark models with complex masses (for two studies, see, e.g., [12, 13]).

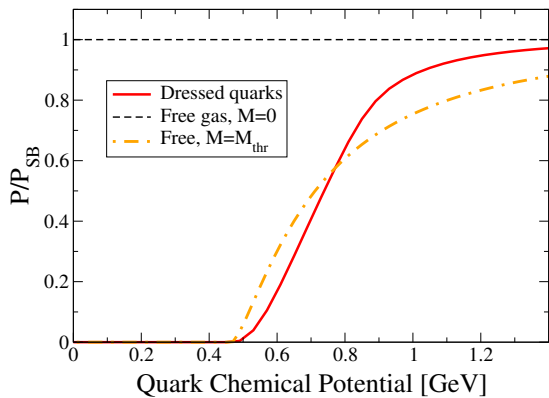


Figure 3. Zero-temperature limit of the pressure as a function of the chemical potential, normalized by the Stefan-Boltzmann limit $P_{SB} = N_c N_f \mu^4 / (12\pi^2)$. Red, solid line: model result. Yellow, dotted-dashed line: free massive gas, with mass $M_{thr} = 0.467 \text{ GeV}$.

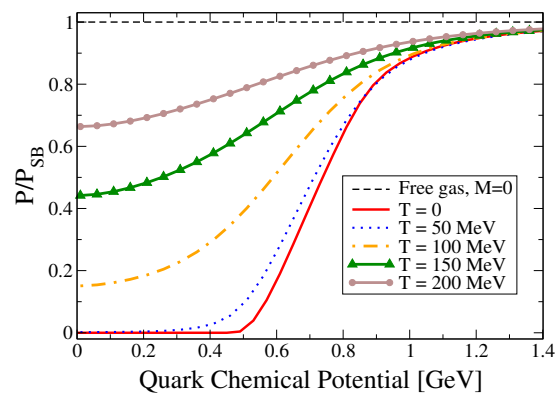


Figure 4. Model pressure as a function of chemical potential for various temperatures. Solid, red: $T = 0$. Dotted, blue: $T = 50 \text{ MeV}$. Dotted-dashed, yellow: $T = 100 \text{ MeV}$. Solid with triangles, green: $T = 150 \text{ MeV}$. Solid with circles, gray: $T = 200 \text{ MeV}$.

5. Summary

In this work, we report on a first exploratory investigation of the thermodynamics of the confining quark model proposed in [14]. We computed exactly the lowest-order partition function of the model at arbitrary temperature and chemical potential. Notice that, although the theory in its lowest nontrivial order is quadratic in the fields, it does not correspond to a free theory. Indeed, the quark propagator arising from the quadratic theory displays violation of reflection positivity and fits quite well lattice results for the quark mass function [24].

Our results for thermodynamical observables such as pressure and trace anomaly show that the microscopic confined degrees of freedom of the model do not correspond to a free gas of massive particles, even at lowest order. This can be understood from the dressing of quarks by the underlying gluons, which is encoded in the momentum-dependent mass function (1). In spite of the effective presence of particles with complex masses in the calculation of the partition function, we have not found any thermodynamical instabilities in the observables computed.

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