

# Correlation diagrams: an intuitive approach to correlations in quantum Hall systems

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**Abstract.** A trial wave function  $\Psi(1, 2, \dots, N)$  of an  $N$  electron system can always be written as the product of an antisymmetric Fermion factor  $\mathcal{F}\{z_{ij}\} = \prod_{i < j} z_{ij}$ , and a symmetric correlation factor  $\mathcal{G}\{z_{ij}\}$ .  $\mathcal{F}$  results from the Pauli principle, and  $\mathcal{G}$  is caused by Coulomb interactions. One can represent  $\mathcal{G}$  diagrammatically [1] by distributing  $N$  points on the circumference of a circle, and drawing appropriate lines representing correlation factors (cfs)  $z_{ij}$  between pairs. Here, of course,  $z_{ij} = z_i - z_j$ , where  $z_i$  is the complex coordinate of the  $i^{\text{th}}$  electron. Laughlin correlations for the  $\nu = 1/3$  filled incompressible quantum liquid (IQL) state contain two cfs connecting each pair  $(i, j)$ . For the Moore-Read state of the half-filled excited Landau level (LL), with  $\nu = 2 + 1/2$ , the even value of  $N$  for the half-filled LL is partitioned into two subsets  $A$  and  $B$ , each containing  $N/2$  electrons [2]. For any one partition  $(A, B)$ , the contribution to  $\mathcal{G}$  is given by  $\mathcal{G}_{AB} = \prod_{i < j \in A} z_{ij}^2 \prod_{k < \ell \in B} z_{k\ell}^2$ . The full  $\mathcal{G}$  is equal to the symmetric sum of contributions  $\mathcal{G}_{AB}$  over all possible partitions of  $N$  into two subsets of equal size. For Jain states at filling factor  $\nu = p/q < 1/2$ , the value of the single particle angular momentum  $\ell$  satisfies the equation  $2\ell = \nu^{-1}N - C_\nu$ , with  $C_\nu = q + 1 - p$ . The values of  $(2\ell, N)$  define the function space of  $\mathcal{G}\{z_{ij}\}$ , which must satisfy a number of conditions. For example, the highest power of any  $z_i$  cannot exceed  $2\ell + 1 - N$ . In addition, the value of the total angular momentum  $L$  of the lowest correlated state must satisfy the equation  $L = (N/2)(2\ell + 1 - N) - K_\mathcal{G}$ , where  $K_\mathcal{G}$  is the degree of the homogeneous polynomial generated by  $\mathcal{G}$ . Knowing the values of  $L$  for IQL states (and for states containing a few quasielectrons or a few quasiholes) from Jain's mean field CF picture allows one to determine  $K_\mathcal{G}$ . The dependence of the pair pseudopotential  $V(L_2)$  on pair angular momentum  $L_2$  suggests a small number of correlation diagrams for a given value of the total angular momentum  $L$ . Correlation diagrams and correlation functions for the Jain state at  $\nu = 2/5$  and for the Moore-Read states will be presented as examples. The generalizations of the method of selecting  $\mathcal{G}$  from small to larger systems will be discussed.

## 1. Introduction

In this paper, we study correlations resulting from Coulomb interactions in fractional quantum Hall systems. We introduce correlation diagrams to guide in the selection of the correlation functions caused by interactions. Electrons are represented by points located at positions  $z_i$  in the complex plane, and there are correlation lines connecting pairs of electrons. A correlation line connecting particles  $i$  and  $j$  represents a correlation factor (cf)  $z_{ij} = z_i - z_j$ . Although our correlation diagrams appear to resemble chemical bonds, they are just the opposite. A factor  $z_{ij}^m$  forbids the pair  $(i, j)$  from having a separation smaller than  $m^{1/2}\lambda$ , where  $\lambda = (\hbar c/eB)^{1/2}$  is the magnetic length. An  $N$  electron system can be partitioned into subsets  $(A, B, C, \dots)$ ; one example is  $(N) \rightarrow (N/2, N/2)$ . There can be different numbers of cfs between pairs belonging to



different subsets, and still different numbers between particles in different subsets. The number of cf lines associated with a particular partition can be determined. The subgroup of the full symmetric group associated with the conjugacy class of the partition is used to obtain the full symmetric correlation function. Our objective is to use correlation diagrams to gain new insights into correlations in strongly interacting many-body systems. New electronic correlation functions are obtained for states containing a few quasielectrons (QEs) in a partially filled QE shell, as well as for the incompressible quantum liquid states containing integrally filled QE shells.

For weakly interacting many-body systems, the interaction Hamiltonian  $H_I$  can be treated as a perturbation acting on energy eigenfunctions of a non-interacting Hamiltonian  $H_0$ . For strongly interacting systems, this standard many-body perturbation approach [3] is not applicable because the interaction energy is much larger than the single particle energy scale. The fractional quantum Hall (FQH) effect [4] is the ultimate example of strongly interacting many-body systems. In determining the energy spectrum of the interacting system at very large values of the applied magnetic field  $B_0$ , there is only one relevant energy scale, the Coulomb scale  $V_c = e^2/\lambda$ , where  $\lambda = (\hbar c/eB_0)^{1/2}$  is the magnetic length. The non-interacting single particle states [5] for an electron confined to the  $x$ - $y$  plane have eigenvalues  $\varepsilon_{nm} = \hbar\omega_c[n + \frac{1}{2}(1 + |m|)]$ , where  $\omega_c = eB_0/\mu c$  is the electron cyclotron frequency,  $m = 0, \pm 1, \pm 2, \dots$ , and  $n$  is a non-negative integer. The lowest energy level (Landau level LL0) has  $n = 0$ , and  $m$  equal to a negative integer or zero. For a disk of finite area  $\mathcal{A}$ , the allowed values of  $m$  for the LL0 are  $\{0, -1, -2, \dots, -N_\phi\}$ , where  $N_\phi = \mathcal{A}B_0(\hbar c/e)^{-1}$  is the number of flux quanta of the applied magnetic field  $B_0$  passing through the sample. Each of the  $N_\phi + 1$  single particle states has the same energy  $\frac{1}{2}\hbar\omega_c$ . The non-interacting eigenfunction can be expressed in terms of a complex coordinate  $z = x - iy$  of the electron as  $\phi(z) \propto z^m$ . Then, for LL0,  $m \in g_0 \equiv \{0, +1, \dots, +N_\phi\}$ . Antisymmetrized products of  $N$  functions  $\phi_m(z)$  selected from the set  $g_0$  form the function space  $(2\ell, N)$  of the LL0. Here we use  $2\ell$  in place of  $N_\phi$  for convenience. Because the particles are Fermions, an  $N$  electron trial wave function can be written as a ubiquitous Gaussian weighting factor  $e^{-\sum_k |z_k|^2/(4\lambda^2)}$ , (which is often not explicitly written but is understood), multiplied by the product of an antisymmetric Fermion factor  $\mathcal{F}\{z_{ij}\} = \prod_{i < j} z_{ij}$  caused by the Pauli exclusion principle, and a symmetric correlation function  $\mathcal{G}\{z_{ij}\}$  caused by Coulomb interactions. Here,  $z_{ij} = z_i - z_j$ , and we often refer to it as a *correlation factor* (cf), even when it is caused by the Pauli principle and not by Coulomb correlations.

## 2. Electron correlations

Laughlin [6] realized that if the interacting electrons could avoid the most strongly repulsive pair states, an incompressible quantum liquid (IQL) state could result. He suggested a trial wave function for a filling factor  $\nu$  (defined as  $\frac{N}{2\ell+1}$ ) equal to the reciprocal of an odd integer  $n$ , in which the correlation function  $\mathcal{G}_n(z_{ij})$  was given by  $\prod_{i < j} z_{ij}^{n-1}$ . This function is symmetric and avoids all pair states with relative pair angular momentum smaller than  $n$  (or all pair separations smaller than  $r_n = n^{1/2}\lambda$ ). One can represent this Laughlin correlation function diagrammatically by distributing  $N$  dots, representing  $N$  electrons on the circumference of a circle, and drawing double lines, representing two correlation factors (cfs) connecting each pair. Thus, there are  $2(N-1)$  cf factors in  $\mathcal{G}\{z_{ij}\}$  emanating from each particle  $i$ . Adding  $(N-1)$  cf factors emanating from each particle due to the Fermion factor  $\mathcal{F}\{z_{ij}\}$  gives a total of  $3(N-1)$  cfs emanating from each particle in the trial wave function  $\Psi$ . This number cannot exceed  $N_\phi = 2\ell$  defining the function space  $(2\ell, N)$  of the LL0.

The other well-known trial wave function is the Moore-Read [7] paired function describing the IQL state of a half filled spin polarized first excited Landau level (LL1). This wave function  $\Psi$  can be written in the form  $\Psi = \mathcal{F} \cdot \mathcal{G}_{\text{MR}}$ , where the correlation function is taken as  $\mathcal{G}_{\text{MR}} = \mathcal{F}\{z_{ij}\}Pf(z_{ij}^{-1})$ . The second factor is called the Pfaffian of  $z_{ij}^{-1}$ . It can be expressed as

[7, 8]

$$Pf(z_{ij}^{-1}) = \hat{\mathcal{A}} \prod_{i=1}^{N/2} (z_{2i-1} - z_{2i})^{-1},$$

where  $\hat{\mathcal{A}}$  is an antisymmetrizing operator and the product is over pairs of electrons. There has been considerable interest in the Moore-Read paired state and its generalizations [9] based on rather formidable conformal field theory. We propose a simple intuitive picture of Moore-Read correlations with the hope that it might lead to new insight into correlations in strongly interacting many-body systems.

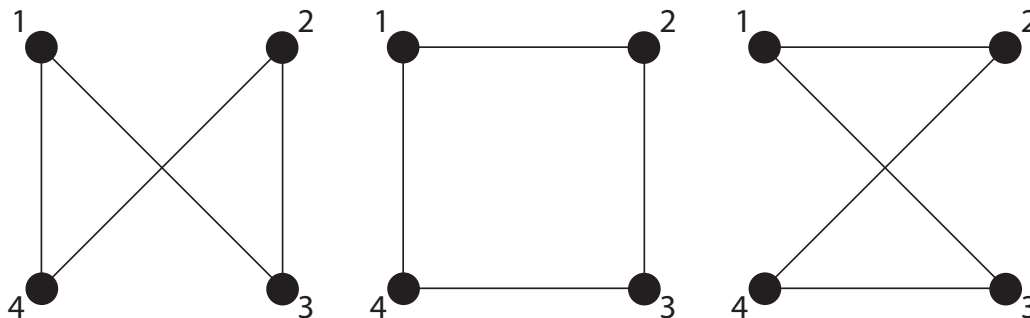
For the simple case of an  $N = 4$  particle system, the Pfaffian can be expressed as

$$Pf(z_{ij}^{-1}) = \hat{\mathcal{A}}\{(z_{12}z_{34})^{-1}\} = [(z_{12}z_{34})^{-1} - (z_{13}z_{24})^{-1} + (z_{14}z_{23})^{-1}]. \quad (1)$$

The product of  $\mathcal{F}\{z_{ij}\}$  and  $Pf(z_{ij}^{-1})$  gives for the Moore-Read correlation function of

$$\mathcal{G}_{\text{MR}}\{z_{ij}\} = z_{13}z_{14}z_{23}z_{24} - z_{12}z_{14}z_{23}z_{34} + z_{12}z_{13}z_{24}z_{34}. \quad (2)$$

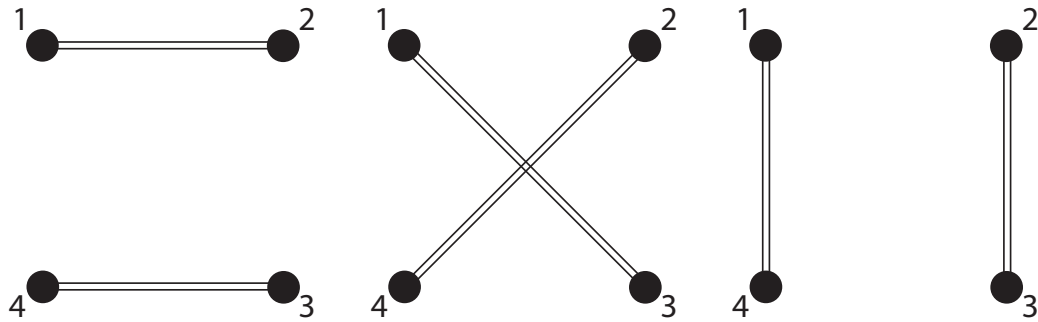
The correlation diagram for  $\mathcal{G}_{\text{MR}}\{z_{ij}\}$  contains four points with a pair of cfs emanating from each particle  $i$  going to different particles  $j$  and  $k$ . There are three distinct diagrams shown in Fig. 1. Note that  $\mathcal{G}_{\text{MR}}$  is symmetric under permutation, as it must be, since it is a product of two antisymmetric functions  $\mathcal{F}\{z_{ij}\}$  and  $Pf\{z_{ij}^{-1}\}$ .



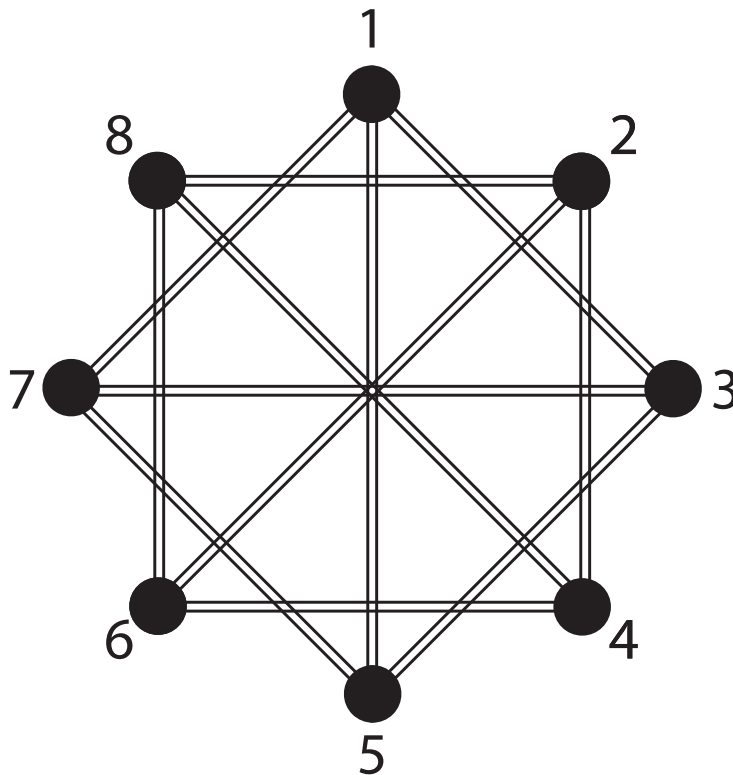
**Figure 1.** Moore-Read correlation diagram for  $N = 4$ . Dots represent particles, and solid lines represent cfs  $z_{ij}$ .  $\mathcal{G}_{\text{MR}}$  is the symmetric sum of the three diagrams and is given by Eq. (2).

A simpler, but seemingly different, correlation is the quadratic function given by  $\mathcal{G}_{\text{Q}} \equiv \hat{\mathcal{S}}(z_{12}^2 z_{34}^2)$ , where  $\hat{\mathcal{S}}$  is a symmetrizing operator. The correlation diagram for  $\mathcal{G}_{\text{Q}}\{z_{ij}\}$  is shown in Fig. 2.  $\mathcal{G}_{\text{MR}}$  and  $\mathcal{G}_{\text{Q}}$  are clearly different. However, when they are expressed as homogeneous polynomials in the independent variables  $z_1$  to  $z_4$  (by simple multiplication), the two polynomials are the same up to normalization constant. The same was true for the an  $N = 6$  particle system, leading to the conjecture that  $\mathcal{G}_{\text{MR}}\{z_{ij}\}$  was equivalent to  $\mathcal{G}_{\text{Q}}\{z_{ij}\}$  for all  $N$ . This conjecture was proved by our group before we discovered that Cappelli et al. [9] had already shown the equivalence.

There are several advantages to the use of  $\mathcal{G}_{\text{Q}}$ . First, it is simpler to partition  $N$  into two subsets of  $N/2$ , e.g.,  $\{1, 2, \dots, N/2\} = A$  and  $\{N/2 + 1, \dots, N\} = B$ , and define  $g_{AB} = g_A g_B = \prod_{i < j \in A} z_{ij}^2 \prod_{k < l \in B} z_{kl}^2$ . Then the full correlation function can be written as  $\hat{\mathcal{S}}_N\{g_{AB}\}$ , where  $\hat{\mathcal{S}}_N$  symmetrizes  $g_{AB}$  over all  $N$  particles. This symmetrization is equivalent to summing  $g_{AB}$  over all possible partitions of  $N$  into two equal size subsets  $A$  and  $B$ . In Fig. 3, we show the contribution to  $\mathcal{G}_{\text{Q}}$  for  $N = 8$  particles for one partition in which  $A = \{1, 3, 5, 7\}$  and  $B = \{2, 4, 6, 8\}$ .



**Figure 2.** Quadratic correlation functions. A double line represents  $z_{ij}^2$ , the square of a cf.  $\mathcal{G}_Q$  is the sum of the contributions from the three diagrams.



**Figure 3.** Correlation diagram for  $\mathcal{G}_Q\{z_{ij}\}$  in an eight electron system due to the partition  $A = \{1, 3, 5, 7\}$  and  $B = \{2, 4, 6, 8\}$ . The full correlation function is the sum over all distinct partitions into subsets  $A$  and  $B$ , each containing  $N/2 = 4$  particles. The trial wave function is  $\Psi_Q(1, 2, \dots, 8) = \mathcal{F}\{z_{ij}\}\mathcal{G}_Q\{z_{ij}\}$ .

### 3. Jain's composite Fermion approach

Jain [10] introduced a composite Fermion (CF) picture by attaching to each electron (via a gauge transformation) a flux tube which carried an even number  $2p$  of magnetic flux quanta. This *Chern-Simons* (CS) flux has no effect on the classical equations of motion since the CS magnetic

field  $\mathbf{b}(\mathbf{r}) = 2p\phi_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \hat{z}$  vanishes at the position of each electron (it is assumed that no electron senses its own CS flux). Here  $\phi_0 = \hbar c/e$  is the quantum of flux, and the sum is over all electron coordinates  $\mathbf{r}_i$ . The classical Lorentz force on the  $i^{\text{th}}$  electron due to the CS magnetic field is  $(-e/c)\mathbf{v}_i \times \mathbf{b}(\mathbf{r}_i)$  and  $\mathbf{b}(\mathbf{r}_i)$  caused by the CS flux on every  $j$  not equal to  $i$  vanishes at the position  $\mathbf{r}_i$ . The CF model results in a much more complicated interaction Hamiltonian, but simplification results from making a mean field (MF) approximation in which the CS flux and the electron charge are uniformly distributed over the entire sample. The average electronic charge  $-eN/\mathcal{A}$  is canceled by the fixed background of positive charge introduced to make the total charge vanish. This MF CF approximation results in a system of  $N$  non-interacting CFs (CF=electron plus attached flux tube) moving in an effective magnetic field  $b^* = \nu b$ . An effective CF filling factor  $\nu^*$  was introduced satisfying the equation

$$\nu^{*-1} = \nu^{-1} - 2p. \quad (3)$$

This resulted in a filled CF level when  $\nu^*$  was equal to an integer ( $\nu^* = n = \pm 1, \pm 2, \dots$ ) and a IQL daughter state at  $\nu = n(1 + 2pn)^{-1}$ . This Jain sequence of states was the most robust set of fractional quantum Hall states observed in experiments.

Making use of Haldane's spherical geometry [11, 12, 13], Chen and Quinn [14] introduced an *effective CF angular momentum*  $\ell^*$  satisfying the relation  $\ell_0^* = \ell - p(N - 1)$ , where  $2p$  is the number of CS flux quanta per electron. The lowest CF Landau level (CF LL0) could hold  $(2\ell^* + 1)$  CFs. There were  $n_{\text{QE}} = N - (2\ell^* + 1)$  composite Fermion QEs of angular momentum  $\ell_{\text{QE}} = \ell^* + 1$  or  $n_{\text{QH}} = (2\ell^* + 1) - N$  CF QHs of angular momentum  $\ell_{\text{QH}} = \ell^*$  if  $2\ell^* + 1$  was not equal to  $N$ . This resulted in a lowest band of quasiparticle (QP) states separated by a gap from the higher energy quasi continuum. This allowed the total angular momentum states in this band to be determined by addition of angular momentum of  $n_{\text{QP}}$  quasiparticles each of angular momentum  $\ell_{\text{QP}}$  using the rules for addition of Fermion angular momenta.

In Table 1, we summarize the results of Jain's MF CF picture of the low energy states of an  $N = 4$  electron system for values of  $2\ell$  equal to 9, 8, 7, and 6. These correspond to the  $\nu = 1/3$  filled IQL states and its excited states containing one, two, and three QEs. The table gives the values of  $\ell$ , the single electron angular momentum, and the resulting values of  $\ell_0^* = \ell - (N - 1)$ , the CF angular momentum;  $n_{\text{QE}} = N - (2\ell_0^* + 1)$ , the number of QEs;  $\ell_{\text{QE}}$ , the QE angular momentum;  $k_M = 2\ell - (N - 1)$ , the maximum number of correlation factor (cf) lines that can emanate from an electron in the correlation function  $\mathcal{G}$ ; and the allowed values of the total angular momentum  $L$  which result.

$\ell$	$\ell_0^*$	$n_{\text{QE}}$	$\ell_{\text{QE}}$	$k_M$	$L$
4.5	1.5	0	2.5	6	0
4	1	1	2	5	2
3.5	0.5	2	1.5	4	$0 \oplus 2$
3	0	3	1	3	0

**Table 1.** Values of  $\ell$  for an  $N = 4$  electron system and the values of  $\ell_0^*$ ,  $n_{\text{QE}}$ ,  $\ell_{\text{QE}}$ ,  $k_M$ , and  $L$  which result.

It might seem surprising that Jain's very simple CF picture correctly predicts the angular momenta in the lowest band of states for any value of  $(2\ell, N)$  which defines the function space of the many-body system. The initial guess that the Chern-Simons gauge interaction and the Coulomb interaction between fluctuations beyond the mean field canceled is certainly not correct. The gauge field interactions are proportional to  $\hbar\omega_c$  which varies linearly with  $B$ , the applied magnetic field. However, the Coulomb interactions are proportional to  $e^2/\lambda$  (where  $\lambda$  is the

magnetic length) and vary as  $B^{1/2}$ . The two energy scales cannot possibly cancel for all values of  $B$ . For very large values of  $B$ , only the Coulomb scale is relevant in determining the low energy band of states. Our group at the University of Tennessee demonstrated that the MF CF picture gave a valid description of the lowest band of states if the pair interaction energy  $V(L_2)$  increased with increasing  $L_2$  faster than the eigenvalue of  $\hat{L}_2^2$ , the square of the pair angular momentum [15].

Knowing this, and the occupancies of CF LLs from Jain's MF CF picture, makes it interesting to explore the correlations among the original electrons. We do this using correlation diagrams for small systems.

#### 4. Correlation diagrams

We have already stated that Laughlin correlation can be described by drawing two cf lines between each pair  $\langle i, j \rangle$ . A cf line between  $i$  and  $j$  represents a correlation factor  $z_{ij}$ . The wave function  $\Psi(1, 2, \dots, N) = \mathcal{F}\{z_{ij}\}\mathcal{G}\{z_{ij}\}$  describing the IQL state at  $\nu = 1/3$  will contain  $3(N-1)$  cf lines emanating from each particle  $i$ .  $(N-1)$  cf lines are associated with  $\mathcal{F}\{z_{ij}\}$ , leaving  $2(N-1)$  cf lines associated with  $\mathcal{G}\{z_{ij}\}$ . The correlation diagram for a Laughlin  $\nu = m^{-1}$  filling factor is simple, because every pair has exactly the same correlations. For other states, like a state with  $n_{\text{QE}}$  quasielectrons, the correlations are more complicated.

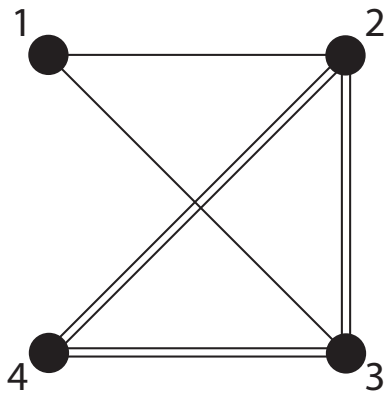
For simplicity, let's use as an example the  $N = 4$  particle system with values of  $2\ell$  in the range  $6 \leq 2\ell \leq 9$ . The values of  $\ell_0^*$ ,  $n_{\text{QE}}$ ,  $\ell_{\text{QE}}$ ,  $k_M$ , and the total angular momentum  $L$  of the lowest energy bands for these states are given in Table 1. We define  $K_{\mathcal{F}} = N(N-1)/2$  as the number of cf lines appearing in the Fermi function  $\mathcal{F}\{z_{ij}\}$ , and  $K_{\mathcal{G}}$  as the number appearing in the correlation function  $\mathcal{G}\{z_{ij}\}$ . Knowing  $N\ell$ ,  $K_{\mathcal{F}}$ , and the allowed values of total angular momentum  $L$ , we can determine  $K_{\mathcal{G}}$  for each of the states listed in Table 1. For  $\ell = 4.5, 4$ , and  $3$  the corresponding values of  $K_{\mathcal{G}}$  are 12, 8, and 6. For  $\ell = 3.5$ , there are two multiplets  $L = 0$  ( $K_{\mathcal{G}} = 8$ ) and  $L = 2$  ( $K_{\mathcal{G}} = 6$ ). We also know  $k_M$  from the table. With this information, we can construct correlation functions which have to be symmetric under permutation of a pair of particles. We show one correlation diagram for each of the values of  $2\ell$ . If it is not symmetric, we must apply  $\hat{S}_4$  on the function to symmetrize over all four particles.

For  $(2\ell, N) = (9, 4)$  there is only a single diagram; it has 2 cfs connecting each pair of particles. For a one QE state, we must partition (4) into (3,1). The single particle  $i$  belongs to subset  $A$  and the other three  $j, k, \ell$  belong to subset  $B$ . The latter subset has Laughlin correlations  $(z_{jk}^2)$  between each pair belonging to  $B$ . Particle  $i$  (in subset  $A$ ) is the QE, and has single cf lines connecting it with two of the three particles in subset  $B$ . Fig. 4 shows one diagram. The diagram corresponds to  $z_{12}z_{13}z_{23}^2z_{24}^2z_{34}^2$ , and this function must be symmetrized by summing over all partitions of (4) into (3,1), i.e., including diagrams in which  $A$  can be 1, 2, 3, or 4. Notice that  $k_M = 5$ ,  $N\ell = 16$ , and  $K_{\mathcal{G}} = 8$ , giving an  $L = 2$  state for the single QE. For the two QE state with  $(2\ell, N) = (7, 4)$ , we partition (4) into (2,2). For example, let one partition be  $A = (1, 2)$  and  $B = (3, 4)$ . One term in the correlation diagram is shown in Fig. 5. This diagram corresponds to  $z_{12}^2z_{14}^2z_{23}^2$ , and it must be symmetrized over all four particles. Notice that  $k_M = 4$ ,  $N\ell = 14$ , and  $K_{\mathcal{G}} = 6$ , giving an  $L = 2$ . To obtain the  $L = 0$  multiplet, we must add two more cfs. Fig. 6 shows one diagram for this case. It corresponds to a contribution  $(z_{12}z_{23}z_{34}z_{41})^2$ , and it must be symmetrized. Now  $K_{\mathcal{G}} = 8$ , and  $L = 0$  results.

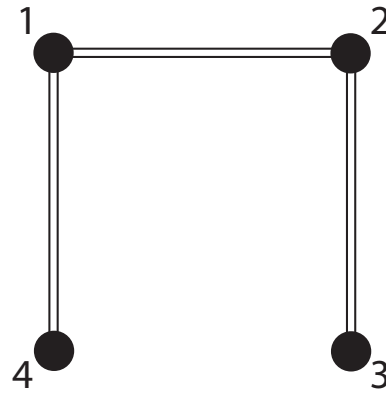
For  $(2\ell, N) = (6, 4)$ , we have three QEs with  $k_M = 3$ , and we can construct the diagram shown in Fig. 7. When symmetrized, it gives

$$\mathcal{G}\{z_{ij}\} = (z_{12}z_{34} + z_{13}z_{24})(z_{13}z_{42} + z_{14}z_{32})(z_{14}z_{23} + z_{23}z_{34}). \quad (4)$$

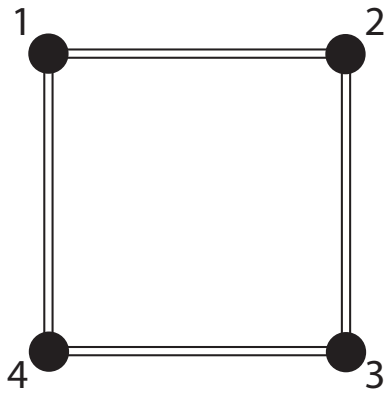
There is only one state of angular momentum  $L = 0$ , and the wave function  $\Psi = \mathcal{F}\mathcal{G}$  obtained using equation (4) agrees exactly with that obtained by standard angular momentum addition.



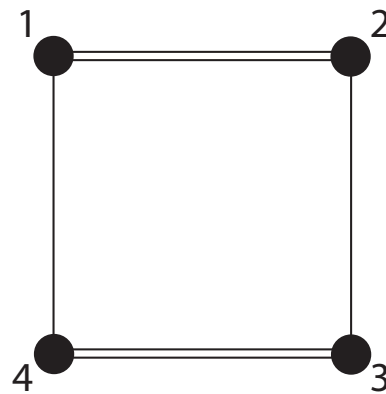
**Figure 4.** One contribution to  $\mathcal{G}$  for  $(2\ell, N) = (8, 4)$ .



**Figure 5.** One contribution to  $\mathcal{G}$  for  $(2\ell, N) = (7, 4)$  that gives  $L = 2$ .



**Figure 6.** One contribution to  $\mathcal{G}$  for  $(2\ell, N) = (7, 4)$  that gives  $L = 0$ .



**Figure 7.** One contribution to  $\mathcal{G}$  for  $(2\ell, N) = (6, 4)$ .

It is worth mentioning that there are three diagrams when Fig. 6 is symmetrized giving three terms  $(z_{12}z_{23}z_{34}z_{41})^2$ ,  $(z_{12}z_{13}z_{24}z_{34})^2$ , and  $(z_{13}z_{14}z_{23}z_{24})^2$ . Their sum gives a symmetric  $\mathcal{G}\{z_{ij}\}$ . For Fig. 7, there are six diagrams giving  $(z_{12}z_{34})^2z_{13}z_{24}$ ,  $(z_{12}z_{34})^2z_{14}z_{23}$ ,  $(z_{13}z_{24})^2z_{12}z_{34}$ ,  $(z_{13}z_{24})^2z_{14}z_{23}$ ,  $(z_{14}z_{23})^2z_{12}z_{34}$ , and  $(z_{14}z_{23})^2z_{13}z_{24}$ . Plus or minus signs must be chosen for each term so that the resulting correlation function  $\mathcal{G}\{z_{ij}\}$  is symmetric.

## 5. Thoughts on larger systems

Though we have used the  $N = 4$  particle system as a simple example, it is not difficult to generalize to the case in which  $N$  is an arbitrary even integer. First, let us consider the Moore-Read state.

For the Moore-Read state,  $2\ell = 2N - 3$ , and we let  $N = 2n$ , where  $n$  is an integer. Then we have

(i)  $2\ell = 4n - 3$



(ii)  $k_M = 2n - 2$

(iii) Take a partition such as  $A = (1, 2, \dots, N/2)$ ,  $B = (N/2 + 1, \dots, N)$ .

Take Laughlin correlations within subsets and write  $g_A = \prod_{i < j \in A} z_{ij}^2$ ,  $g_B = \prod_{k < \ell \in B} z_{k\ell}^2$ , and

$$\mathcal{G} = \sum_{\text{all partitions}} g_A g_B. \quad (5)$$

Note that  $N\ell = n(4n - 3)$ ,  $K_{\mathcal{F}} = n(2n - 1)$ , and  $K_{\mathcal{G}} = n(2n - 2)$  giving  $L = 0$ . There are Laughlin correlations among the  $n$  particles in  $A$  and among the  $n$  particles in  $B$ , but no correlations between particles in different subsets. This is a simple intuitive way of fitting  $N$  particles into the function space  $(2\ell, N)$  with maximum avoidance of the most repulsive pair states (ones with pair angular momentum  $L_2 < 2\ell - 3$ ).

For the Jain state at  $\nu = 2/5$ , we can apply the same technique. In that case, we have

(i)  $2\ell = 5n - 4$

(ii)  $k_M = 3n - 3$

(iii) Partition into two subsets  $(A, B)$ , as with the Moore-Read case.

Take  $g_A$  and  $g_B$  exactly as in that case.

(iv) Add a factor due to intersubset correlations to increase  $K_{\mathcal{G}}$  so that an  $L = 0$  state is produced despite the increase in the value of  $2\ell$ . For a partition such as  $A = (1, 2, \dots, n)$ ,  $B = (n + 1, \dots, 2n)$ , define the intersubset correlation function

$$g_{AB} = \left( \prod_{i \in A} \prod_{j \in B} z_{ij} \right) \sum_{\sigma \in S_n} \prod_{i=1}^n z_{\sigma(i), n+i}^{-1}. \quad (6)$$

In (6), the first factor gives a product of  $n^2$  correlation factors  $z_{ij}$ . The second factor is a sum of products of  $n$  factors of  $z_{ij}^{-1}$ . Define  $G_{AB} = g_A g_B g_{AB}$  for a given partition  $(A, B)$ . Then the full correlation function  $\mathcal{G}$  in this case is the sum of the  $G_{AB}$  taken over all possible partitions.

## 6. Summary and conclusions

We have introduced a novel intuitive way of looking at electron correlations. For the very simple case of  $N = 4$  particles, it gives, without numerical calculation, the exact ground state wave function of the Jain  $\nu = 2/5$  filled state. For  $2n = 6$  particles, the correlation function  $\mathcal{G}\{z_{ij}\}$  is given by the sum over all partitions of  $(2n)$  into two equivalent subsets  $(n, n)$  of  $G_{AB}$  (the contribution for the particular partition  $(A, B)$ ). We have demonstrated that the trial wave function constructed with this correlation function has almost 99% overlap with the wave function [16] obtained by exact numerical diagonalization of the Coulomb interaction  $H_I = \sum_{i < j} e^2 / |z_{ij}|$ . This result was presented as a poster at the International Conference on High Magnetic Fields in Semiconductors in 2014, and appeared in an unpublished workbook proceedings of the conference (available on the conference website).

Our method of obtaining the correlation functions for the Moore-Read and the Jain states is far more simple and intuitive than that of conformal field theory. It should be noted that the contributions of all partitions to the wave function are essentially the same since the wave function itself is antisymmetric. One need only count the number  $n_j$  of pairs which have  $j$  correlation factor lines and be sure that  $\sum_j n_j = N(N - 1)/2$  and  $\sum_j j n_j = K_{\mathcal{G}}$ .



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- [12] There is a one-to-one correspondence between  $N$  electrons on a plane described by coordinates  $(r, \phi)$  and  $N$  electrons on a sphere described by  $(\ell, \ell_z)$ . For the plane, the  $z$ -component of angular momentum takes on the values  $m = 0, 1, \dots, N_\phi$  and the total  $z$  component of angular momentum is  $M = \sum_{i=1}^N m_i$ , where  $m_i$  is the  $z$ -component of angular momentum of a particle ( $i = 1, 2, \dots, N$ ).  $M$  is the sum of the *relative* angular momentum  $M_R$  and the center of mass angular momentum  $M_{CM}$ . On a sphere, the  $z$ -component of the single particle angular momentum is written as  $\ell_z$ , and  $|\ell_z| \leq \ell$ , where  $\ell$  is the angular momentum in the shell (or Landau level). The total angular momentum  $L$  is determined by addition of the angular momenta of  $N$  Fermions, each with angular momentum  $\ell$ .  $N$  electron states are designated by  $|L, L_z, \alpha\rangle$ , where  $\alpha$  is used to label different multiplets with the same value of  $L$ . It is apparent that  $M = N\ell + L_z$ , and one can show that  $M_R = N\ell - L$  and  $M_{CM} = L + L_z$ . Therefore, for a state of angular momentum  $L = 0$ ,  $M_R$  must be equal to  $N\ell$ . In general, the value of  $L$  for a given correlation function is given by the equation  $L = N\ell - K_{\mathcal{F}} - K_{\mathcal{G}}$ , where  $K_{\mathcal{F}} = N(N-1)/2$  is the number of cf lines appearing in the Fermi function  $\mathcal{F}$  and  $K_{\mathcal{G}}$  is the number of cf lines in the correlation function  $\mathcal{G}$ .
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