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On the absence of higher generations of incompressible daughter states of composite Fermion quasiparticles

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Abstract. Jain [1] introduced a simple mean-field (MF) composite Fermion (CF) picture by attaching to each electron in a quantum Hall system a flux tube producing a Chern- Simons magnetic field $\mathbf{b}(\mathbf{r}) = 2p\phi_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i)\hat{z}$. Here $\phi_0 = hc/e$ is the quantum of flux, and the sum is over all electron coordinates $\mathbf{r}_i$. He then averaged the total flux and the total charge (electronic plus positive background) over the entire sample. This MF picture gave a system of noninteracting CFs in an effective magnetic field $B_0^* = \nu B_0$. It predicted incompressible quantum liquid (IQL) states at filling factors $\nu = n(1+2pn)^{-1}$ for integral values of n. Chen and Quinn [2] demonstrated that Jain's MF CF picture predicted the total angular momentum values of the lowest energy band of states for any value of the applied magnetic field B_0. Justification of when the MFCF picture was valid was given by Wojs and Quinn [3], who extended the CF hierarchy scheme of Sitko et al. [4]. The CF hierarchy gave the Jain states for integrally filled CF Landau levels (CF LLs), and the Haldane hierarchy of all odd denominator fractions when quasiparticles in the highest (partially filled) CF angular momentum shell had interactions sufficiently similar to the Coulomb interactions of electrons in the lowest Landau level. Sitko et al. showed that the predictions of the CF hierarchy scheme were not always correct. By using a simple pair angular momentum identity and the concept of *fractional grandparentage*, Wojs and Quinn showed that higher generations of CFs could result from the interactions of the original CF quasiparticles only if their interaction energy $V_{\rm QP}(L_2)$ as a function of their pair angular momentum L_2 increased with increasing L_2 faster than $L_2(L_2+1)$. For Laughlin quasielectrons of the $\nu = 1/3$ IQL state this condition was not satisfied. Therefore, no second generation of CFs could occur. The observed IQL at electron filling factor $\nu = 4/11$ can not be attributed to a daughter IQL state at $\nu_{\rm QE} = 1/3$ in a totally spin polarized system. Rather, it is an IQL state of electron pairs with pair angular momentum $\ell_{\rm p} = 2\ell - 1$. A description of the IQL state of electron pairs and its low energy excitations will be presented, and the possibility of a spin flip quantum phase transition as a function of well width and applied magnetic field will be discussed.

1. Introduction

For a system of noninteracting electrons confined to a 2D surface in the presence of a strong perpendicular magnetic field B_0, the density of states, $g(\varepsilon)$, consists of a series of degenerate levels [5] at $\varepsilon_n = \hbar\omega_{\rm c}(n + 1/2)$ called Landau levels (LLs). For a finite sample of surface area A, the lowest LL can accommodate $(N_\phi + 1)$ electrons, where $N_\phi = B_0 A(hc/e)^{-1}$ is the number of magnetic flux quanta passing through the surface area A. The energy gaps between LLs give

rise to the integral quantum Hall (IQH) effect whenever the filling factor $\nu = N(N_\phi + 1)^{-1}$ is an integer [6].

It is convenient to take the 2D surface on which the electrons reside to be a sphere of radius $R = (A/2\pi)^{1/2}$ with a magnetic monopole of strength $2Q$ flux quanta at its center causing a radial magnetic field of magnitude $B_0 = 2Q(hc/e)/(4\pi R^2)$. This spherical geometry [7] has the advantage of a finite surface area with full rotational symmetry. The single particle angular momentum ℓ has a projection ℓ_z satisfying $-\ell \leq \ell_z \leq \ell$. On the plane $z = 0$, the allowed values of m_z, the z-component of angular momentum, must belong to the set $g_0 = \{0, 1, 2, \cdots, 2\ell\}$. The total angular momentum L of the N particle system has a projection L_z on the sphere. On the plane, M is defined as the sum over all electrons of the value of m_z for each electron. Because ℓ_z and m_z differ by ℓ, it is clear $L_z = M - N\ell$. The eigenstates on the sphere can be written as $|L, L_z\rangle$ and on the plane as $|M_{\rm R}, M_{\rm CM}\rangle$ where $M = M_{\rm R} + M_{\rm CM}$ is the sum of relative and center of mass angular momenta. Interaction energies depend only on L but not L_z, and only on $M_{\rm R}$ but not $M_{\rm CM}$. [8] It is apparent that $M_{\rm R} = N\ell - L$ and $M_{\rm CM} = L + L_z$. To construct an N electron product state of angular momentum $L = 0$, a linear combination of product states with $L_z = 0$ is required, implying that $L = N\ell - M$.

The fractional quantum Hall (FQH) effect discovered by Tsui et al. [9] displayed behavior similar to the IQH effect at specific fractional values of the filling factor ν. The energy gaps giving rise to this FQH effect could only be caused by interactions between the electrons, making it a quantum mechanical many-body problem with no exact solution for $N > 2$. In fact, for $\hbar\omega_{\rm c}$ much larger than $V_c = e^2/\lambda$, where $\lambda = (\hbar c/eB_0)^{1/2}$ is the magnetic length, the Coulomb scale V_c is the only relevant energy for determining the low energy states. No standard methods based on diagrammatic perturbation theory are valid.

Laughlin [10] suggested that an incompressible quantum liquid (IQL) state occurred whenever it became possible for every pair of electrons to avoid pair states with pair angular momentum, L_2, larger than $2\ell - 3$, the pair state with the largest Coulomb repulsion. The function space of this N-body problem depends on $N_\phi = 2\ell$, which sets the number of single particle functions, and on N, the number of electrons, which must fill a fraction $\nu = N/N_\phi$ of these states. Laughlin suggested a trial wave function $\Psi(1, 2, \cdots, N)$ which was proportional to $\mathcal{F}\{z_{ij}\}\mathcal{G}\{z_{ij}\}$. Here $\mathcal{F} = \prod_{ij} z_{ij}$ is an antisymmetric Fermion factor caused by the Pauli principle, and $\mathcal{G}\{z_{ij}\}$ is a symmetric correlation function caused by the Coulomb interactions. The complex coordinates of the $i^{\rm th}$ electron is denoted by $z_i (= x_i - iy_i)$ and $z_{ij} = z_i - z_j$. Laughlin also showed that the elementary excitations of the $\nu = 1/3$ IQL state were quasielectrons (QEs) and quasiholes (QHs) of charge $\mp e/3$. He proposed for the IQL at $\nu = 1/3$ that $\mathcal{G}\{z_{ij}\} = \prod_{i<j} z_{ij}^2$.

Haldane [7] suggested that the problem of putting $N_{\rm QP}$ fractionally charged quasiparticles (QPs) into a QP LL was essentially the same as the original problem of putting N electrons into the electron LL, if the interaction $V_{\rm QP}(L_2)$ between QPs was sufficiently similar to the Coulomb interaction between electrons. This assumption generated a hierarchy of FQH states at all filling factors equal to odd denominator fractions.

2. Composite fermion approach

Jain [1, 11] introduced a composite Fermion (CF) picture by attaching to each electron (via a gauge transformation) a flux tube which carried an even number, $2p$, of magnetic flux quanta. This *Chern-Simons*(CS) flux has no effect on the classical equations of motion, since the Chern-Simons magnetic field $\mathbf{b}(\mathbf{r}) = -2p\phi_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i)\hat{z}$ vanishes at the position of each electron (it is assumed that no electron senses its own Chern-Simons flux). Here $\phi_0 = hc/e$ is the quantum of flux, and the sum is over all electron coordinates $\mathbf{r}_i$. In a mean-field (MF) approximation, the CS flux and the electron charge are uniformly distributed over the entire area of the sample, giving a system of N composite Fermions (electrons plus CS flux) without interactions (the average charge of $-eN/A$ is canceled by the fixed uniform positive charge background). The effective

composite Fermion filling factor ν^* satisfies Jain's equation [1, 11] $\nu^{*-1} = \nu^{-1} - 2p$. When ν^* is an integer $\nu^* = n$, an integral number of CF levels are filled giving an $L = 0$ daughter state at $\nu = n(1 + 2pn)^{-1}$. This Jain sequence of MF CF states (with $n = \pm 1, \pm 2, \cdots$) is the most robust set of fractional quantum Hall states observed experimentally.

Chen and Quinn [2] introduced an *effective CF angular momentum* ℓ^* satisfying the relation $\ell_0^* = \ell - p(N-1)$, where $2p$ is the number of CS flux quanta attached to each electron to form composite Fermions. Then Jain's simple MF CF picture predicted not only the value of 2ℓ at which an IQL state occurred, but also the total angular momentum values for the lowest band of energy states at any value of the applied magnetic field. This lowest band occurs when the minimum number of QPs consistent with the value of $(2\ell, N)$, defining the function space, is present. The allowed values of the QH and QE angular momenta are $\ell_{\rm QH} = \ell_0^*$ and $\ell_{\rm QE} = \ell_0^* + 1$. The values of the total angular momentum in the lowest band are obtained by angular momentum addition of $n_{\rm QP}$ QP angular momenta $\ell_{\rm QP}$, treating the QPs as identical Fermions. The justification of the simple Jain MF CF picture has been given by Wojs and Quinn [12] and by Benjamin et al. [13] and is reviewed briefly in a later section of this paper.

As an example, consider the 10 electron systems defined by $(2\ell, N) = (25, 10)$ and (29,10). The former case has two QEs, each with $\ell_{\rm QE} = 9/2$. The latter has two QHs, each with $\ell_{\rm QH} = 11/2$. For two identical Fermions, each with angular momentum ℓ, the allowed values of the pair angular momentum are given by $L_2 = 2\ell - j$, where j is an odd integer. This gives [14, 15] the two QE band of $L = 0 \oplus +2 \oplus +4 \oplus +6 \oplus +8$, and the two QH band with $L = 0 \oplus +2 \oplus +4 \oplus +6 \oplus +8 \oplus 10$. In Jain's simple MF picture, the QPs are noninteracting, so that the energies are the same for each value of L_2 ($E = 2\varepsilon_{\rm QE}$ or $E = 2\varepsilon_{\rm QH}$ for the two different cases, where $\varepsilon_{\rm QE}$ and $\varepsilon_{\rm QH}$ are the single QE and QH energies, respectively). Numerical diagonalization studies [14, 15] show that this degeneracy does not occur. In Fig. 1, we reproduce numerical

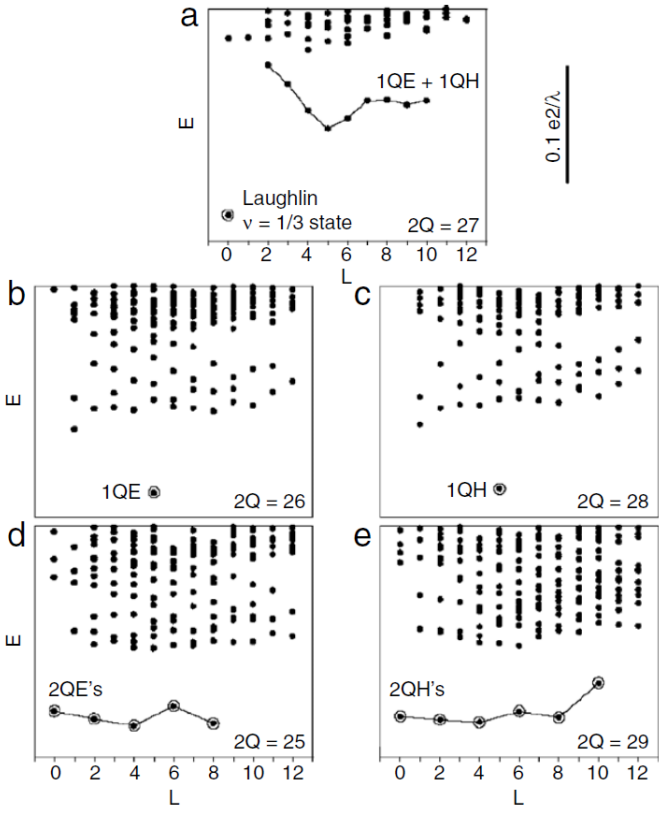

Figure 1. The spectra of 10 electrons in the lowest Landau level calculated on a Haldane sphere with $2Q$ between 25 and 29. The open circles and solid lines mark the lowest energy bands with the fewest composite fermion quasiparticles.(See [15].)

results obtained earlier [15] for an $N = 10$ electron system in the lowest LL (LL0). Frames (d) and (e) contain 2 QEs and 2 QHs, respectively. From the numerical results one can extract $V_{\rm QE}(L_2)$ and $V_{\rm QH}(L_2)$, the interaction energies of a pair of QEs (QHs) as a function of the QP pair angular momentum. These interaction energies (or pseudopotentials) are obtained up to an overall constant which has no effect on correlations. In frame (d) there are two QEs each with $\ell_{\rm QE} = 9/2$, and in frame (d) there are two QHs each with $\ell_{\rm QH} = 11/2$. The lowest energy bands, separated from a quasi-continuum of higher states by a gap, gives us $V_{\rm QE}(L_2)$ for $L = 0 \oplus +2 \oplus +4 \oplus +6 \oplus +8$, and $V_{\rm QH}(L_2)$ for $L = 0 \oplus +2 \oplus +4 \oplus +6 \oplus +8 \oplus 10$. $V_{\rm QE}(L_2)$ has a maximum at $L_2 = 6$ and minima at $L_2 = 8$ and $L_2 = 4$. $V_{\rm QH}(L_2)$ has maxima at $L_2 = 10$ and $L_2 = 6$ and minima at $L_2 = 8$ and $L_2 = 4$. This behavior is quite different from the electron pseudopotential in the LL0 which increases monotonically with increasing L_2.

For large systems (e.g. $N > 14$) numerical diagonalization of the electron-electron interactions becomes difficult, so we have investigated the low lying energy states by determining the number of QEs or QHs ($n_{\rm QE}$ or $n_{\rm QH}$), their angular momenta $\ell_{\rm QE}$ and $\ell_{\rm QH}$, and their interaction energies $V_{\rm QE}(L_2)$ and $V_{\rm QH}(L_2)$. Since $n_{\rm QE}$ (or $n_{\rm QH}$) is much smaller than N, and $\ell_{\rm QE}$ (and $\ell_{\rm QH}$) much smaller than ℓ, the electron angular momentum, we can easily diagonalize these smaller systems. One example [16] is shown in Fig. 2 for the case $(2\ell, N) = (29, 12)$ which corresponds to $(2\ell_{\rm QE}, n_{\rm QE}) = (9, 4)$. The low lying states of the electron system are very close to those of the four QE system, suggesting that description in terms of QP excitations interacting via $V_{\rm QP}(L_2)$ is reasonable.

3. The composite Fermion hierarchy

Sitko et al. [4] introduced a very simple CF hierarchy picture in an attempt to understand Haldane's hierarchy of Laughlin correlated daughter states and Jain's sequence of IQL states with integrally filled CF LLs. Jain's MF CF picture neglected interactions between QPs. The gaps causing incompressibility were energy separations between the filled and lowest empty single particle CF LLs. Not all odd denominator fractions occurred in the Jain sequence $\nu = n(2pn \pm 1)^{-1}$, where n and p are non-negative integers. The missing IQL states were ones with partially filled CF QP shells. The energy gap causing their incompressibility resulted from *residual interactions* between the CF QPs. For an initial electron filling factor ν_0, the

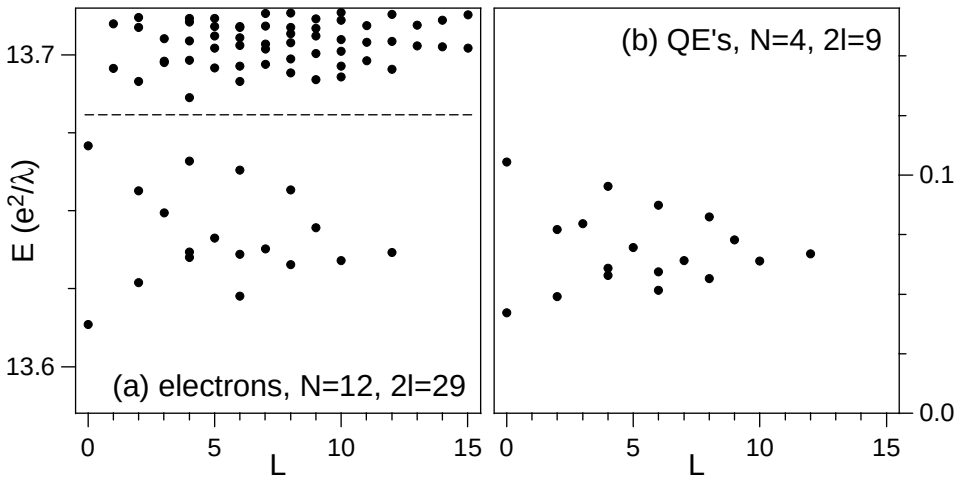

Figure 2. Energy spectra for $N = 12$ electrons in LL0 with $2\ell = 29$, and for $N = 4$ QEs in CF LL1 with $2\ell = 9$. The energy scales are the same, but the QE spectrum was determined using $V_{\rm QE}(\mathcal{R})$ as the pair pseudopotential (up to an arbitrary constant). (See [16].)

relation between ν_0 and ν_0^*, the effective CF filling factor, satisfied $\nu_0^{-1} = (\nu_0^*)^{-1} + 2p_0$, and gave rise to the Jain states when ν_0^* was equal to an integer n. What happens if ν_0^* is not an integer? It was suggested [4] that then one could write $\nu_0^* = n_1 + \nu_1$, where n_1 was an integer and ν_1 represented the filling factor of the partially filled CF QP shell. If Haldane's assumption that the pair interaction energy $V_{\rm QP}(L_2)$, as a function of the angular momentum L_2 of the QP pair, was sufficiently similar to $V_0(L_2)$, the interaction energy of the electrons in the LL0, then one could reapply the CF transformation to the CF QPs by writing $(\nu_1^*)^{-1} = \nu_1^{-1} - 2p_1$. Here ν_1 is the CF QP filling factor and $2p_1$ is the number of CS flux quanta added to the original CF QPs to produce a second generation of CFs. For $\nu_1^* = n_2$, an integer, this results in $\nu_1 = n_2(2p_1n_2 \pm 1)^{-1}$, and a daughter IQL state at $\nu_0^{-1} = 2p_1 + \left[n_1 + n_2(2p_1n_2+1)^{-1}\right]^{-1}$. This new odd denominator fraction does not belong to the Jain sequence. If ν_1^* is not an integer, then set $\nu_1^* = n_2 + \nu_2$ and reapply the CF transformation to the CF QE in the new QP shell of filling factor ν_2. In general one finds at the $\ell^{\rm th}$ level of the CF hierarchy $\nu_\ell^{-1} = 2p_\ell + (n_{\ell+1} + \nu_{\ell+1})^{-1}$. When $\nu_{\ell+1} = 0$, there is a filled CF shell at the $\ell^{\rm th}$ generation of the CF hierarchy. This procedure generates Haldane's continued fraction leading to IQL states at all odd denominator fractional electron fillings. The Jain sequence is a special case in which $\nu_0^* = n$ gives an integral filling of the first CF QP shell, and the gap is the separation between the last filled and first empty CF levels.

The CF hierarchy picture was tested by Sitko et al. for the simple case of $(2\ell, N) = (18, 8)$ for LL0 by comparing its prediction to the result obtained through exact numerical diagonalization. For this case $2\ell_0^* = 2\ell - 2(N-1) = 18 - 2(7) = 4$. Therefore CF LL0 can accommodate $2\ell_0^*+1 = 5$ CFs. The three remaining CFs must go into CF LL1 as CF QEs of angular momentum $\ell_{\rm QE} = \ell_0^*+1 = 3$. This generates a band of states with $L = 0\oplus+2\oplus+3\oplus+4\oplus+6$. This is exactly what is found for the lowest energy band of states obtained by numerical diagonalization shown in Fig. 3. Reapplying the CF transformation to the first generation of CF QEs would generate $2\ell_1^* = 2\ell_0^* - 2(N_{\rm QE} - 1) = 4 - 2(2) = 0$, giving an $L = 0$ daughter IQL state if the CF hierarchy were correct. Clearly the lowest energy state obtained in the numerical diagonalization does not have angular momentum $L = 0$ as predicted by the CF hierarchy. The $L = 0$ and $L = 3$ multiplets clearly have higher energies than the other three multiplets. Sitko et al. conjectured that this must have resulted because the pseudopotential $V_{\rm QE}(L_2)$ was not sufficiently similar to that of electrons in LL0 to support Laughlin correlations. Laughlin correlations are essential for forming a next generation of CFs.

The QEs and QHs have residual interactions that are more complicated than the simple Coulomb interaction in LL0. We have already seen from Fig. 1(d) and (e), that we can obtain $V_{\rm QP}(L_2)$ up to an overall constant from numerical diagonalization of N-electron systems in LL0. More careful estimates of $V_{\rm QE}(\mathcal{R})$ and $V_{\rm QH}(\mathcal{R})$ (where $\mathcal{R} = 2\ell - L_2$, and L_2 is the pair angular momentum) are shown in Fig. 4. We define a pseudopotential to be *harmonic* if it increases with L_2 as $V_{\rm H}(L_2) = A + BL_2(L_2+1)$, where A and B are constants. For LL0, the actual pseudopotential $V(L_2)$ always increases with L_2 more rapidly than $V_{\rm H}(L_2)$. For QEs in CF LL1, the pseudopotential $V_{\rm QE}(L_2)$ has minima at $L_2 = 2\ell - 1$ and at $L_2 = 2\ell - 5$, and a maximum at $L_2 = 2\ell - 3$. This oscillatory behavior of the interaction energy of a QE pair must be responsible for the failure of the CF hierarchy prediction of an $L = 0$ IQL state.

4. Justification of the CF approach

Pan et al. [17] found IQL states of electrons in LL0 that do not belong to the Jain sequence of integrally filled CF states. One example is the $\nu = 4/11$ filling factor of a state that is assumed to be fully spin polarized. Numerical diagonalization studies of fully spin polarized systems did not find an $L = 0$ IQL ground state at $\nu_{\rm QE} = 1/3$, which would result in an IQL state at electron filling factor of $\nu = 4/11$. In addition, Pan et al. found strong minima in ρ_{xx} at even denominator filling factors ($\nu = 3/8$ and $\nu = 3/10$) suggesting the existence of IQL states that can't be part of the CF hierarchy. Our research group has made an important contribution to

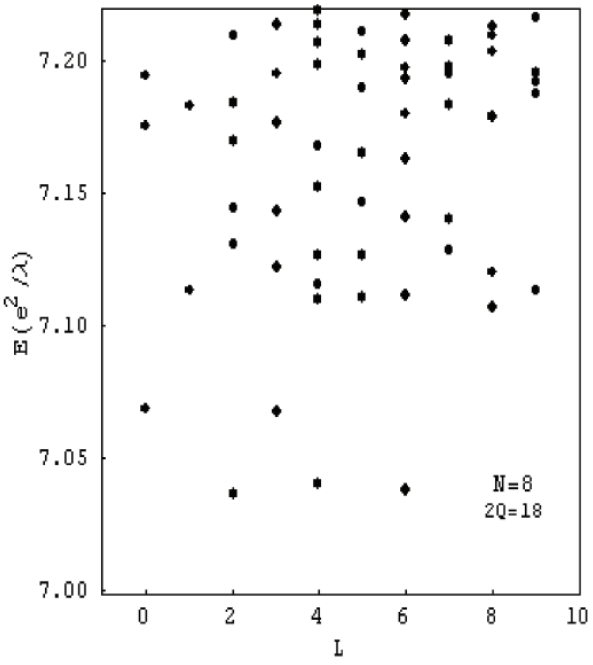

Figure 3. Low energy spectrum of 8 electrons at $2\ell = 18$. The lowest band contains 3 QEs each with $\ell_{\rm QE} = 3$. Reapplying the CS mean-field approximation to these QEs would predict an $L = 0$ daughter state corresponding to $\nu = 4/11$. The data makes it clear that this is not valid. (See [16].)

this field, by rigorously proving [14, 15, 16] under which conditions Jain's elegant CF approach correctly predicts the angular momentum multiplets belonging to the lowest energy sector of the spectrum for any value of the applied magnetic field. Because there is no small parameter in this strongly interacting many body system, our proof does not involve treating fluctuations beyond the MF by a perturbation expansion. It involves proving some rigorous mathematical theorems and applying them, together with well-known concepts frequently used in atomic and nuclear physics. We outline these theorems and give references to earlier publications for proofs. [16]

Theorem 1: Let $\hat{L} = \sum_j \hat{L}_j$ be the total angular momentum operator of an N Fermion system, and $\hat{L}_{ij} = \hat{l}_i + \hat{l}_j$ be the angular momentum operator of the pair $\langle i, j\rangle$. Then

$$\hat{L}^2 + N(N-2)\hat{l}^2 - \sum_{\langle i,j\rangle} \hat{L}_{ij}^2 = 0. \tag{1}$$

where the sum is over all pairs. Here we assume N Fermions are in a shell of angular momentum ℓ, so each Fermion has angular momentum ℓ and projection ℓ_z, with $-\ell \le \ell_z \le \ell$.

Theorem 2: The antisymmetric angular momentum multiplets formed from N Fermions in a shell of angular momentum ℓ can be written as

$$|l^N; L\alpha\rangle = \sum_{L'\alpha'} \sum_{L_2} G_{L\alpha,L'\alpha'}(L_2)|l^2, L_2; l^{N-2}, L'\alpha'; L\rangle \tag{2}$$

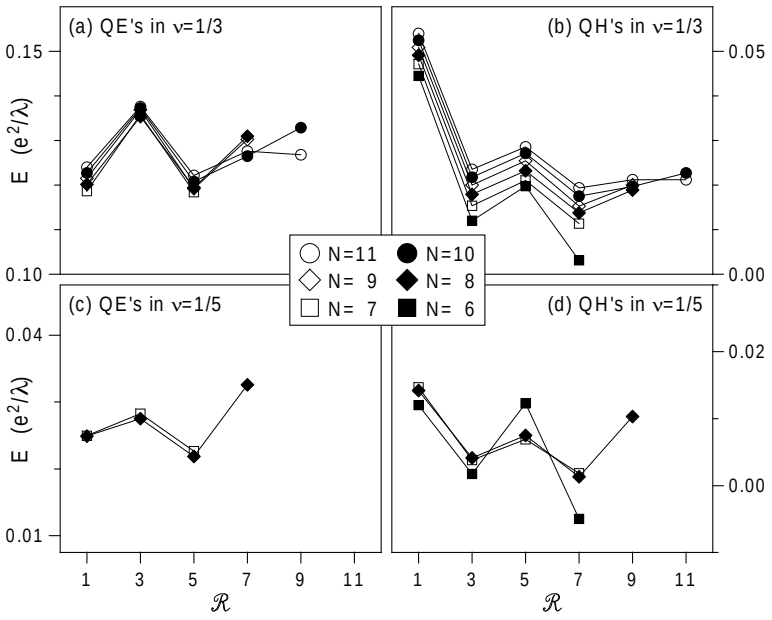

Figure 4. The pseudopotentials of a pair of quasielectrons (left) and quasiholes (right) in Laughlin $\nu = 1/3$ (top) and$\nu = 1/5$ (bottom) states, as a function of relative angular momentum $\mathcal{R}$. Different symbols mark data obtained in the diagonalization of between 6 and 11 electrons.(See [3].)

Here the index α labels different multiplets with the same value of L, and $|l^2, L_2; l^{N-2}, L'\alpha'; L\rangle$ is an N Fermion multiplet of angular momentum L formed from a multiplet $|\ell^{N-2}; L'\alpha'\rangle$ and a pair wave function $|\ell^2, L_2\rangle$. It is antisymmetric with respect to the interchange of indices i and j when both belong to the set (1,2) or when both belong to the set $(3, 4, \cdots, N)$. However, the coefficient $G_{L\alpha,L'\alpha'}(L_2)$, called the coefficient of fractional parentage (CFP) can be chosen so that $|l^N; L\alpha\rangle$ is totally antisymmetric. All that we need to know about the CFP is that

$$\sum_{L'\alpha'} |G_{L\alpha,L'\alpha'}(L_2)|^2 = \mathcal{P}_{L\alpha}(L_2), \tag{3}$$

where $\mathcal{P}_{L\alpha}(L_2)$ is the probability that multiplet $|l^N; L\alpha\rangle$ has pairs with pair angular momentum L_2.

Theorem 3: Taking the expectation value of the operator identity (in Theorem 1) in the state $|l^N; L\alpha\rangle$ *gives*

$$\langle l^N; L\alpha| \sum_{\langle i,j\rangle} \hat{L}_{ij}^2 |l^N; L\alpha\rangle = \frac{1}{2} N(N-1) \sum_{L_2} L_2(L_2+1)\mathcal{P}_{L\alpha}(L_2). \tag{4}$$

This makes use of the fact that $|l^N; L\alpha\rangle$ is totally antisymmetric, so that the sum over all pairs can be replaced by a sum over the allowed values of the pair angular momentum L_2 for any one pair multiplied by the number of pairs, $\frac{1}{2}N(N-1)$.

Theorem 4: If the pseudopotential is harmonic by which we mean $V(L_2) = V_{\rm H}(L_2) = A + BL_2(L_2+1)$ *where* A *and* B *are constants, then every multiplet* α *with the same total*

angular momentum L has the same energy, given by

$$E_\alpha(L) = N\left[\frac{1}{2}N(N-1)A + B(N-2)\ell(\ell+1)\right] + BL(L+1). \quad (5)$$

This means that the degeneracy of the angular momentum multiplets of noninteracting Fermions is not removed by a harmonic pseudopotential for different multiplets having the same value of L. [14, 15]

Theorem 5: If $G_{N\ell}(L)$ is the number of independent multiplets of total angular momentum L that can be formed from N Fermions in a shell of angular momentum ℓ, then $G_{N\ell^*}(L) \le G_{N\ell}(L)$ for every L, if $\ell^* = \ell - (N-1)$. [13]

Theorem 6: The subset $G_{N\ell^*}(L)$ of angular momentum multiplets of the set $G_{N\ell}(L)$ avoids the largest allowed pair angular momentum $L_2 = 2\ell - 1$, which for LL0, corresponds to the largest pair repulsion.

This is obvious for $N = 2$ where $L_2^{\rm MAX} = 2\ell - 1$ and $L_2^{*\rm MAX} = 2\ell^* - 1 = 2\ell - 3$, but it is true for arbitrary N. This theorem means that the set of states selected by Jain's MF CF picture (where ℓ^* plays the role of the effective CF angular momentum) is subset of $G_{N\ell}(L)$. This subset avoids pair states with $L_2 = 2\ell - 1$, and contains multiplets with low angular momentum and low energy.

Theorem 7: By adding an integral number, α, of Chern-Simons flux quanta (oriented opposite to the applied magnetic field) to the Hamiltonian for N electrons, not via a gauge transformation, but adiabatically, the pair eigenstate (in the planar geometry) $\Psi_{nm} = e^{im\phi}u_{n,m}(r)$, where $u_{n,m}(r)$ is the radial wave function, transforms to $\tilde{\Psi}_{nm} = e^{im\phi}u_{n,m+\alpha}(r)$.

These theorems justify Jain's CF picture when applied to LL0. Is this important? In our opinion, Jain's MF CF picture is a brilliant success. It is used very often to interpret experimental data. However, because Coulomb and CS gauge interactions beyond the MF involve two entirely different energy scales ($\hbar\omega_{\rm c} = \nu B$ and $e^2/\lambda \propto \sqrt{B}$), these two interactions between fluctuations beyond MF cannot possibly cancel for all values of B. In addition, treating the interactions by standard many-body perturbation theory can't be rigorously justified because there is no small parameter to assume the convergence of the perturbation series.

Because correlations, that is, the lifting of the degeneracy of the angular momentum multiplets $|l^N; L\alpha\rangle$ of noninteracting electrons in partially filled LL0 depend on the deviation of the actual pseudopotential from the harmonic behavior (i.e. on $\Delta V(L_2) = V(L_2) - V_{\rm H}(L_2)$), it is interesting to explore the simplest possible anharmonicity. The simplest *anharmonic* contribution to the pseudopotential can be taken as

$$\Delta V(L_2) \equiv V(L_2) - V_{\rm H}(L_2) = k\ \delta(L_2, 2\ell - 1). \quad (6)$$

If $k > 0$ it is apparent that the lowest energy multiplet for each value of total angular momentum L is the one which avoids (to the maximum possible extent) having pairs with $L_2 = L_2^{\rm MAX} = 2\ell - 1$. This is exactly what is meant by Laughlin correlations. Complete avoidance of pairs with $L_2 = L_2^{\rm MAX}$ cannot occur unless $2\ell \ge 3(N-1)$. In the limit of large systems this corresponds to filling factor $\nu \le 1/3$. If $k < 0$, then the lowest energy state for each value of L is the one with $\mathcal{P}_{L\alpha}(L_2^{\rm MAX})$ a maximum probability. This corresponds to forming pairs with pair angular momentum $\ell_{\rm p} = 2\ell - 1$.

It is important to emphasize that Laughlin correlations occur only if $V(\mathcal{R})$ (where $\mathcal{R} = 2\ell - 1$) is *superharmonic* at $\mathcal{R} = 1$. By this we mean that $V(\mathcal{R})$ increases faster than $L_2(L_2+1)$ as L_2 approaches its maximum value of $L_2 = 2\ell - 1$. This is not true of $V_{\rm QE}(\mathcal{R})$, so we do not expect

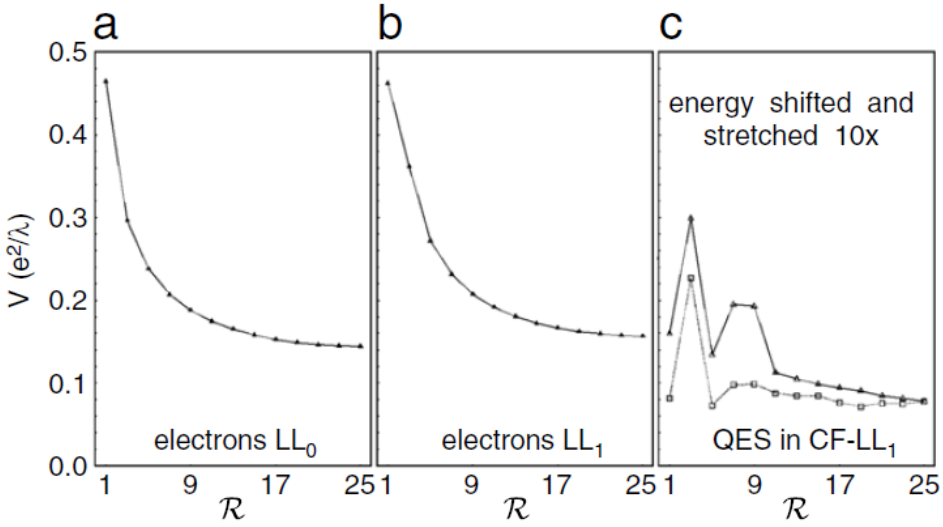

Figure 5. Pair interaction pseudopotentials as a function of relative angular momentum $\mathcal{R}$ for electrons in the LL0 (a), LL1 (b) and for the QEs of the Laughlin $\nu = 1/3$ state by Lee et al.[19] (squares) and by Wojs et al. [20] (triangles). (See [16].)

Laughlin correlations among the CF QEs of the $\nu = 1/3$ Laughlin state at $\nu_{\rm QE} = 1/3$. This agrees with the numerical result of Sitko et al. [4] Now, however, we understand why the CF hierarchy fails for a spin polarized system. There have been a number of papers [18] suggesting that novel IQL states found by Pan et al. [17] (like the $\nu = 4/11$ state) can be understood as a second generation of CFs forming an $L = 0$ daughter state due to QE interactions. This can't be correct for a spin polarized system with $\nu_{\rm QE} = 1/3$, because $V_{\rm QE}(\mathcal{R} = 1)$ will not support Laughlin correlations among the CF QEs.

5. Role of pairing in novel incompressible quantum liquid states

How can the novel IQL states observed by Pan et al. be explained if there is no second generation of CFs formed by Laughlin correlations among the CFs of the first generation? We have shown that if the pseudopotential describing the interaction of a pair of Fermions as a function of the pair angular momentum L_2 is *subharmonic* (i.e. rises with increasing L_2 more slowly than the harmonic pseudopotential $V_{\rm H}$), the Fermions tend to form pairs with pair angular momentum $L_2 = 2\ell - 1$ instead of being Laughlin correlated. In Fig. 5, we display the pseudopotentials $V^{(0)}(\mathcal{R})$ for electrons in the LL0, $V^{(1)}(\mathcal{R})$ for electrons in LL1, and $V_{\rm QE}(\mathcal{R})$ for quasielectrons in CF LL1, the first excited CF level.[19, 20] The pseudopotentials are displayed as a function of $\mathcal{R} = 2\ell - L_2$, the *relative pair angular momentum*. $V^{(0)}(\mathcal{R})$ is *superharmonic* at all values of $\mathcal{R}$; $V^{(1)}(\mathcal{R})$ is *superharmonic* for $\mathcal{R} > 3$ but is harmonic or very weakly superharmonic for $\mathcal{R} = 1$. The QE pseudopotential $V_{\rm QE}(\mathcal{R})$ is not even monotonic; it has strong maxima at $\mathcal{R} = 3$ and 7, and minima at $\mathcal{R} = 1$ and 5.

Moore and Read [21] proposed a *paired wave function* for the observed IQL state at $\nu = 2+\nu_1 = 5/2$, the half-filled state of a spin polarized excited state LL (LL1), sitting above LL0 filled for both spin up and spin down states. The Moore-Read paired wave function (also called the Pfaffian wave function) made use of a correlator based on conformal field theory. Greiter et al. [22] showed that this Phaffian state is an exact solution to a special Hamiltonian, which is large and repulsive when three electrons form single droplets (with total three particle angular momentum $L_3 = 3\ell - 3$ (or relative three particle angular momentum $\mathcal{R}_3 = 3\ell - L_3 = 3$) and

zero otherwise. For the Pfaffian wave function at $\nu_1 = 1/2$ in LL1, 2ℓ is given by $2\ell = 2N - 3$ in agreement with numerical diagonalization results. It is worth noting that Laughlin-Jain states in LL0 at filling factors $\nu_1 = n/q < 1/2$ occur when $2\ell = \nu^{-1}N - (q+1-n)$. No even denominator fractional fillings occur in the Laughlin-Jain sequence.

The simplest way of picturing paired states is to introduce a pair angular momentum $\ell_{\rm p} = 2\ell - 1$ and form $N_{\rm p} = N/2$ such pairs. The pairs cannot get too close to one another without violating the Pauli principle. One would normally think of pairs of Fermions as Bosons, but in two dimensional systems we can alter the particle statistics by using a Chern-Simons transformation. We introduce a Fermion pair (FP) angular momentum $\ell_{\rm FP}$ satisfying the equation

$$2\ell_{\rm FP} = 2\ell_{\rm p} - \gamma_{\rm F}(N_{\rm p} - 1). \tag{7}$$

For a single pair $\ell_{\rm FP} = 2\ell - 1$. As $N_{\rm p}$ increases the allowed values of the total angular momentum are restricted to values less than or equal to $2\ell_{\rm FP}$. The value of the constant $\gamma_{\rm F}$ is determined by requiring that the FP filling factor be equal to unity when the single Fermion filling factor has an appropriate value. For $\ell_{\rm p} = 2\ell - 1$, this value corresponds to single Fermion filling $\nu = 1$. Setting $\nu_{\rm FP}^{-1} = (2\ell_{\rm FP} + 1)/N_{\rm p}$, $\nu^{-1} = (2\ell + 1)/N$, and $N_{\rm p} = N/2$ gives $\nu_{\rm FP}^{-1} = 4\nu^{-1} - 3$, (i.e. $\gamma_{\rm F} = 3$), so that $\nu_{\rm FP} = 1$ when $\nu = 1$. The factor of 4 multiplying ν^{-1} results from the pairs having a charge of $-2e$, and $N_{\rm p}$ being equal to $N/2$. This procedure allows the Fermion pairs to be Laughlin correlated instead of the individual electrons being so. It predicts that an IQL state at $\nu_1 = 1/2$ occurs when $2\ell = 2N - 3$ as found by Moore and Read.

We shall apply the same idea to the quasielectrons and quasiholes of the Laughlin $\nu = 1/3$ state, and to quasiholes of the Laughlin-Jain $\nu = 2/5$ state. QHs of $\nu = 1/3$ state reside in CF LL0, but both QEs of the $\nu = 1/3$ state and QHs of the $\nu = 2/5$ state reside in CF LL1.

The pseudopotentials of these quasiparticles are shown in Fig. 6 for a limited range of values of the relative angular momentum $\mathcal{R}$. Note that QEs of the $\nu = 1/3$ state and QHs of the $\nu = 2/5$ state have maxima at $\mathcal{R} = 3$ and minima at $\mathcal{R} = 1$, while QHs of the $\nu = 1/3$ state has maxima at $\mathcal{R} = 1$ and 5 and a minimum at $\mathcal{R} = 3$. The behavior of QHs of the Laughlin $\nu = 1/3$ state leads to avoidance of $\mathcal{R} = 1$ and 5, so we assume the formation of pairs in this case with $\ell_{\rm p} = 2\ell - 3$ instead of $2\ell - 1$.

If we assume that the QEs form pairs and treat the pairs as Fermions, then Eq. (7) gives the relation between the *effective FP angular momentum* $\ell_{\rm FP}$, and the QE angular momentum ℓ, and the relation between the *effective FP filling factor* $\nu_{\rm FP}$, and the QE filling factor $\nu_{\rm QE}$. If

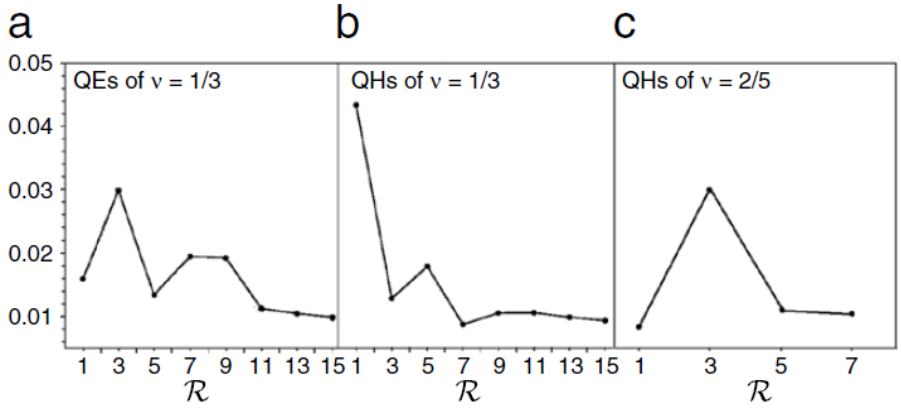

Figure 6. The pseudopotentials $V_{\rm QE}(\mathcal{R})$ and $V_{\rm QH}(\mathcal{R})$ for (a) QEs of $\nu = 1/3$ state, (b) QHs of $\nu = 1/3$, and (c) QHs of $\nu = 2/5$ state. (See [16].)

Table 1. Values of $\nu_{\rm FP}^{-1} = m$ for $m = 3$, 5, 7, and 9 and the resulting values of $\nu_{\rm QE}$, $\nu_{\rm QH}$, and the electron filling factor that they generate.

$\nu_{\rm FP}^{-1}$	3	5	7	9
$\nu_{\rm QE}$	2/3	1/2	2/5	1/3
ν	5/13	3/8	7/19	4/11
$\nu_{\rm QH}$(CF LL1)	2/3	1/2	2/5	1/3
ν	4/11	3/8	8/21	5/13

Table 2. Values of $\nu_{\rm QH}$ satisfying $1/3 > \nu_{\rm QH} \geq 1/5$ and the resulting electron filling factors ν for Laughlin correlated QP_2s with $\nu_{\rm FP}^{-1}$ =7, 9, 11, and 13.

$\nu_{\rm FP}^{-1}$	7	9	11	13
$\nu_{\rm QH}$	2/7	1/4	2/9	1/5
ν	5/17	3/10	7/23	4/13

we take $\nu_{\rm FP} = m^{-1}$, where m is an odd integer, we can obtain the value of $\nu_{\rm QE}$ corresponding to the Laughlin correlated state of FPs (pairs of quasielectrons with $\ell_{\rm p} = 2\ell - 1$). Exactly the same procedure can be applied to QHs in CF LL1 since $V_{\rm QE}(\mathcal{R})$ and $V_{\rm QH}(\mathcal{R})$ are qualitatively similar at small values of $\mathcal{R}$. Here we are assuming that $V_{\rm QE}(\mathcal{R})$ and $V_{\rm QH}(\mathcal{R})$ are dominated by their short range behavior $\mathcal{R} \leq 5$. The QH pseudopotential is not as well determined for $\mathcal{R} > 5$ because it requires larger N electron systems than we can treat numerically. The electron filling factor is given by $\nu^{-1} = 2 + (1 + \nu_{\rm QE})^{-1}$ or by $\nu^{-1} = 2 + (2 - \nu_{\rm QH})^{-1}$. This results in the values of ν shown in Table 1 for $2/3 \geq \nu_{\rm QE} \geq 1/3$.

The states generated at the values of $\nu_{\rm QE}$ and $\nu_{\rm QH}$ equal to 2/5 have not been observed. Clear IQL states were observed by Pan et al.[17] at $\nu = 3/8$ and $\nu = 4/11$. A somehow weaker IQL state at $\nu = 5/13$ is also observed.

The daughter states generated by QPs in CF LL1 (QEs of the parent $\nu = 1/3$ Laughlin state or QHs of the parent $\nu = 2/5$ Jain state) give rise to filling factors for the electron system with $\nu > 1/3$. QHs of the $\nu = 1/3$ Laughlin state (residing in CF LL0) can also form daughter states, and they result in an electron filling factor ν in the range $1/3 > \nu \geq 1/5$. The pseudopotential for these QHs is *superharmonic* at $\mathcal{R} = 1$ and has a strong minimum at $\mathcal{R} = 3$. Because of this, if they form pairs, the pairs must have angular momentum $\ell_{\rm p} = 2\ell - 3$ (instead of $\ell_{\rm p} = 2\ell - 1$ for QE pairs). Then, Eq.(7) must be modified. We replace $2\ell - 1$ in the definition of $2\ell_{\rm FP}$ by $2\ell - 3$, and $\gamma_{\rm F}$ by $\tilde{\gamma}_{\rm F}$. The value of $\tilde{\gamma}_{\rm F}$ is determined by requiring that $\nu_{\rm FP} = 1$ when $\nu_{\rm QH} = 1/2$. This condition results from the fact that the pairs are formed by two QHs separated by two filled CF states. The resulting value of $\tilde{\gamma}_{\rm F}$ is 7, so that $\nu_{\rm FP}^{-1} = 4\nu_{\rm QE}^{-1} - 3$ is replaced by $\nu_{\rm FP}^{-1} = 4\nu_{\rm QH}^{-1} - 7$. The QH daughter states resulting from Laughlin correlated QH_2 (pairs of QHs of the $\nu = 1/3$ state) and the electron filling factor satisfying $\nu^{-1} = 2 + (1 - \nu_{\rm QH})^{-1}$ are given in Table 2.

All of the electron filling factors except $\nu = 7/19$ and $\nu = 8/21$ have been observed with minima in ρ_{xx} (or plateaus in ρ_{xy} in many cases). Numerical diagonalization studies seem to support the occurrence of IQL states. However, recent studies by Samkharadze et al.[23] at very low temperatures raise questions about the even denominator fractions.

6. Summary and conclusions

In this paper we have reviewed exact numerical diagonalization of small systems within the Hilbert subspace of a single partially occupied LL. The numerical results are thought of as *numerical experiments*, and simple intuitive models fitting the numerical data are sought, to better understand the underlying correlations. We describe calculations for N electrons confined to a Haldane spherical surface, and present simple results at different values of the LL degeneracy $g = 2\ell + 1$. We demonstrate that Jain's remarkable CF picture predicts not only the values of 2ℓ at which incompressible quantum liquid ground states occur for different values of N, but

also predicts the angular momenta L of the lowest band of multiplets for any value of 2ℓ in a very simple way. We emphasize that Jain's CF picture is valid, not because of some magical cancellations of Coulomb and Chern-Simons gauge interactions beyond mean-field, but because it introduces Laughlin correlations by avoiding pair states with the lowest allowed relative angular momentum $\mathcal{R} = 2\ell - L_2$. The allowed angular momentum multiplets which avoid pair states with $\mathcal{R} = 1$ form a subset of the set of multiplets $G_{N\ell}(L)$ that can be formed from N Fermions in a shell of angular momentum ℓ. This subset avoids the largest repulsion and has the lowest energy. Adiabatic addition of Chern-Simons flux introduces Laughlin correlations without the necessity of introducing an irrelevant mean-field energy scale $\hbar\omega_{\rm c} = \nu\hbar\omega_{\rm c}$.

Jain's sequence of filled CF shells does not require an interaction between CF quasiparticles. The incompressibility results from the energy required to create a QE-QH pair in the integrally filled CF state. Haldane's hierarchy of IQL states was based on the implicit assumption that the residual interaction between QPs was sufficiently similar to the Coulomb interaction between electrons in LL0 that the QPs would form their own Laughlin correlated daughter states.

The experiment of Pan et al. showed that neither Jain's CF picture nor Haldane's hierarchy was the whole story. Residual pair interactions between QPs had been determined up to an overall constant (unimportant for QP correlations). This pseudopotential $V_{\rm QP}(L_2)$ could be used to determine the spectrum of daughter states containing $N_{\rm QP}$ quasiparticles in a partially filled QP shell. Qualitatively correct results can be expected when $V_{\rm QP}(L_2)$ is small compared to the energy necessary to create a QE-QH pair in the IQL state. When the CF picture was reapplied to the QPs, the Haldane hierarchy of all odd denominator fractions resulted. Numerical calculations demonstrated that this CF hierarchy scheme of Laughlin correlated QPs at each level did not always work, because $V_{\rm QP}(L_2)$ was not sufficiently similar to $V_0(L_2)$, the pseudopotential for electrons in LL0.

Our study of when the MF CF picture is valid justifies Jain's ideas about integrally filled CF QP levels, but shows that higher generations of CF states resulting from CF QP interactions do not always support Laughlin correlations. Laughlin correlation among QPs is necessary to obtain a higher generation of CFs.

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