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Signature of the many-particle localization transition in the entanglement spectrum

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Abstract. We use the statistical properties of the entanglement spectrum of an interacting disordered system, in order to track the localized to extended transition as function of interaction strength and excitation energy expected from the many-body localization transition. We show that such a transition is indeed observed, although an interesting saturation behavior in the intermediate excitation energy is seen.

1. Introduction

Ideas and measures from the field of quantum information have been recently applied to condensed mater physics [1, 2], mainly in order to identify quantum phase transition (QPT). Entanglement of a many particle system in a pure state, divided into two regions A and B is quantified by different measures (e.g., entanglement entropy, Rényi entropy and entanglement spectrum (ES)) derived from the reduced density matrix of area A, ρ_A or B, ρ_B.

For example, the entanglement entropy measures the von-Neumann (Shannon) entropy of the eigenvalues λ_i^A of ρ_A, such that: $S_A = -\sum_i \lambda_i^A \ln \lambda_i^A$. A different measure is known as the ES corresponding a transformation of the eigenvalues $\varepsilon_i^A = -\ln \lambda_i^A$. Li and Haldane [3] showed that the ES of a partitioned fractional quantum Hall state at $\nu = 5/2$ state resembled the edge excitation spectrum, and thus established a connection between the properties of the ES and the topological order of this state. Expanding on these insights, several authors suggested that the low-energy ES distribution shows some correspondence to the true many-body excitations (MBE) of the partitioned segment (region A), for example the statistical properties of the MBE and ES of disordered systems are the same [4, 5, 6]. The logic behind this proposal is that the reduced density matrix of a region encodes the influence of the connection to the rest of the system on the local MBE of the disconnected region. For low-lying excitations, the phase space is rather low, and therefore the reduced density matrix corresponds rather well to the local MBE spectrum.

Recently, the concept of many-body localization (MBL) [7] has emerged: Under certain conditions the quantum many-body states of the system are localized in the Hilbert (Fock) space resembling the celebrated Anderson localization [8] of a single particle states in a random potential. MBL implies that even after an arbitrary long evolution a state of a decoupled system remains dependent on the initial conditions. It means that time averaging does not result in equipartition distribution and the entropy never reaches its thermodynamic value. In other

words the time average is not equivalent to the ensemble average, i.e. the ergodic hypothesis fails.

Numerical studies of many-body models such as the Heisenberg chains of spins $\frac{1}{2}$ in a random field [9] or one-dimensional Josephson arrays [10] provide evidence in favor of the MBL. Recently there appeared a mathematical proof of the existence of MBL transition [11]. It is commonly believed [12] that from the point of view of MBL a generic many-body system behaves similarly to a one-particle problem on the Bethe lattice (random regular graph or disordered Cayley tree [13]). This allows one to develop some intuition about the properties of the system not far from the MBL transition

Here we use the statistical properties of the ES, in order to numerically study the MBL transition. Essentially, we seek a transition from the localized statistics to extended statistics as function of increased interaction strength, or excitation energy. This actually follow in the footsteps of previous studies seeking a signature of the MBL in the MBE spectrum [9, 14, 15], although it has the advantage of being able to address much larger systems due to the use of ES instead of MBE spectrum. It is also connected to recent ideas on the connection between the ES statistics and irreversability [16].

For the disordered single particle spectrum there is an extensive literature on the statistical properties of the energy spectrum in the localized, critical, diffusive and chaotic regimes. Different energy spectrum and wave function statistics, depending on whether the disordered system has time reversal symmetry (Gaussian orthogonal ensemble (GOE)), broken time reversal symmetry (Gaussian unitary statistics (GUE)), spin-orbit interactions (Gaussian symplectic statistics (GSE)) [17, 18, 19]. The energy spectrum statistics can be used to identify the Anderson localization transition [20].

The statistics of MBE in disordered interacting systems have an interesting twist. For non-interacting many-particle systems the level spacing MBE distribution is expected to follow the Poisson distribution for excitation energies above the second spacing, without depending on the single-level distribution [21]. On the other hand, once repulsive interactions between the particles are considered, a transition to the Wigner distribution for higher excitations is observed [22, 23, 24, 25, 9]. This transition may be interpreted as a signature of the MBL transition[9, 14, 15]. One difficulty in studying this transition is that exact diagonalization needed to study excited states is limited to very small systems. Here, the Li and Haldanes' conjecture can come to the rescue, since as we shall demonstrate below, one may extract the ES up to a few tens of states. Thus, a significant number of low lying excitations of a rather large many-body systems are numerically available.

Here we demonstrate that using the ES can actually provide us with some interesting insights into the MBL transition. By monitoring the transition of the ES statistics as function of the interaction strength and level number one can follow the degree to which the excitation is localized or extended. Generally, one expects that the MBL mobility edge will occur at a certain excitation energy corresponding to a level number. As interaction increases the mobility edge should move down to the Fermi energy. Thus, for low-lying levels and weak interactions we expect that the behavior of the ES statistics will follow Poisson, while for higher level numbers or interaction strength it should follow GOE statistics. Thus, by tracking the crossover in the ES statistics we hope to learn about the MBL transition.

2. Model

To study the MBL transition we consider a spinless 1D electrons system of size L with repulsive nearest-neighbor interactions and on-site disordered potential. The Hamiltonian is given by:

$$H = \sum_{j=1}^{L} \epsilon_j \hat{c}_j^\dagger \hat{c}_j - t \sum_{j=1}^{L-1} (\hat{c}_j^\dagger \hat{c}_{j+1} + h.c.) + U \sum_{j=1}^{L-1} (\hat{c}_j^\dagger \hat{c}_j - \frac{1}{2})(\hat{c}_{j+1}^\dagger \hat{c}_{j+1} - \frac{1}{2}), \quad (1)$$

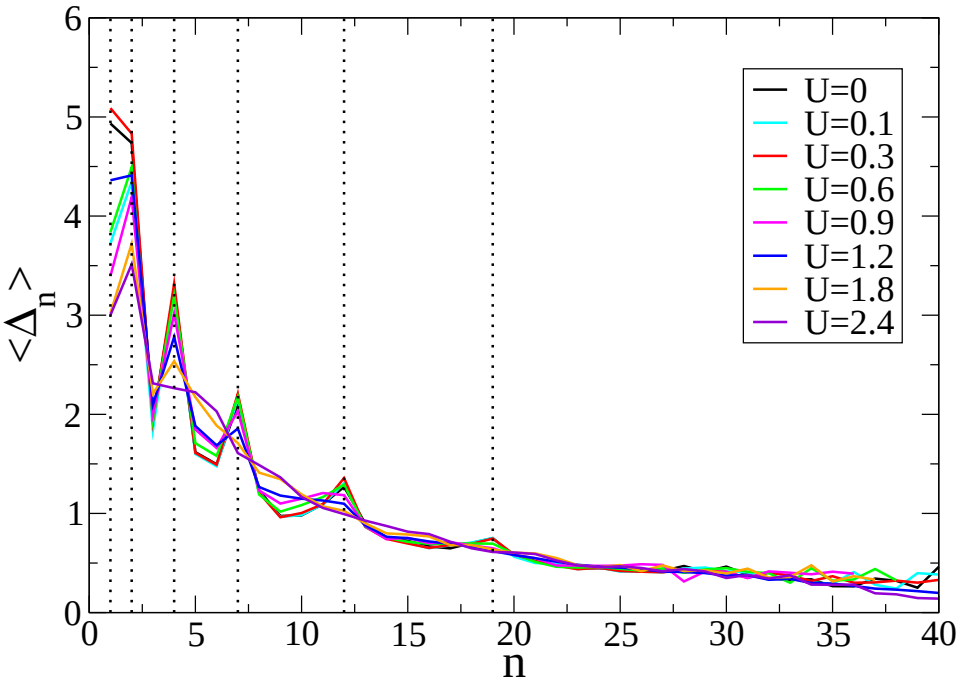

Figure 1. The average ES spacing as function of the level number n for different values of interaction strengths U. The dashed vertical lines correspond to $n = 1, 2, 4, 7, 12, 19$ for which larger spacings are expected [28].

where ϵ_j is the on-site energy drawn from a uniform distribution $[-W/2, W/2]$, $\hat{c}_j^\dagger$ is the creation operator of an electron at site j, and $t = 1$ is the hopping matrix element. The repulsive interaction strength is depicted by $U \geq 0$, and a positive background is considered.

The density matrix renormalization group (DMRG) [26, 27]. is a very accurate numerical method for calculating the ground state of disordered interacting 1D system and for the calculation of the reduced density matrix. We calculate the eigenvalues of ρ_A for a system of length $L = 700$ at half-filling (i.e., the number of electron in the system is $N = L/2 = 350$) and different values of $L_A = 100, 110, \ldots, L - 100$, for at least 20 realizations of disorder. The disorder strength $W = 3.5$ corresponds to a localization length $\xi_0 \approx 105/W^2 \sim 8.5$. The ES is calculated using the relation $\varepsilon_i^A = -\ln \lambda_i^A$.

3. Entanglement spectrum level statistics

The first step in studying the statistical properties of the ES, is to realize that when one bisects the system into regions A and B the number of particles in each region N_A and $N_B = N - N_A$ are still a good quantum number. Since reduced density matrix eigenvalues of different values of N^A do not couple, one can denote the eigenvalues as, $\lambda_n^{N_A}$, which translate into the ES by [3]: $\varepsilon_n^{N_A} = -\ln(\lambda_n^{N_A})$. Thus, for each number sector N_A, the spacing of the ES, $\Delta_n^{N_A} = \varepsilon_{n+1}^{N_A} - \varepsilon_n^{N_A}$, and the average over different lengths L_A, number of particles in the region N_A and realizations depicted by $\langle \Delta_n \rangle$ may be calculated. The results for different strength of interaction U are presented in Fig. 1. Since $\langle \Delta_n \rangle$ should correspond to the many-particle level spacing, it is expected to fall of exponentially as function of n, which is indeed seen. Another interesting feature is the appearance for the non-interacting case of peaks (i.e., larger spacings) for $n = 1, 2, 4, 7, 12, 19$. As discussed elsewhere [28], the origin of these peaks in the

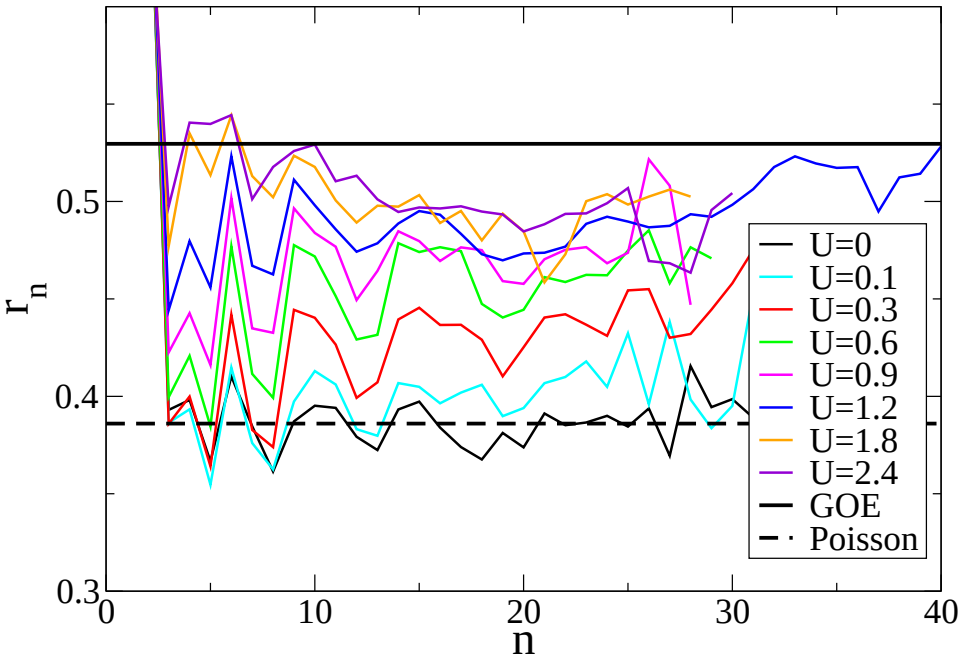

Figure 2. Average r_n as function of level number for different strengths of interaction U. The expected value for r_n in the localized (Poisson) regime is indicated by the heavy dashed black line and in the extended (GOE) regime by a continuous heavy black line.

non-interacting case is essentially combinatoric, where peaks appear when the only possibility of reaching the next excitation energy is by a single-electron transition to a higher energy, contrasted to transitions of several electron, some up some down. These existence of such peaks are an indication of MBL, since they cannot survive the coupling between the many-particle states in the extended states. Indeed, as interaction increase the peaks of Δ_n are erased, and for the highest interactions ($U = 2.4$) only the lowest peaks (at $n = 1, 2$) remain. This is in line with our expectations that as interactions increase the mobility edge goes down, and the localized regime shrinks to the immediate vicinity of the Fermi energy. Nevertheless, this is not an exact method to extract the mobility edge.

In order to get a more quantitative estimation of the localized-extended transition we shall use a statistical measure of the behavior of the spacings between successive ES energies. Following Ref. [15] we define a dimensionless parameter r_n that captures the correlations between successive gaps in the spectrum:

$$\langle r_n \rangle = \frac{\min(\Delta_{\rm n}^{\rm N_A}, \Delta_{\rm n+1}^{\rm N_A})}{\max(\Delta_{\rm n}^{\rm N_A}, \Delta_{\rm n+1}^{\rm N_A})}, \tag{2}$$

where $\langle \ldots \rangle$ stands for average over different values of N_A, L_A and realizations of disorder. For the Poisson statistics, $r_n = 2\ln(2) - 1 \sim 0.386$, while for the GOE (Wigner) statistics $r_n \sim .53$. The numerical results of r_n as function of the level number n for different strengths of interaction U are presented in Fig. 2. Let's begin by discussing the non-interacting case $U = 0$. Here we expect that since $\xi \ll L$ all the excited states will show Poisson statistics, i.e., $r_n \sim 0.386$. Indeed, except for the first few values of r_n (especially r_2), which correspond to large level spacings peaks seen in Fig. 1, the values of r_n corresponds very well to the expected GOE value. As U increases, r_n also rises, climbing closer to the GOE value of $r_n \sim .53$. For lower values

of interaction ($U < 1$), the larger n is (i.e., the higher the excitation energy) the larger is the increase in r_n. This is in line with the expectation from a mobility edge at higher energy, and with it moving down in energy as interaction increases. At higher interactions ($U > 1$), r_n for higher excitations ($n > 10$) saturates at a value close, but significantly below the GOE value, while r_n for low excitation ($n < 10$) shoots up, reaching the GOE value around $U = 2$. Thus for the highest U we have studied, there is an *intermediate* range of excitations ($10 < n < 30$) for which the statistics of r_n is not quite GOE although it fits very well anywhere else.

4. Discussion

Thus, there is a clear crossover from the Poisson to GOE statistics for the ES statistics in line with the MBL transition. Nevertheless, in order to prove a QPT, we will need to demonstrate finite size behavior corresponding to a mobility edge. There is also the question of the curious saturation of r_n at intermediate excitation numbers. For the ground state of a disordered interacting system interactions drive the system to stronger localization with shorter localization length [29]. On the other hand for the first few excited state, metallic behavior is seen for strong enough interaction [30]. Thus, one naively would expect this metallic behavior to continue for higher excitation, which obviously from Fig. 2 does not occur. This might be connected to the non-ergodic behavior expected in the MBL transition [31], but this needs further study.

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