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Regularization of derivatives on non-differentiable points

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Abstract. The notion of derivative is limited only to the idealization of linear growth. The paper presents a formal regularization procedure for the derivatives of functions using the fractional velocity, i.e. a local fractional derivation operation. The approach can be applied in strongly non-linear or fractal settings, such as singular functions, Brownian motion, quantum paths, etc.

1. Fractional approach to derivation

A central notion of physics is the rate of change. It can be argued that this perception inspired Newton and Leibniz to develop the apparatus of differential calculus. However, the notion of *derivative* is limited only to the idealization of linear growth. Classical physical variables, such as velocity or acceleration, are considered to be differentiable functions of position. On the other hand, quantum mechanical paths were found to be non-differentiable and stochastic in simulations [2]. The relaxation of the differentiability assumption could open new avenues in describing physical phenomena, for example, using the **scale relativity theory** developed by Nottale [1], which assumes fractality of quantum-mechanical trajectories. Cherbit [4] introduced the notion of α *fractional (fractal) velocity* as the limit of the fractional difference quotient. His main application was the study of fractal phenomena and physical processes for which the instantaneous velocity was not well defined. This concept can be extended to mixed-orders as follows [5, 6] :

Definition 1 (Mixed-order velocity). *Define the fractional velocity of mixed order $n+\beta$ of function $f(x) \in \mathbb{C}^n$ as*

$$\upsilon_{\pm}^{n+\beta} f(x) := (\pm 1)^{n+1} (n+1)! \lim_{\epsilon \to 0} \frac{f(x \pm \epsilon) - T_n(x, \pm\epsilon)}{\epsilon^{n+\beta}} \tag{1}$$

where $n \in \mathbb{N}$; $\epsilon > 0$, $0 < \beta \leq 1$ are real parameters and $T_n(x, \epsilon)$ is the Taylor polynomial $T_n(x,\epsilon) = f(x) + \sum\limits_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \epsilon^k$.

Some of the most peculiar properties of α-velocity are that it takes discrete values, it is discontinuous and non-vanishing only at points where the usual derivative is unbounded[3, 5].

The **fractional velocity** is able to characterize growth of functions varying on fractal sets. This can be exemplified by its application to singular functions, such as the Cantor's function.

Example 1. *Cantor's function obeys the functional equation*

$$C(x) := \begin{cases} 0, & x = 0 \\ \frac{1}{2}C(3x), & 0 \le x < 1/3 \\ \frac{1}{2} & 1/3 \le x \le 2/3 \\ \frac{1}{2} + \frac{1}{2}C(3x-2), & 2/3 \le x \le 1 \\ 1, & x = 1 \end{cases}$$

Therefore $\upsilon_{+}^{\alpha}C\,(x)$ *=1 if x belongs to the Cantor set* $\mathbb{C}_{1,3}$ *and* $\upsilon_{+}^{\alpha}C\,(x)$ *=0 otherwise for* $\alpha = \frac{\log 2}{\log 3}$. *The* ***Cantor's set*** *in turn is given by* $\mathbb{C}_{1,3} = \left\{x : x = \overline{0.d_1 \ldots d_n},\ d \in \{0,2\}\right\}$ *in ternary number representation.*

2. Regularized Taylor expansions

In other applications it transpires that **fractional velocity** acts as an auxiliary object with regard to integer-order derivatives. Paradoxically, the irregularity of the fractional velocity can be used to regularize the usual derivatives at singular points. This can be demonstrated in the regularization procedure for the derivatives of Hölder functions, which allows for removal of the weak singularity in the derivative caused by strong non-linearities:

Definition 2. *Regularized derivative of a function is defined as:*

$$\frac{\bar{d}^{\ \beta\pm}}{dx} f(x) := \lim_{\epsilon \to 0} \frac{\Delta_{\epsilon}^{\pm}[f](x) - \upsilon_{+}^{\beta} f\,(x)\,\epsilon^{\beta}}{\epsilon}$$

We will require as usual that the forward and backward regularized derivatives be equal for a uniformly continuous function.

Then the following statement holds: Let $f(t, w) \in \mathbb{C}^2$ be composition with $w(x)$, a Hölder function $\mathbb{H}^{1/q}$ at x, then

$$\frac{\bar{d}^{\ \pm}}{dx} f(x, w) = \frac{\partial f}{\partial x} + \frac{\bar{d}^{\ \pm}}{dx} w(x) \cdot \frac{\partial f}{\partial w} + \frac{(\pm 1)^q}{q!}[w^q]^{\pm} \cdot \frac{\partial^q f}{\partial w^q} \tag{2}$$

where

$$[w^q]^{\pm} = \lim_{\epsilon \to 0} \left(\upsilon_{1/q}^{\pm\epsilon}\,[w]\,(x)\right)^q$$

is the fractal q-adic (co-)variation. Possible applications of presented approach are regularizations of quantum mechanical paths and Brownian motion trajectories, which are Holder 1/2.

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