Contextually in a Peres–Mermin square using arbitrary operators

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Abstract. The contextuality of quantum mechanics can be shown by the violation of inequalities based on measurements of well chosen observables. These inequalities have been designed separately for both discrete and continuous variables. Here we unify both strategies by introducing general conditions to demonstrate the contextuality of quantum mechanics from measurements of observables of arbitrary dimensions. Among the consequences of our results is the impossibility of having a maximal violation of contextuality in the Peres–Mermin scenario with discrete observables of odd dimensions. In addition, we show how to construct a large class of observables with a continuous spectrum enabling the realization of contextuality tests both in the Gaussian and non-Gaussian regimes.

In a non-contextual theory the measurement result of an observable depends only on the state of the system and the observable A being measured. In particular, it does not depend on the compatible observables that are measured alongside. This corresponds to the classical view in which the system is in a well defined state with well defined properties at all times. It was first shown by Kochen and Specker that quantum mechanics is a contextual theory and that it is impossible to assign pre-determined values to all (compatible) observables describing a system [1]. Later on, Peres and Mermin derived a simple quantitative argument [2, 3] to test contextuality that also lead to the discovery of state independent non-contextual inequalities [4].

Here, we briefly recall the principles of the Peres–Mermin scenario (PMS) and show how it can be generalized to arbitrary operators. Using this approach we show that in the discrete case a state independent violation of non-contextuality with qudits of odd dimension is impossible. In the continuous case we will prove the existence of a whole class of operators, Gaussian and Non-Gaussian enabling the demonstration of the contextuality of quantum mechanics.

To see that quantum mechanics is contextual, one can consider a set of 9 dichotomic observables A_{ij}, with outcome $+1$ and -1, such that each observables sharing a common subscript are compatible (commute). From this set of observables, one can form the quantity :

$$\langle X\rangle = \langle A_{11}A_{12}A_{13}\rangle + \langle A_{21}A_{22}A_{23}\rangle + \langle A_{31}A_{32}A_{33}\rangle + \langle A_{11}A_{21}A_{31}\rangle + \langle A_{12}A_{22}A_{32}\rangle - \langle A_{13}A_{23}A_{33}\rangle\,. \quad (1)$$

In a non-contextual theory, one is able to assign pre-determined values $+1$ or -1 to every observable in 1, thus yielding the noncontextual bound of 4. However, in the quatum case, it is possible to find combinations of Pauli measurements such that $\langle X \rangle_{QM} = 6$ for every quantum state [4].

Instead of considering observables for the A_{ij}, we can consider arbitrary unitary operators $U_{ij} = A^R_{ij} + iA^I_{ij}$ where A^R_{ij} and A^I_{ij} are the real and imaginary hermitian parts. In this case X is not an observable anymore and the quantity to consider is the real part of X, made of all A^R_{ij} and A^I_{ij}. Under the unitarity condition, it is possible to prove that this quantity is bounded by $3\sqrt{3}$ (see [5]). In order to construct a PMS, one needs that the observables sharing common subscripts are compatibles. Also, in order to maximally violate the inequality, one needs that the product of the observables in each bracket is either $\mathbb{1}$ or $-\mathbb{1}$ (see [6] for details). These two conditions lead to the constraints:

$$\{\hat{U}_i, \hat{U}_j\} = 2\delta_{ij}\hat{U}_i^2, \tag{2}$$

$$[\hat{U}_i, \hat{U}_j] = \pm 2i\epsilon_{ijk}\hat{U}_k^\dagger. \tag{3}$$

A point worth discussing, but that is out of the scope of the present Letter, is the case of state dependent inequalities, as the ones of the CHSH type [7] and the relation between the obtained condition and fault tolerant quantum computation with continuous variables [8]: we can notice that a common feature to all these results is the necessity of unitary operations defined in continuous variables to obey the commutation relations (3), providing insight on the relationship between contextuality and quantum computation.

In the discrete case these conditions can be satisfied only if the dimension of the system is even. Examples of operators satisfying these relations are given by the rotation operators :

$$\hat{R}_1 = e^{i\hat{S}_x\pi}, \quad \hat{R}_2 = e^{i\hat{S}_y\pi} \quad \text{and} \quad \hat{R}_3 = e^{i\hat{S}_z\pi} \tag{4}$$

where $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are the three components of a half integer spin S.

In the continuous case it has been shown [5] that it is possible to use displacement operators $\mathcal{D}(\alpha_i) = \exp(\alpha_i\hat{a}^\dagger - \alpha_i * \hat{a})$ where the α_i obey $\mathrm{Im}(\alpha_i\alpha_j^*) = \pm\pi/2$. We further show that any set of operators of the form $\hat{V}\mathcal{D}(\alpha_i)\hat{V}^\dagger$, where $\hat{V}$ is an arbitrary unitary operator, is suitable for a contextuality test. In particular, using unitaries that are polynomials of order higher than 3 in the canonical variables, we can create a family of non-Gaussian operators enabling the violation of non-contextuality.

We have shown that that it is possible to derive a general non-contextual inequality in terms of complex observables that can be applied to discrete and continuous variable measurements. This allowed us to derive general conditions that must be fulfilled by the operators in order to have a maximal violation of the contextual inequality. In particular, we showed that in the case of discrete system it is possible to have a maximal state independent violation of the contextual inequality only for system of even dimension. In the case of continuous variables we can derive a family of operators, Gaussian and non-Gaussian, suitable for contextuality tests.

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