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Structure Process, Weak Values and Local Momentum

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Abstract. We explain how weak values and the local momentum can be better understood in terms of Bohm's notion of structure process. The basic ideas of this approach can be expressed in a fully algebraic way, generalising Heisenberg's original matrix mechanics. This approach leads to questions that are now being experimentally investigated by our group at University College London.

1. Introduction

1.1. Act not Fact

It is almost a century since quantum mechanics was discovered but there remains the unresolved problem of the collapse of the wave function. Could it be that we have been focusing on the wrong mathematical structure to provide an adequate understanding of quantum phenomena? David Finkelstein [1] thinks so and argues that "to speak about the *wave function of the system* is a syntactic error". I agree, so let me explain what he means.

When we set up a Stern-Gerlach magnet with its axis in a certain direction and find the spin is, say up, the wave function contains information about what the experimenter is *doing* to get such a result. Reorienting the magnet will produce a different result on an identically produced system. This involves a double act, Act 1, as Finkelstein calls it, is the preparation and Act 2, is mistakenly called the 'measurement'. However there is no difference between the two Acts, they are just carried out in a different order. Act 2 could be used for the preparation of another experiment and therefore could be called 'preparation 2', but today we use the terminology 'pre- and post-selection'.

Before embarking on a development of a formal algebraic approach to quantum phenomena it is important to realise that the context in which we are working involves a radical reappraisal of underlying physical ideas forming the basis of our approach. These ideas go back to the 1960s when David Bohm, Roger Penrose and I were exploring the possibility of changing the geometrical picture of the space-time structure to accommodate quantum processes. Bohm's proposal [2] was that we should not assume that these quantum processes can be analysed as particles moving in an *a priori* given space-time, but rather *all* is a modern form of Heraclitian flux so that what is *is* the process of becoming itself. All objects and particles are then ultimately quasi-stable, quasi-local forms that can be abstracted from this underlying process which we called the *holomovement* and hence they are themselves inseparable from the underlying activity. In a little known paper, Bohm [3] showed that this activity could be described by a set of

elemental structure-processes (see Figure 1) whose meaning we will bring out as we go along. This structure of elemental processes undergoes a discrete or continuous movement as it unfolds in a process of development. Penrose explored these ideas in his theory of *spin networks* [4], which ultimately led to the development of twistors [5]. Bohm and I developed a more general algebraic structure-process, originally based on orthogonal Clifford algebras, but now extended to symplectic Clifford algebras [6, 7].

Within this context, let me develop the mathematics in a heuristic manner and show that the formal mathematical structure mentioned above is closely related to the conventional formalism but sufficiently different to offer new insights. To do this, let us make use of a formalism already introduced by Dirac [8], but generally ignored.

We will denote Act 2 by $\psi\rangle$ and Act 1 by $\phi\rangle$; we are then looking at the process $\phi\rangle \rightarrow \psi\rangle$[1]. Following Feynman, we define the transition probability amplitude, [TPA], for the process,

$$TPA = \langle\psi|\phi\rangle. \tag{1}$$

But this brings the process to an end by forming a complex number. Indeed forming $\langle x|\psi\rangle = \psi(x)$, produces the wave function. But this is simply another transition probability amplitude, $\psi\rangle \rightarrow x\rangle$, which tells you more about the *act* of producing an effect, rather than telling us about the *state of the system.*

In the standard approach eigenvalues are assumed to be associated with *properties* of the system, giving these properties a *value* which is *assumed* to describe the *state of the system.* We have argued above that the wave function is a TPA and so how are we to understand eigenvalues in terms of TPAs? They are simply TPAs in which Act 2 is the same as Act 1, $\psi\rangle \rightarrow \psi\rangle$, that is the pre- and post-selections are identical. Therefore they are values that persist unchanging when you subject the system to the *same process* – nothing more, nothing less. How then are we to develop a mathematics of *process*?

1.2. Clifford Algebras and Process

As I was exploring the origins of Clifford algebras and their relations to quaternions (essentially Pauli spinors), my attention was drawn to a lecture by Hamilton [9] in which he was discussing the metaphysical aspects of algebra. He writes:

> In algebra, the relations .... are between successive states of some *changing thing or thought.* Numbers are the names or nouns of the algebra; they are the marks or signs, by which one of these successive states may be remembered and distinguished from another..... relations between successive thoughts thus viewed as successive states of one or more general changing thought, are the primary relations of algebra.

In attempting to generalise the quaternions to higher dimensional space, Clifford [10] points out that there are two sides to the notion of a product. The relation 2×3 equals 6, may be regarded as the product of the two numbers 2 and 3; or it may be regarded as the act of doubling the number 3. In the latter, 3 is a number but 2 is an operation, and the two factors play very distinct parts. Thus Hamilton's quaternions are likewise operations which transform one vector into another but now, when we a dealing with quantum phenomena, we have an important new feature, namely, non-commutativity, a feature which plays a crucial role in distinguishing between classical and quantum mechanics. For more details see Hiley [7, 11].

Let us return to the quaternion and consider its action of rotating and stretching a vector ρ to produce a vector σ,

$$q\rho = \sigma.$$

[1] This is not a typo. The absence of '|' will be explained below.

But this multiplication can be non-commutative, so that we may find that $\rho q = \eta \neq \sigma$. The precise relationships between the various elements appearing here are determined, not individually, but by the overall structure of the algebra.

We want to effect a further change and make the vectors themselves part of the algebra; we therefore introduce ρ^*, a dual to ρ so that

$$q = \sigma\rho^* \quad \text{with} \quad \rho\rho^* = 1. \tag{2}$$

Notice that q is an *action* and therefore the product $\sigma\rho^*$ must also be an *action*.

Let us consider re-writing equation (2) in the form

$$q = \sigma\rangle\langle\rho. \tag{3}$$

Now we are in a position to explain the *standard ket*, a new notation first introduced by Dirac [8] and subsquently discussed in more detail in the 3rd edition of his classic text on quantum mechanics [12]. Originally the symbol, | , in front of the ket | $\rangle$ was introduced to separate operators from the labels used to specify the wave function. But the content of the wave function is determined by the operators themselves, simply by replacing the operators by their eigenvalues. Dirac thereby shows that nothing is lost by removing the distinction between operator and wave function.

What Dirac had effectively done was to remove the distinction between an element of the operator algebra and the wave function without losing any information about the content of what is carried by the wave function. Thus everything now is contained in the algebra itself. This is exactly what we require if we want to obtain an algebraic description of process. In fact Dirac had formed an element of a left ideal in the algebra by the symbol $\rangle$. Only multiplication from the left is allowed. He called the object $\rangle$ the *standard ket* and its dual $\langle$ the *standard bra*. We then have a way of distinguishing between left and right translations, a distinction that is essential in non-commutative geometry.

We have now reached the position in which all elements are within the same algebra except one, namely $\rangle\langle$. To identify this symbol, notice that in a non-nilpotent algebra, there always exists an idempotent, i.e., an element ϵ such that $\epsilon^2 = \epsilon$. Then we can always write an element of a left ideal in the form $\Psi_L = \psi\epsilon$ where ϵ is an idempotent satisfying $\epsilon^2 = \epsilon$. Thus $\rangle\langle$ is a symbol for an idempotent, which is, unfortunately, not in the nilpotent Heisenberg algebra and therefore must be added.

The fact that $\rangle\langle$ is an idempotent has already been noticed by Kauffman [13] and its role has been discussed in more detail in Hiley [7]. There we point out that in order to introduce spin and relativity, we must introduce the orthogonal Clifford algebra. That algebra is not nilpotent and many idempotents exist. They are used to describe the break in the space-time symmetry when a homogeneous magnetic field is introduced by the experimenter.

In a series of insightful papers on quantum geometry, Schönberg [14] has already introduced the required idempotent to complete the Heisenberg algebra through the defining relations

$$E^2 = E, \quad PE = EX = 0$$

where $E \equiv \rangle\langle$ and X and P are elements of the Heisenberg algebra. To make the role of E clearer, Frescura and Hiley [15] showed that in the usual transformation to the bosonic $a, a^\dagger$ algebra, E is replaced by the projection onto the vacuum state $V = |0\rangle\langle 0|$ so that $aV = Va^\dagger = 0$.

What we have done can easily be understood in terms of a matrix representation. A general way of writing a rank one matrix, q, is to product a column matrix, σ with a row matrix ρ^*, which is done in equation (2). Thus we have returned to a "matrix mechanics", where our matrices now have an actual physical meaning in terms of a structure-process. In fact we do

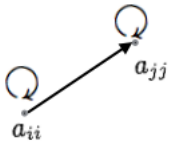

Figure 1. Elementary Process

not need to resort to a matrix representation once we realise that a simple element of a general algebra can be written as a product of an element of a left ideal Ψ_L and an element of a right ideal Ψ_R.

It should be noted that the mathematical structure we have highlighted here is of physical significance in the context of structure process. Furthermore it can be shown that this algebraic structure can be *mapped* isomorphically onto the standard Hilbert space formalism. However if that is done and the standard interpretation is adopted, we are right back into the measurement problem and interpretational difficulties.

2. Structure-Process

2.1. The Elementary Process and Groupoids

Let us take a step back and try to examine in more detail the mathematical description we have introduced heuristically in the previous section. Recall that *becoming* was taken as primary, whereas *being*, a natural primary term in the classical paradigm, is a special case of becoming, a repeated becoming that produces no change. In order to put these words into a firmer mathematical context, let us represent becoming by an element a_{ij} and take multiplication to be the *order of succession*. Each process has a source i and a target j. The sources and targets can be regarded as the idempotents a_{ii} and a_{jj}. A useful image is shown in Figure 1.

To give the abstract notion of an elementary process more physical meaning, recall that in relativity we are forced to take the notion of a *point event* as basic. Then the elementary process would be the relation between a pair of events, the primary connection being a light ray. This is the basic idea that Penrose [5] adopts in his twistor theory. However the meaning of the symbol is not directly relevant. It is only the relationships and the operations in which these symbols take part that gives the whole structure its meaning. In other words the single algebraic symbol is similar to a word, in the sense that its implicit meaning only comes out in the way in which the language as a whole is used [2].

If a process g_{ij} is followed by another process $g'_{j'k}$, then 'succession' is defined only when $j = j'$. In other words we have assumed that our structure process forms a groupoid. In symbols

$$g, g' \in G \text{ and } j = j', \quad \text{then} \quad g_{ij} g'_{j'k} = g''_{ik}$$

where G is some group.

What then is a 'being', a particle, in a process philosophy? Eddington [16] has suggested that we do not want to introduce a particle as a "lump of matter", matter being a metaphysical term that has no place in a process philosophy. Rather, as we have hinted above, a natural structural concept of existence can be represented by an idempotent, $a_{ii}a_{ii} = a_{ii}$. This is a process that continuously turns into itself. This idempotent could, itself, be a complicated substructure of processes, a notion that is exactly what is needed for the concept of a particle in relativity, where there is a fundamental difficulty with the idea of extended matter. In relativity, an extended object must be represented by a world tube of a complex structure of events in space-time.

The assumption that structure process can be described by a groupoid fits in naturally with Heisenberg's original idea of replacing each of position and momentum by two-indexed symbols in order to explain the Ritz-Rydberg combination principle $\nu(nm) = \nu(nk) + \nu(km)$. Then if we replace the position q by $q(nm)$ and write it in the form

$$q \to q(nm) \to a(nm)\exp[2\pi i\nu(nm)t]$$

we see immediately that the product produces the correct combination of frequencies since $h\nu(nm) = (E_n - E_m)$.

Heisenberg actually went further by writing $E(nm) = E_n\delta_{nm}$ so that

$$Eq = \sum_k E(mk)q(kn) = \sum_k E_m\delta_{mk}q(kn) = E_m\delta_{mn}$$
$$qE = \sum_k q(mk)E(kn) = \sum_k q(mk)E_k\delta_{kn} = E_m\delta_{mn}$$

giving Heisenberg's equation of motion,

$$\dot{q} = \frac{i}{\hbar}(Eq - qE).$$

Thus by introducing addition to capture the *order of coexistence*, we are led to a non-commutative algebra, which is at the heart of quantum mechanics. Our approach is not merely to accept formalism as an abstract structure but to provide a physical structure lying behind the formalism, leading to the need for a clearer understanding of non-commutative geometry.

2.2. New Concept of Motion

Notice that in a non-commutative structure, a key factor is the recognition that we must distinguish between left and right translations. In other words, *order is vital*. The distinction gives rise to the possibility of a new type of motion, namely, that an old *structure* may evolve into a new *structure* via the inner automorphism:

$$e' = M_1^{-1}(\tau)eM_2(\tau) \tag{4}$$

where τ parameterises the order of succession. This is what Bohm [2] calls an enfolding-unfolding movement that lies behind the path integral method introduced by Feynman [17]. We will assume that $M_1 = M_2 = \exp[iH\tau]$ so that for small τ, equation (4) can be written in the form

$$e' = (1 - iH\tau)e(1 + iH\tau) \quad \Rightarrow \quad i\frac{(e' - e)}{\tau} = He - eH. \tag{5}$$

In the limit this becomes the Heisenberg equation of motion, provided we identify τ with time. Thus we directly make contact with the historical starting point of quantum mechanics in terms of structure-process.

We can say more. e and e' are elements of the algebra so that they can be written in the form $e = |\psi\rangle\langle\phi|$ in the way we have suggested above[2]. We find equation (5) becomes

$$i\frac{d|\psi\rangle}{d\tau}\langle\phi| + i|\psi\rangle\frac{d\langle\phi|}{d\tau} = (H|\psi\rangle)\langle\phi| - |\psi\rangle(\langle\phi|H)$$

[2] We retain the | in order to make immediate contact with the usual notation.

which splits into two equations

$$i\frac{d|\psi\rangle}{d\tau} = H|\psi\rangle \quad \text{and} \quad -i\frac{d\langle\phi|}{d\tau} = \langle\phi|H.$$

So the Schrödinger equation *and* its dual emerge from this process [18]. Since the dual is considered as the complex conjugate of the wave function, it is usually assumed that this adds no new content and therefore the dual equation is of no consequence. However approaching through the algebraic point of view where we are distinguishing between the left and right translation, it does have a significance. For more details see Hiley [11].

2.3. Equations of Motion from the Lagrangian

Notice however that we also obtain two equations from the Heisenberg Lagrangian

$$\mathcal{L} = -\frac{1}{2m}\nabla\psi^* \cdot \nabla\psi + \frac{i}{2}[(\partial_t\psi)\psi^* - (\partial_t\psi^*)\psi] - V\psi^*\psi.$$

We then use the Euler-Lagrange equations treating ψ and ψ^* as independent so that

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi_\mu}\right) = 0 \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\psi^*} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^*_\mu}\right) = 0$$

where $\psi_\mu = \partial_\mu\psi$ and $\psi^*_\mu = \partial_\mu\psi^*$. Evaluating the two equations of motion, we find

$$i\hbar\partial_t\psi = H\psi \quad \text{and} \quad -i\hbar\partial_t\psi^* = H\psi^*$$

or more suggestively

$$i\hbar\overrightarrow{\partial}_t|\psi\rangle = \overrightarrow{H}|\psi\rangle \quad \text{and} \quad -i\hbar\langle\psi|\overleftarrow{\partial}_t = \langle\psi|\overleftarrow{H}.$$

Now we can rewrite the Lagrangian in terms of the two real fields R and S where as usual $\psi = R\exp^{iS/\hbar}$ so that

$$\mathcal{L} = -R^2\left(\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + \frac{\hbar^2}{2m}\frac{(\nabla R)^2}{R^2} + V\right). \tag{6}$$

Treating R and S as independent variables, we find the two Euler-Lagrange equations now become

$$\partial_t\rho + \nabla.(\rho\nabla S/m) = 0$$

and

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}(\nabla^2 R)/R + V(x) = 0. \tag{7}$$

These equations are, of course, the equations central to the Bohm approach but notice they are also central to the quantum formalism.

Returning to the Lagrangian (6), we see that there are two kinetic energy terms, $(\nabla S)^2/2m$ and $\hbar^2(\nabla R)^2/2mR^2$. This is confirmed by noting that the diagonal terms in the usual expression for the energy-momentum tensor, evaluated using the Lagrangian given in equation (6), can be written as

$$T^{kk} - \mathcal{L}\delta^{kk} = \frac{R^2}{2m}\left[(\partial^{x_k}S)^2 + \hbar^2\frac{(\partial^{x_k}R)^2}{R^2}\right] + VR^2.$$

Furthermore we find two canonical momenta, namely,

$$p_1(x,t) = \nabla S(x,t) \quad \text{and} \quad p_2(x,t) = \hbar \nabla R(x,t)/R(x,t). \tag{8}$$

The first term is simply the real part of the local momentum, sometimes called the Bohm momentum, while the second term is called the 'osmotic' momentum after it appeared in Nelson's [19, 20] stochastic model used in his derivation of the Schrödinger equation from Brownian motion. However we are talking about a single particle, and therefore such an approach requires the existence of some form of sub-quantum medium, an old idea which has dropped out of favour.

In our approach the two momenta arise because the order of succession is vital and gives rise to a non-commutative structure. In such a structure it is necessary to distinguish between left and right translations and it is this difference that gives rise to the 'strange' features of quantum processes.

In an early important paper by Hirschfelder *et al* [21] it is shown that

$$\frac{\langle x|P|\psi\rangle}{\langle x|\psi\rangle} = p_1 + ip_2 \tag{9}$$

where p_1 and p_2 are exactly the same terms as appear in equation (8). But the LHS of equation (9) is what is now called the weak value of the momentum for a post-selected state that is the final position.

Our approach to the behaviour of what we can call a single particle is dictated by equation (7) which ensures the conservation of energy provided we regard the term

$$Q = -\frac{\hbar^2}{2m}(\nabla^2 R)/R$$

as a new quality of energy arising from the additional 'osmotic' process. Traditionally this has been called the quantum potential energy because it has, at times, been regarded as giving rise to an 'internal force'. This is not the way we regard it in the structure process approach. Rather it is an internal energy arising from an extra degree of freedom in which the process acts.

In the above equations, the properties are assumed to be local, hence the term 'local momentum'. If we regard equation (7) as a local particle equation, then where are the *global* properties which are required if the experimental conditions are to play a key role as Bohr has claimed? They are encoded in the quantum potential energy which enables us to work with a *local* model.

This is in contrast to standard quantum field theory which works with *global* properties of both energy and momentum with the momentum of the particle defined through

$$P^j = \int T^{0j}(x,t)d^3x \quad \text{and} \quad E = \int T^{00}(x,t)d^3x.$$

Thus we are left with the question of whether we should regard a quantum particle as a global or local object, a conflict that Colosi and Rovelli [22] have already discussed at some length and so we will not repeat their arguments here but strongly recommend their paper to the interested reader.

3. The Problem of Stationary States

In order to illustrate one of the advantages of the new analysis through structure-process, let us return to the objection that Einstein [23] raised against the original Bohm proposals. In a

stationary state, the spatial part of the wave function can be real so that $p_B = \nabla S = 0$. This implies that the 'particle' is stationary, a result that violates our physical intuition.

A way to see what is going on is to analyse a simple example, namely the eigenstate of a particle in a one-dimensional box with infinite potential delta function walls. Here, as every undergraduate learns, the eigenfunctions, $\psi_n(x,t)$, are all real

$$\psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{iE_n t}{\hbar}}$$

where a is the length of the box. In terms of a polar decomposition of the wave function

$$R_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{and} \quad S_n = -E_n t.$$

Then

$$\partial_x S_n = 0; \quad \partial_t S_n = -E_n \quad \text{and} \quad Q_n = -\frac{\hbar^2}{2m}\nabla^2 R_n/R_n = \frac{n^2\hbar^2\pi^2}{2ma^2}.$$

Thus $p_1 = 0$. However notice that equation (8) gives $p_2 \neq 0$ so there is activity present even in the stationary state, a point that was not apparent in the original Bohm model. Thus the appearance of the quantum potential energy seemed mysterious, simply arising in the real part of the Schrödinger equation, a feature that led Heisenberg [24] to declare that the potential was *ad hoc*. However Hirschfelder *et al* [21] constructed the local value (ie weak value) of the kinetic energy showing

$$(2m\psi)^{-1}\hat{\boldsymbol{p}}.\hat{\boldsymbol{p}}\psi = (2\psi)^{-1}\hat{\boldsymbol{p}}.(\boldsymbol{v}_1 + i\boldsymbol{v}_2)\psi = mv_1^2/2 + Q + i(m\boldsymbol{v_1}.\boldsymbol{v_2} - \hbar\boldsymbol{\nabla}.\boldsymbol{v_2}/2).$$

Then we find

$$2Q = -mv_2^2 + \hbar\boldsymbol{\nabla}.\boldsymbol{v}_2 = -\frac{\hbar^2}{m}\frac{\nabla^2 R}{R}. \tag{10}$$

This gives a very different insight as to the origins of the quantum potential energy. It also shows that the kinetic energy of the particle cannot be negative, thus contradicting an earlier result of Aharonov and Rohrlich [25] who concluded from an argument using weak values that the kinetic energy must be negative.

In terms of our model of structure process, the particle is an invariant feature of an underlying process, which comprises two components arising from the non-commutative aspect of the process. What this shows is the condition $p_1 = 0$ does not imply that there is no activity in the box. On the contrary there is still activity present, but it is in the form of what has been called 'osmotic' activity. This term originates from early attempts to understand quantum processes in terms of some underlying Brownian-type process induced by a sub-quantum medium subjecting the particle to random stochastic forces. Nelson showed that such stochastic background processes could lead to the Schrödinger equation provided the diffusion parameter was replaced by a term including Planck's constant.

The main step in his argument that is of relevance to our work involves the two derivatives he introduces, the 'backward derivative' and the 'forward derivative'. These are defined through the following simplified equations

$$b_*(x,t) = \frac{x(t) - x(t-\Delta t)}{\Delta t} \quad \text{and} \quad b(x,t) = \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

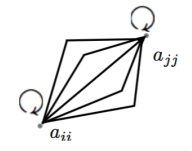

Figure 2. Many Elementary Processes between two Idempotents

These are the analogues of the left and right translations referred to above. In the appropriate limit, these form derivates from which we can construct two velocities

$$v_1(x,t) = [b(x,t) + b_*(x,t)]/2 \quad \text{and} \quad v_2(x,t) = [b(x,t) - b_*(x,t)]/2.$$

Nelson [19] also shows that the resulting Fokker-Planck equation gives rise to a continuity equation

$$\partial_t \rho + \boldsymbol{\nabla}.(\boldsymbol{v}\rho) = 0$$

where ρ is the density of the fluid. This equation is the analogue of the conservation of probability equation, the imaginary part of the Schrödinger equation under polar decomposition of the wave function. Thus we can identify $m\boldsymbol{v}$ with the local momentum, the Bohm momentum p_1 of equation (8).

4. Weak Values

4.1. Statistics

The key equation in the argument of Hirschfelder *at al* [21] is (9). There we see that the LHS is a particular example of what Aharonov *et al* [27] called a 'weak value'. In our approach, however, the form of the expression tells us that it is a ratio of two transition probability amplitudes. So the question is how are the various TPAs related to each other?

Before we can answer that question we need to take a closer look at how statistics enters the algebraic structure. There may be a number of different ways to connect a pair of idempotents as shown in Figure 2. In a simple algebra there is a Wedderburn theorem which can be written in the form

$$\epsilon A \epsilon = \langle A \rangle \epsilon$$

where $\langle A \rangle$ is the expectation value. We can see this result very quickly if we write

$$\epsilon = |\psi\rangle\langle\psi| \quad \text{so that} \quad |\psi\rangle\langle\psi|A|\psi\rangle\langle\psi| = \langle A\rangle|\psi\rangle\langle\psi|.$$

This can also be written as

$$\langle A \rangle = tr(\rho A) \quad \text{where} \quad \rho = |\psi\rangle\langle\psi| = \Psi_L \Psi_R = \psi_L \epsilon \psi_R.$$

Thus probability arises naturally in the context of structure-process.

4.2. Relationship of Weak Values to Expectation Values

To see how the weak value fits into our structure-process approach, we write

$$tr(A\rho) = \sum_{\phi_j} \rho(\phi_j) \frac{\langle \phi_j | A | \psi \rangle}{\langle \phi_j | \psi \rangle} = \sum_{\phi_j} \rho(\phi_j) \langle A_{(\phi_j;\psi)} \rangle_w$$

with $\rho(\phi_j) = |\langle\phi_j|\psi\rangle|^2$. If we choose the following basis

$$|\phi_j\rangle = \sum_{j,n} d_{jn}|a_n\rangle; \quad |\psi\rangle = \sum_k c_k|a_k\rangle \quad \text{and} \quad A|a_n\rangle = a_n|a_n\rangle,$$

we find that

$$\langle A_{(\phi_j;\psi)}\rangle_w = \frac{\langle\phi_j|A|\psi\rangle}{\langle\phi_j|\psi\rangle} = \frac{\sum_{n,k} d^*_{jn}c_k\langle a_n|A|a_k\rangle}{\sum_{n.k} d^*_{jn}c_k\langle a_n|a_k\rangle} = \frac{\sum_n d^*_{jn}c_n a_n}{\sum_n d^*_{jn}c_n}.$$

Thus in a two-state system, if we choose the following coefficients

$$a_1 = -1,\ a_2 = 1,\ d_{j1} = -99,\ d_{j2} = 101, \quad \text{and} \quad c_1 = c_2 = 1,$$

we find

$$\langle A_{(\phi_j;\psi)}\rangle_w = \frac{\langle\phi_j|A|\psi\rangle}{\langle\phi_j|\psi\rangle} = 100.$$

So a two-state system with eigenvalues ±1 can easily give rise to a weak value of 100. Thus there is nothing surprising about the result we have obtained, once we realise that we are talking about TPAs and *not* about properties the *system possesses*.

The usual result focussed on in the standard interpretation is

$$\langle\psi|A|\psi\rangle = \sum_n \rho'_n a_n,$$

with $\rho'_n = |\langle a_n|\psi\rangle|^2$. This can be obtained from the weak value by first choosing $|\phi_j\rangle = |a_j\rangle$ and $|\psi\rangle = \sum_k c_k|a_k\rangle$ and finding

$$\langle A_{(\phi_j;\psi)}\rangle_w = \frac{\sum_k c_k\langle a_j|A|a_k\rangle}{\sum_k c_k\langle a_j|a_k\rangle} = \frac{c_j a_j}{c_j} = a_j.$$

Thus we see that the eigenvalue is a special case of the weak value.

The example we have chosen might seem somewhat artificial but it is easily reproduced by a more realistic system for the weak value of the z-component of a spin-half system defined by

$$\langle\sigma_{z(\phi;\psi)}\rangle_w = \frac{\langle\phi|\sigma_z|\psi\rangle}{\langle\phi|\psi\rangle}.$$

This gives a whole spectrum of weak values from $0 \to \infty$ simply by noting that with $\langle\phi| = \langle\uparrow_z| + \langle\downarrow_z|$ and $\sigma_z|\psi\rangle = d_+|\uparrow_z\rangle + d_-|\downarrow_z\rangle\rangle$ and choosing

$$\sqrt{2}d_+ = \cos\alpha/2 + \sin\alpha/2 \quad \text{and} \quad \sqrt{2}d_- = \cos\alpha/2 - \sin\alpha/2,$$

we find $\langle\sigma_{z(\phi;\psi)}\rangle_w = \tan\alpha/2$, a result that has already been obtained by Duck *et al* [26]. Note that $\langle\sigma_{z(\phi;\psi)}\rangle_w = \tan\alpha/2$ gives the eigenvalues $+1$ or -1 when $\alpha = \pi/2$ or $\alpha = 3\pi/2$.

By highlighting what appears to be an anomalous result, Aharonov, Albert and Vaidman [27] have drawn attention to a weakness in the standard interpretation. It is clear that the important change in interpretation can now be tested experimentally in a number of ways [28, 29].

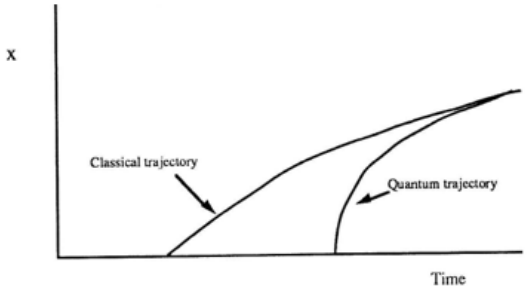

Figure 3. Pair of Merging Trajectories

4.3. Why doesn't the 'Osmotic' Velocity show up in the Standard Approach?
Where does the osmotic velocity appear in the standard approach? Consider the momentum expectation values defined by

$$\langle\psi|P|\psi\rangle = \int \left[\rho\nabla S + \frac{\hbar}{2}\nabla\rho\right] d^3x = \int \left[\rho\nabla S + \frac{\hbar\rho}{2}\frac{\nabla\rho}{\rho}\right] d^3x$$
$$= m\int \left[\rho\boldsymbol{v}_1 + \rho\boldsymbol{v}_2\right] d^3x.$$

Since $\rho \to 0$ as $x \to \pm\infty$, the second term, which contains the osmotic component, vanishes as $\int \nabla\rho \, d^3x = 0$. Thus only the local velocity contributes to the mean momentum.

Does this mean that there are no consequences of the existence of the osmotic component in the standard approach? As we have already seen from equation (10), it is the osmotic component that gives rise to the quantum potential energy which already appears in the real part of the Schrödinger equation. In terms of the present discussion we should note that the mean value of the kinetic energy is

$$\langle\psi|P^2|\psi\rangle = \int \rho[(mv)^2 + 2mQ]d^3x,$$

while the weak value of the kinetic energy is

$$\langle P^2_{x,\psi}(x,t)\rangle_w/2m = mv^2/2 + Q.$$

Thus once again we cannot escape the appearance of the quantum potential energy. It is an essential component of quantum processes; it is essential for the conservation of energy as seen in equation (7).

4.4. Emergence of Classical Mechanics
The local momentum has been used to calculate sets of streamlines [21, 30, 31] which were then interpreted as 'particle trajectories'. This has always been a contentious intrepretation. The claim originates from the similarity of the form of equation (7) and the classical Hamilton-Jacobi equation. If $Q \to 0$ and $S \to S_a$, the classical action, one has the possibility of a smooth transition of a quantum trajectory to a classical trajectory. When will such a transition take place?

It is possible to illustrate how this could happen using a toy-model. Noticing that Guth and Pi [32] had constructed a simple model to explain an inflationary universe using an up-side-down harmonic potential, Hiley and Mufti [33] showed that by starting with a Gaussian wave

packet on the top of the potential, a set of trajectories which started out as 'quantum' smoothly transformed into a set of 'classical' trajectories as the 'particles' accelerated down the potential. A pair of converging trajectories is shown in Figure 3.

The explanation is simple at one level. As the particle accelerates down the potential the quantum potential becomes negligible compared with the kinetic energy so that the classical motion results. However in the context of our approach, where the 'rock-like' image of the particle is being seriously challenged, all we can conclude is that this suggests that it is not inconsistent to regard the flow lines as particle trajectories.

Within this simplified context, we see a novel form of energy flow

$$\frac{1}{2m}(\nabla S)^2 \leftrightarrow \frac{1}{2m}\left[(\nabla S)^2 - \hbar^2\frac{\nabla^2 R}{R}\right].$$

As a metaphor, one could liken this process to the KERS device used in F-1 car racing where internal energy can be stored before being released into the classical world!

5. Conclusions

The concept of a local momentum has been around for many years, but it seems to have been ignored and dismissed as not being meaningful in a quantum context. Indeed Fritz London [34] as long ago as 1945 wrote:

> The local mean velocity has no true quantum mechanical significance since it cannot be expressed as the expectation value of any linear operator.

In other words it has been excluded on the grounds that it does not fit into standard quantum mechanics. In this paper we have shown that the standard quantum formalism is a fragment of a larger non-commutative algebraic structure. Further, it is within this wider context that the local momentum can be given a physical meaning.

More importantly, weak values can be measured experimentally, as has been shown by Ritchie, Story and Hulet [35], Parkes, Cullen and Stoudt [36] and Kocsis *et al* [37]. This last experiment has been the most relevant for us since it measured the local momentum of photons in a two-slit experiment from which photon 'trajectories' can be constructed. The exact meaning of these momentum flow lines is still openly debated. The Bohm approach treats photons very differently from finite rest mass particles such as atoms. For example in the Bohm theory applied to photons [38] there are no photon trajectories. This makes the experimental determination of local momenta using atoms of vital significance. Such an experiment is being developed by our group and is reported in the paper by Morley, Edmunds and Baker, [39].

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