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Isospectral Trigonometric Pöschl-Teller Potentials with Position Dependent Mass Generated by Supersymmetry

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Abstract. In this work a position dependent mass Hamiltonian with the same spectrum of the trigonometric Pöschl-Teller one was constructed by means of the underlying potential algebra. The corresponding wave functions are determined by using the factorization method. A new family of isospectral potentials are constructed by applying a Darboux transformation. An example is presented in order to illustrate the formalism.

1. Introduction

In recent years the study of systems with position-dependent mass (PDM) has gain much interest form both, theoretical and experimental points of views, due to its multiple applications in different areas of physics and engineering [1–8]. The very concept of a PDM system is far from being exhaustively discussed. To this respect, many contributions to the fundamental understanding of the problem have been carried out from different points of view [9–13]. The factorization method [14–16] has been considered also in this context. Its application in the generation of exactly solvable potentials in the classical as well as in the quantum mechanical regimes has been discussed, *e. g.*, in [17–24]. The aim of this work is the construction of position dependent mass Hamiltonians, having the same spectrum of the trigonometric Pöschl-Teller (TPT) potential, by means of the potential algebra of the constant mass case. In section 2 the solution of the constant mass TPT potential by means of the factorization method is presented. In section 3 it is used the constant mass potential algebra to construct position-dependent mass potentials with the TPT spectrum, and the corresponding wave functions are determined. In section 4 these Hamiltonians are used to generate a new family of PDM potentials isospectral to the TPT one by applying a Darboux transformation. Finally, some concluding remarks are presented.

2. The constant mass trigonometric Pöschl-Teller potential revisited

In proper units, the constant mass TPT potential Hamiltonian is

$$\mathcal{H}_\ell = \frac{p^2}{2} + \frac{\ell\,(\ell-1)}{2\cos^2 x}, \tag{1}$$

where

$$p = -i\frac{d}{dx}$$

stands for the momentum operator and ℓ is an integer number such that $\ell > 1$. This Hamiltonian can be factorized in the form [25]

$$\mathcal{H}_\ell = a_\ell^+ a_\ell^- + \epsilon_\ell = a_{\ell-1}^- a_{\ell-1}^+ + \epsilon_{\ell-1} \tag{2}$$

with

$$a_\ell^\pm = \frac{1}{\sqrt{2}}\left(\mp ip + \ell \tan x\right), \quad \epsilon_\ell = \frac{1}{2}\ell^2. \tag{3}$$

The factorizations (2) lead to the following intertwining relations

$$a_\ell^- \mathcal{H}_\ell = \mathcal{H}_{\ell+1} a_\ell^-, \quad a_{\ell-1}^+ \mathcal{H}_\ell = \mathcal{H}_{\ell-1} a_{\ell-1}^+, \tag{4}$$

meaning that the operators $a_\ell^\pm$ allow to construct the wave functions of $\mathcal{H}_{\ell-1}$ and $\mathcal{H}_{\ell+1}$ in terms of those of $\mathcal{H}_\ell$. Indeed if $\mathcal{H}_\ell \psi_\ell^n = E_\ell^n \psi_\ell^n$, then $a_{\ell-1}^+ \psi_\ell^n$ is the eigenfunction of $\mathcal{H}_\ell$ with eigenvalue $E_{\ell-1}^{n+1} = E_\ell^n$ and $a_\ell^- \psi_\ell^n$ is the eigenfunction of $\mathcal{H}_\ell$ with eigenvalue $E_{\ell+1}^{n-1} = E_\ell^n$. Thus

$$\psi_{\ell-1}^{n+1} \propto a_{\ell-1}^+ \psi_\ell^n, \qquad \psi_{\ell+1}^{n-1} \propto a_\ell^- \psi_\ell^n. \tag{5}$$

The ground state ψ_ℓ^0 may be constructed as a square integrable function which is annihilated by a_ℓ^-, *i.e.*,

$$a_\ell^- \psi_\ell^0 = 0. \tag{6}$$

In this way, the normalized ground state wave function is

$$\psi_\ell^0(x) = \sqrt{\frac{\ell!}{\pi^{1/2}\Gamma(\ell + \frac{1}{2})}}\,(\cos x)^\ell, \tag{7}$$

and corresponds to the eigenvalue $E_\ell^0 = \epsilon_\ell = \frac{1}{2}\ell^2$. The state ψ_ℓ^n can be obtained by subsequent applications of the corresponding operators a_ℓ^+, indeed

$$\psi_\ell^n(x) = N_\ell^n a_\ell^+ a_{\ell+1}^+ \cdots a_{\ell+n-1}^+ \psi_{\ell+n}^0, \tag{8}$$

with N_ℓ^n a normalization coefficient. Since the operators $a_\ell^\pm$ can be expressed in the form

$$a_\ell^\pm = \mp\frac{1}{\sqrt{2}}(\cos x)^{\mp\ell}\frac{d}{dx}(\cos x)^{\pm\ell}, \tag{9}$$

the equation (8) leads to

$$\psi_\ell^n(x) = \sqrt{\frac{2^{2\ell-1}(\ell+n)n!\left(\Gamma(\ell)\right)^2}{\pi\Gamma(2\ell+n)}}\,(\cos x)^\ell\, C_n^{(\ell)}(\sin x), \tag{10}$$

where $C_n^{(\ell)}(z)$ are the Gegenbauer polynomials. The corresponding spectrum, defined by the intertwining relations (4), is given by

$$Sp(\mathcal{H}_\ell) = \left\{E_\ell^n = E_{\ell+n}^0 = \frac{1}{2}\left(\ell+n\right)^2,\ \ell = 2, 3, \ldots,\ n = 0, 1, \ldots\right\}.$$

3. Trigonometric Pöschl-Teller potentials with position-dependent mass

In order to construct TPT potentials with PDM let us consider the Hamiltonian

$$H_\ell = \frac{1}{2} m^a p m^{2b} p m^a + V_\ell, \qquad 2a + 2b = -1, \tag{11}$$

where $m(x)$ is a given function and the potential $V_\ell(x)$ will be determined in such a way that H_ℓ has the same spectrum as the constant mass TPT potential. To this end, let us assume that H_ℓ admit the following factorizations (compare to (2))

$$H_\ell = A_\ell^+ A_\ell^- + \epsilon_\ell = A_{\ell-1}^- A_{\ell-1}^+ + \epsilon_{\ell-1}, \tag{12}$$

in terms of two first order operators of the form

$$A_\ell^+ = \frac{1}{\sqrt{2}} \left(-i m^a p m^b + W_\ell \right), \qquad A_\ell^- = \frac{1}{\sqrt{2}} \left(i m^b p m^a + W_\ell \right), \tag{13}$$

with W_ℓ a function of the position to be determined and $\epsilon_\ell = \frac{1}{2}\ell^2$. These factorizations imply the following set of equations from which W_ℓ and V_ℓ may be constructed

$$-DW_\ell + 4\left(a + \frac{1}{4}\right)(D \ln J) W_\ell + W_\ell^2 = 2\left(V_\ell - \epsilon_\ell\right) \tag{14}$$

$$DW_{\ell-1} + 4\left(a + \frac{1}{4}\right)(D \ln J) W_{\ell-1} + W_{\ell-1}^2 + 4\left(a + \frac{1}{4}\right)\left(D^2 \ln J\right) = 2\left(V_\ell - \epsilon_{\ell-1}\right), \tag{15}$$

with

$$J(x) = \sqrt{m(x)}, \quad D = \frac{1}{J}\frac{d}{dx}.$$

A first solution to the system (14)-(15) can be set in the form

$$W_\ell = \ell \tan \int_{x_0}^x J(r) dr - 2\left(a + \frac{1}{4}\right)(D \ln J) \tag{16}$$

$$V_\ell = \frac{1}{2}\ell\left(\ell - 1\right) \sec^2 \int_{x_0}^x J(r) dr - 2\left(a + \frac{1}{4}\right)^2 (D \ln J)^2 + \left(a + \frac{1}{4}\right)\left(D^2 \ln J\right), \tag{17}$$

with x_0 an integration constant with length units. As in the constant mass case, the factorization (12) imply the following intertwining relations

$$A_\ell^- H_\ell = H_{\ell+1} A_\ell^-, \qquad A_{\ell-1}^+ H_\ell = H_{\ell-1} A_{\ell-1}^+, \tag{18}$$

meaning that one can determine the eigenfunctions of $H_{\ell+1}$ and $H_{\ell-1}$ in terms of those of H_ℓ, and that $Sp(H_\ell) = Sp(\mathcal{H}_\ell)$. In fact, if $\Psi_\ell^n(x)$ denote the eigenfunctions of H_ℓ, *i.e.*,

$$H_\ell \Psi_\ell^n(x) = E_\ell^n \Psi_\ell^n(x), \tag{19}$$

then, the action of the operators $A_\ell^\pm$ on Ψ_ℓ^n is given by

$$A_\ell^- \Psi_\ell^n = \frac{1}{\sqrt{2}}\sqrt{n\left(2\ell + n\right)}\Psi_{\ell+1}^{n-1}, \qquad A_{\ell-1}^+ \Psi_\ell^n = \frac{1}{\sqrt{2}}\sqrt{(n+1)\left(2\ell + n - 1\right)}\Psi_{\ell-1}^{n+1}. \tag{20}$$

Accordingly, the general wave function of H_ℓ can be written in the form

$$\Psi_\ell^n(x) = \sqrt{\frac{2^n \Gamma(2\ell + n)}{n! \Gamma(2(\ell + n))}} A_\ell^+ A_{\ell+1}^+ \cdots A_{\ell+n-1}^+ \Psi_{\ell+n}^0, \tag{21}$$

where the state Ψ_ℓ^0 is the ground state of the Hamiltonian H_ℓ given by

$$A_\ell^- \Psi_\ell^0 = 0. \tag{22}$$

Now consider the following change of variable

$$y(x) = \int_{x_0}^{x} J(r)dr. \tag{23}$$

It is not difficult to show that [21]

$$A_\ell^\pm J^{1/2} = J^{1/2}\tilde{a}_\ell^\pm \tag{24}$$

with

$$\tilde{a}_\ell^\pm = \frac{1}{\sqrt{2}}\left(\pm D + \ell \tan\int_{x_0}^{x} J dr\right) = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dy} + \ell\tan y\right). \tag{25}$$

Note that the operators $\tilde{a}_\ell^\pm$ correspond to the factor operators of the constant mass TPT Hamiltonian in the $y-$configuration space. Expressing the ground state Ψ_ℓ^0 in the form

$$\Psi_\ell^0(x) = J^{1/2}(x)\psi_\ell^0(y(x)) \tag{26}$$

then

$$A_\ell^-\Psi_\ell^0 = A_\ell^- J^{1/2}\psi_\ell^0 = J^{1/2}\tilde{a}_\ell^-\psi_\ell^0 = 0.$$

It is clear, thus, that $\psi_\ell^0(y)$ is nothing but the ground state of the constant mass TPT potential evaluated in the variable $y(x)$ defined by the choice of the mass function. Hence

$$\Psi_\ell^0(x) = \sqrt{\frac{\ell!}{\pi^{1/2}\Gamma(\ell+1/2)}}J^{1/2}(x)\left(\cos\int_{x_0}^{x} J(r)dr\right)^\ell. \tag{27}$$

Now the expression (21), together with (26) and (24) allows the construction of the wave functions Ψ_ℓ^n as

$$\Psi_\ell^n(x) = J^{1/2}\psi_\ell^n(y(x)), \tag{28}$$

where $\psi_\ell^n(y)$ is given by

$$\psi_\ell^n(y) = \sqrt{\frac{2^n\Gamma(2\ell+n)}{n!\Gamma(2(\ell+n))}}\tilde{a}_\ell^+\tilde{a}_{\ell+1}^+\cdots\tilde{a}_{\ell+n-1}^+\psi_{\ell+n}^0(y), \tag{29}$$

from which it is clear that ψ_ℓ^n are the constant mass TPT potential wave functions evaluated in the variable y. Finally, we can immediately write the normalized wave functions for the TPT potential with position-dependent mass as

$$\Psi_\ell^n(x) = \sqrt{\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+2)}}\Gamma(\ell)J^{1/2}(x)\left(\cos\int_{x_0}^{x} J(r)dr\right)^\ell C_n^{(\ell)}\left(\sin\int_{x_0}^{x} J(r)dr\right). \tag{30}$$

This expression is consistent with the point canonical transformation discussed, *e.g.*, in [11].

In order to illustrate our results, we consider the mass [23]

$$m(x) = \frac{1-\beta(\lambda x)^2}{1-(\lambda x)^2}, \tag{31}$$

with $\beta < 0$ a dimensionless parameter and λ a constant with length units. This mass has been used in molecular dynamics to determine the inversion potential of the NH_3 molecule by means

of the density functional theory approach (see [4]). Observe that this function has singularities at the points $x = \pm 1/\lambda$, meaning that the choice of λ will be determinant in fixing the domain of V_ℓ. Note also that in the limit as $\lambda \to 0$ we recover the constant mass case with a unit mass $m = 1$. The expression (31) leads to

$$y(x) = \int_0^x \sqrt{\frac{1-\beta(\lambda r)^2}{1-(\lambda r)^2}}\, dr = \frac{1}{\lambda} E_{\text{int}}\left(\arcsin \lambda x, \beta^{1/2}\right), \tag{32}$$

with $E_{\text{int}}(\phi, k)$ the incomplete elliptic integral of second kind [23, 26]. The substitution of (31)-(32) in (17) and (30) leads us to the explicit expressions for the PDM TPT potential and its corresponding wave functions. In Figure 1 we show the potential and some wave functions for particular values of the parameters ℓ, a and β.

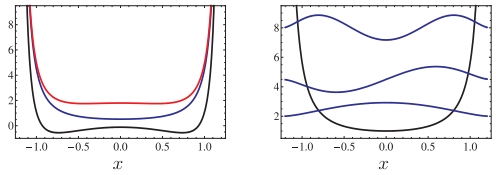

Figure 1. Left: Position-dependent mass TPT potentials $V_\ell(x)$ for the mass (31). In this plot $\ell = 2$, $\lambda = 0.7$, $\beta = -0.3$ and $a = -2$ (black), $a = -1$ (blue) and $a = 1$ (red). Right: Position-dependent mass TPT potential and its first three wave functions for $\ell = 2$, $\lambda = 0.7$, $\beta = -0.3$ and $a = -\frac{1}{4}$.

4. New position-dependent mass Hamiltonians with the TPT potential spectrum

It is well known that, in the constant mass case, the operators $a_\ell^\pm$ factorizing the TPT Hamiltonian (1) are not unique [14]. The same is true for the PDM Hamiltonian (11). Let us assume that

$$H_\ell = B_{\ell-1}^- B_{\ell-1}^+ + \epsilon_{\ell-1}, \tag{33}$$

with

$$B_\ell^+ = \frac{1}{\sqrt{2}}\left(-im^a p m^b + U_\ell\right), \qquad B_\ell^- = \frac{1}{\sqrt{2}}\left(im^b p m^a + U_\ell\right). \tag{34}$$

The function $U_\ell(x)$ must then fulfill the Riccati equation (15) that may be rewritten as

$$DU_\ell + 4\left(a+\frac{1}{4}\right)(D\ln J)\, U_\ell + U_\ell^2 + 4\left(a+\frac{1}{4}\right)\left(D^2 \ln J\right) = 2\left(V_{\ell+1} - \epsilon_\ell\right)$$

where $V_{\ell+1}$ is given by (17). In order to construct the solution let us propose, as usual [14, 27],

$$U_\ell(x) = W_\ell(x) + v(x),$$

where $v(x)$, in turn, must hold

$$Dv(x) + 2\ell v(x) \int_{x_0}^x J(r) dr + v^2(x) = 0,$$

leading to

$$v(x) = D\ln\left(\int_{x_0}^{y(x)} (\cos s)^{2\ell}\, ds + \gamma\right),$$

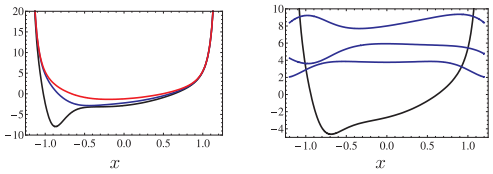

Figure 2. Left: Position-dependent mass supersymmetric TPT potentials $\widetilde{V}_\ell(x,\gamma)$ for the mass (31) with $\ell=2$, $\lambda=0.7$, $\beta=-0.3$, $a=-\frac{1}{4}$ and $\gamma=0.62$ (black), $\gamma=0.9$ (blue), $\gamma=1.2$ (red). Right: Position-dependent mass supersymmetric TPT potential and its first three wave functions for $\ell=2$, $\lambda=0.7$, $\beta=-0.3$ and $a=-\frac{1}{4}$ and $\gamma=0.7$.

with γ an integration constant with the constrain

$$|\gamma| > \max \int_{x_0}^{y(x)} \left(\cos \int_{x_0}^{r} J(s)ds\right)^{2\ell} J(r)dr$$

in order to avoid singularities.

The function U_ℓ can be readily written

$$U_\ell(x) = \ell \tan \int_{x_0}^{x} J(r)dr + D \ln \left[\int_{x_0}^{y(x)} \left(\cos \int_{x_0}^{r} J(s)ds\right)^{2\ell} J(r)dr + \gamma\right] - 2\left(a+\frac{1}{4}\right) D \ln J(x), \tag{35}$$

Following [14, 22], we may construct a new family of PDM Hamiltonians with the spectrum of the TPT one by means of a supersymmetric transformation

$$\widetilde{H}_\ell(\gamma) = B_\ell^+ B_\ell^- + \epsilon_\ell = \frac{1}{2} m^a p m^{2b} p m^a + \widetilde{V}_\ell(x,\gamma) \tag{36}$$

where

$$\widetilde{V}_\ell(x,\gamma) = V_\ell(x) - D^2 \ln \left[\int_{x_0}^{y(x)} \left(\cos \int_{x_0}^{r} J(s)ds\right)^{2\ell} J(r)dr + \gamma\right]. \tag{37}$$

The set of operators $\left\{B_\ell^\pm, H_\ell, \widetilde{H}_\ell\right\}$ fulfill the intertwining relations

$$B_\ell^- \widetilde{H}_\ell = H_{\ell+1} B_\ell^-, \quad B_\ell^+ H_{\ell+1} = \widetilde{H}_\ell B_\ell^+, \tag{38}$$

which means that it is possible to construct the wave functions of $\widetilde{H}_\ell$ in terms of those of H_ℓ. Indeed, if $\widetilde{H}_\ell \Theta_\ell^n = E_\ell^n \Theta_\ell^n$, the wave functions Θ_ℓ^n are related to the eigenfunctions of H_ℓ by

$$\Theta_\ell^n \propto B_\ell^+ \Psi_\ell^{n-1}, \qquad n = 1, 2, \ldots. \tag{39}$$

It may happen that the set $\{\Theta_\ell^n,\ \ n=1,2,\ldots\}$ do not span the whole space of states of $\widetilde{H}_\ell$. In that case there is an isolated eigenstate $\Theta_\ell^0(x)$, orthogonal to all Θ_ℓ^n $n=1,2,\ldots$, non connected to the set $\{\psi_\ell^n,\ \ n=0,1,\ldots\}$, given by

$$B_\ell^- \Theta_\ell^0 = 0, \tag{40}$$

and corresponding to the eigenvalue $E_\ell^0 = \epsilon_\ell$. Figure 2 shows some members of the family of potentials $\widetilde{V}_\ell(x,\gamma)$ (Left) and the first wave functions Θ_ℓ^n for $\gamma = 0.7$ (Right), using the mass (31) for some values of the parameters ℓ, a and β.

Concluding remarks

A PDM Hamiltonian having the spectrum of the constant mass TPT one was constructed by means of the factorization method. It was shown that there exist intertwining relations between the constant mass and the PDM factor operators. This fact allows to construct the corresponding PDM wave functions in terms of that of its constant mass counterpart. The method is consistent with the well known point canonical transformation approach. New isospectral Hamiltonians were generated by applying a Darboux transformation and their corresponding eigenfunctions were determined. This method can be generalized to constant mass Hamiltonians with different underlying algebraic structure. Results on the matter are in progress.

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References

[1] Richstone D O and Potter M D 1982 *Astrophys. J* **254** 451
[2] Cveticanin L 1993 *Trans. ASME J. Appl. Mech.* **60** 954
[3] Bastard G 1998 *Wave mechanics applied to semiconductor heterostructures* (Paris: Les Ulis Editions de Physique)
[4] Aquino N, Campoy G and Yee-Madeira H 1998 *Chem. Phys. Lett.* **296** 111
[5] Wolf K B 2004 *Geometric optics on phase space* (Berlin: Springer-Verlag)
[6] Cariñena J F, Perelomov A M, Rañada M and Santander M 2008 *J. Phys. A: Math. Theor.* **41** 085301
[7] Ballesteros A, Enciso A, Herranz J F, Ragnisco O and Riglioni D 2011 *Int. J. Theor. Phys.* **50** 2268
[8] Kordhad R 2011 *Physica* B **406** 3911-3916
[9] Mathiews P M and Lakshmanan M 1974 *Quart. Appl. Math.* **32** 215
[10] Lévy-Leblond J M 1995 *Phys. Rev.* A **52** 1845
[11] Alhaidari A D 2002 *Phys. Rev.* A **66** 042116
[12] Ganguly A, Kuru Ş, Negro J and Nieto L M 2006 *Phys. Lett.* A **360** 228
[13] Musielak Z E 2008 *J. Phys. A: Math. Theor.* **41** 055205
[14] Mielnik B 1984 *J. Math. Phys.* **25** 3387
[15] Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **37** 10007
[16] Kuru Ş and Negro J 2008 *Ann. Phys.* **323** 413
[17] Milanovic V and Ikonic Z 1999 *J. Phys. A: Math. Gen.* **32** 7001
[18] Plastino A R, Rigo A, Casas M, Gracias F and Plastino A 1999 *Phys. Rev.* A **60** 4318
[19] Quesne C 2006 *Ann. Phys.* **321** 1221
[20] Cruz y Cruz S, Negro J and Nieto L M 2007 *Phys. Lett.* A **369** 400
[21] Cruz y Cruz S and Rosas-Ortiz O 2009 *J. Phys. A: Math. Theor.* **42** 185205
[22] Cruz y Cruz S 2013 Factorization method and the position-dependent mass problem *Trends in Mathematics* (Springer Basel) pp 229-237
[23] Cruz y Cruz S and Rosas-Ortiz O 2013 *SIGMA* **9** 004
[24] Cruz y Cruz S and Santiago-Cruz C 2014 *J. Phys. Conf. Ser.* **538** 012006
[25] Kuru Ş and Negro J 2009 *Ann. Phys.* **324** 2548
[26] Gradshstein I S and Ryzhik I M 2000 *Tables of integrals, series and products* (Academic Press Inc. San Diego CA)
[27] Ince E L 1956 *Ordinary differential equations* (New York: Dover)