

# Propagation of Ultra-Short Higher-Order Solitons in a Photonic Crystal Fiber

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**Abstract.** We study the propagation of ultra-short optical pulses ( $\sim 10$  fs) traveling through a photonic crystal fiber. For this, we develop a realistic model of the fiber dispersion by numerical solution and a rational fit. Then, we model the laser pulses as higher-order solitons, and we show that they present rich dynamics along their propagation inside a fiber from where the steepest pulses are of great interest in the search of optical analogues of Hawking radiation.

## 1. Introduction

Optical solitons are localized waves that are formed by the opposite effects of nonlinearity and dispersion in optical fibers. The understanding of such pulses has come through theoretical studies and numerical simulations mainly from the nonlinear Schrödinger equation (NLSE) [1, 2], which has been successful at explaining some features of nonlinear pulse dynamics, including the emission of dispersive waves [3], soliton fission [4, 5] and supercontinuum generation [6]. Nevertheless, if we consider ultra-short pulses ( $\sim 10$ -100 fs in the visible spectrum), several higher-order nonlinear and dispersive effects come into play and the approximations used to derive the NLSE are not valid anymore [7]. Additionally, if we consider higher-order solitons, the nonlinear effects take further importance, specially the soliton fission, which immediately brakes down the pulses by emission of dispersive waves (DW), also known as non-solitonic radiation (NSR) or Cherenkov radiation [3].

On the other hand, photonic crystal fibers (PCFs) or microstructured fibers, have a solid core surrounded by a regular array of air holes in the propagation direction that changes the dispersion relation of the fiber core. These holes can be fabricated with certain degree of freedom to engineer this relation beyond material dispersion [7-9]. The PCFs made of silica have been very successful at shedding *supercontinuum* light that has found a wide range of applications [6].

Another feature of the PCFs is that the ability to support a wide range of frequencies opens the possibility of creating very steep pulses, close to the single-cycle regime. These pulses are useful in the search of optical analogues of Hawking radiation because the effective temperature and the emission rate depend crucially on the rate of change of the refractive index due to the Kerr effect, which ultimately depends on the steepness of the pulses. See, e.g., Refs. [10, 11].

In this work, we will study the propagation of ultra-short higher-order solutions inside PCFs. In Section 2 we will present the equations that model the pulse propagation. Then, in Section 3 we will study the PCFs and obtain a rational approximation for its dispersion. In Section 4 we will write the definitions concerning higher-order solitons. Then, in Section 5 we will be able to



present a study of the propagation by analyzing four slightly different pulses with three degrees of approximation in the dispersion. Finally, in Section 6 we will present our conclusions.

## 2. Model equations

We consider a single-mode fiber using the scalar approximation for pulses, i.e., the field structure in the radial direction is fixed and the signal  $E(z, t)$  only changes in the propagation direction  $z$ . We describe the system in the co-moving frame  $\tau = t - \beta_1 z$ , where  $\beta_1$  is the inverse of the group velocity  $v_g$ . The basic model for the pulse propagation in fibers is given by the NLSE:

$$i\partial_z \mathcal{E} + \frac{\beta_2}{2}(i\partial_\tau)^2 \mathcal{E} + \gamma |\mathcal{E}|^2 \mathcal{E} = 0, \quad (1)$$

where we have introduced the analytic signal  $\mathcal{E}$  of the electric field  $E = \text{Re}(\mathcal{E})$ ,  $\beta_2$  is the group velocity dispersion (GVD),  $\gamma$  is proportional to the third-order nonlinear susceptibility [1]. Eq. (1) is also known as cubic Schrödinger equation or Gross-Pitaevskii equation.

Nevertheless, we are interested in describing the pulse dynamics in a real fiber, in order to do that, we have to include higher-order dispersion terms, i.e., we have to replace the second term by the full-dispersion operator, usually written as a Taylor expansion as:

$$\mathcal{D}(i\partial_\tau) \mathcal{E} = \sum_{m=2}^M \frac{\beta_m}{m!} (i\partial_\tau)^m \mathcal{E}. \quad (2)$$

In this case, the pulse carrier frequency is shifted and not well-defined. Furthermore, it is convenient to consider the so-called *self-steepening effect* and the nonlocal terms or, more generally, we can include the dispersion inside the nonlinear term of the NLSE:

$$i\partial_z \mathcal{E} + \mathcal{D}(i\partial_\tau) \mathcal{E} + \gamma \frac{\beta_0}{\omega_0^2} \frac{(i\partial_\tau)^2}{\beta(i\partial_\tau)} |\mathcal{E}|^2 \mathcal{E} = 0. \quad (3)$$

We can study this equation in the Fourier-transformed space of frequencies  $\omega$ , i.e.,

$$i\partial_z \mathcal{E}_\omega + \mathcal{D}(\omega) \mathcal{E}_\omega + \gamma \frac{\beta_0}{\omega_0^2} \frac{\omega^2}{\beta(\omega)} (|\mathcal{E}|^2 \mathcal{E})_\omega = 0, \quad (4)$$

where  $\mathcal{E}_\omega$  and  $(|\mathcal{E}|^2 \mathcal{E})_\omega$  are signals in the Fourier space. It is not clear if we can ignore the effects of the stimulated Raman scattering (SRS) even if we are using ultra-short pulses because we are interested in the temporal shape of the pulse and not in its spectrum. Therefore, we included it in Fourier space by replacing:

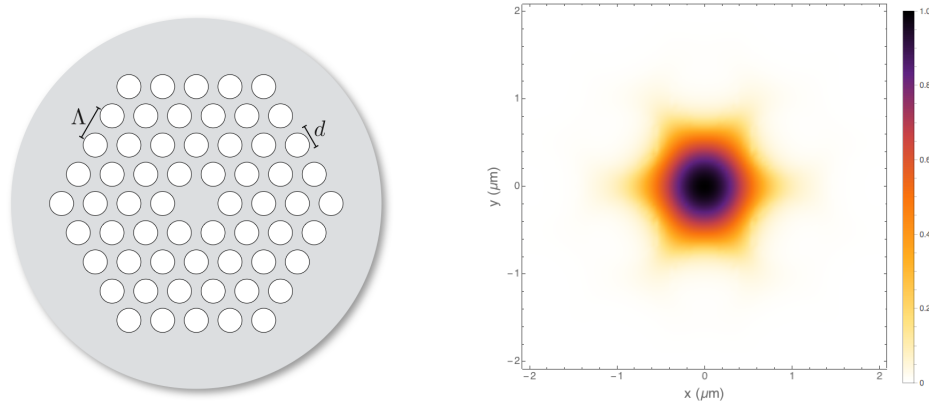
$$(|\mathcal{E}|^2 \mathcal{E})_\omega \rightarrow \mathcal{E}_\omega \int_{-\infty}^{\tau} R(\tau - \tau') |\mathcal{E}_\omega|^2 d\tau' = 0, \quad (5)$$

where  $R(\tau)$  is called the *response function*, which comes from delayed material vibrations caused by the optical field. In the case of silica fibers, an analytic approximation for this function has been recently developed for ultra-short pulses [1, 12]:

$$R(\tau) = (1 - f_2 - f_3) \delta(\tau) + f_2 \frac{\tau_1^2 + \tau_2^2}{\tau_1 \tau_2^2} \exp(-\tau/\tau_2) \sin(\tau/\tau_1) + f_3 \frac{2\tau_3 - \tau}{\tau_3^2} \exp(-\tau/\tau_3), \quad (6)$$

where  $f_2 = 0.194$  is the fractional contribution of the delayed Raman response to the nonlinear polarization and  $f_3 = 0.051$  is the fractional contribution of a boson peak,  $\tau_1 = 12.2$  fs,  $\tau_2 = 32$  fs and  $\tau_3 = 96$  fs are time parameters to fit  $R(\tau)$  with experimental data.

In the case of ultra-short pulses propagating in PCFs, the usual approximation for the dispersion given by Eq. (1) is not enough but some generalization is needed. Usually, the model given by Eqs. (4-6) works well for ultra-short pulses, but it depends critically on the shape of the dispersion  $\beta(\omega)$ . In the next section we will address this issue.



**Figure 1.** (Color online.) Left: Diagram of a typical PCF with hexagonal geometry. Gray represents the fiber material (silica) and the white circles are air holes. Right: The transversal mode in  $\lambda=800$  nm, most of the energy goes inside the solid core, air-holes affect more longer wavelengths.

### 3. Dispersion of the photonic crystal fiber

One difficulty in the study of ultra-short pulses in real materials is the usual approximation of its dispersion relation with few terms of its Taylor expansion around a central frequency. The reason is that in this case, the width of the involved frequencies is comparable with the central frequency and, therefore, this approximation becomes invalid. Furthermore, this cannot be solved by taking more terms of the dispersion relation (see Amiranishvili et al. [2, 13]). Here we take a different approach, i.e., instead of the polynomial expansion we consider a rational one as it was done in Ref. [13], but in our case, we approximate the *phase index*  $n_p(\omega)$  and our fit function is:

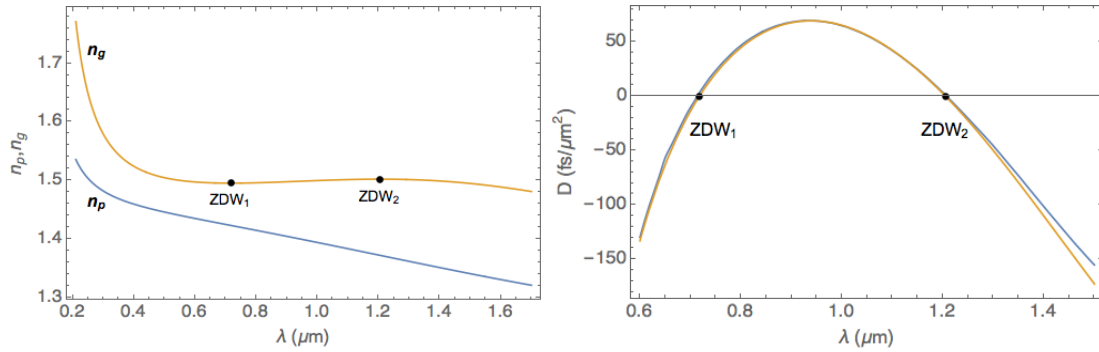
$$n_p(\lambda) = \frac{\lambda}{\lambda^2 - \lambda_r^2} \frac{\sum_{k=0}^4 a_k \lambda^k}{\sum_{k=0}^4 b_k \lambda^k}, \quad (7)$$

where  $\lambda_r = 116.2414$  nm is a known resonance for silica and the ten coefficients  $a_k$  and  $b_k$  are the fitting parameters, which should have the correct units in order to obtain a dimensionless quantity  $n_p$ . We choose this model because the resonances of  $n_p(\lambda)$  are known experimentally with high precision, which helps fixing the fit function in Eq. (7). Then we can obtain  $\beta(\omega)$  for Eq. (4) through

$$\beta(\omega) = \frac{\omega n_p(\omega)}{c}, \quad \omega = \frac{2\pi c}{\lambda}. \quad (8)$$

The PCFs that generate the shortest optical pulses in the visible spectrum are the solid-core silica fibers with hexagonal geometry, see Fig. 1. For this work, we studied a commercial fiber from NKT Photonics: NL-1.5-590 [14], which is advertised to have zero dispersion wavelengths (ZDWs) around 590 nm and 1410 nm.

We obtain a numerical solution of  $n_p(\lambda)$  through finite-element method using a FemSIM, a commercial software [15]. We obtain a very accurate result based on the material dispersion for air and silica (which are known with very high precision) and the two geometrical parameters of a PCF: the pitch  $\Lambda$  and the diameter  $d$  of the air holes (see Fig. 1). These two parameters were obtained through an electronic microscope photography of the fiber as  $\Lambda = 1079$  nm,  $d = 763$  nm. In the right-hand side of Fig. 1 we plot the amplitude of the electric field of the fundamental mode with  $\lambda = 800$  nm, where we can see that most of its energy travels inside the core, as needed in order to use the scalar approximation in Section 2. For longer wavelengths, the amount of energy traveling in the region with air holes increases, therefore, the dispersion is different from the material dispersion of silica in the infrared region. In Fig. 2 we plot the phase and group



**Figure 2.** (Color online.) Left: Phase (blue) and group (orange) indexes obtained by our numerical solution using Eq. (7). Right: Dispersion function  $D$  from Eq. (9) (orange) compared with the one a function reconstructed from the data in Ref. [8] (blue). The ZDWs for our solution are marked in both plots.

indexes that we obtained numerically and the related function  $D$ , defined by

$$D(\lambda) = \frac{\partial^2 \beta}{\partial \lambda \partial \omega}, \quad (9)$$

(not to be confused with the full dispersion  $\mathcal{D}$  in Eq. (2)). The ZDWs of our fit function are in 713 nm and 1207 nm, different from the advertised values. Between these two points the fiber presents anomalous GVD. The full dispersion data is not available from the company but it was also found by numerical means in Ref. [8], we compare it with our result in Fig. 2.

#### 4. Higher order solitons

For a pulse in the anomalous GVD region of the spectrum, the nonlinearity can counter the dispersive effects, resulting in optical solitons or *solitary waves*. For the NLSE with second-order dispersion given in Eq. (1) there are localized solutions that either do not change (fundamental soliton) or change periodically (higher-order solitons) along the fiber. They are given by:

$$\mathcal{E}(\tau) = N \mathcal{E}_{\text{FS}} \text{sech}(\tau/\tau_0) \exp(-i\omega_0 \tau) \quad (10)$$

where  $\mathcal{E}_{\text{FS}}$  is the amplitude of the fundamental soliton and  $N$  is known as the *soliton number*:

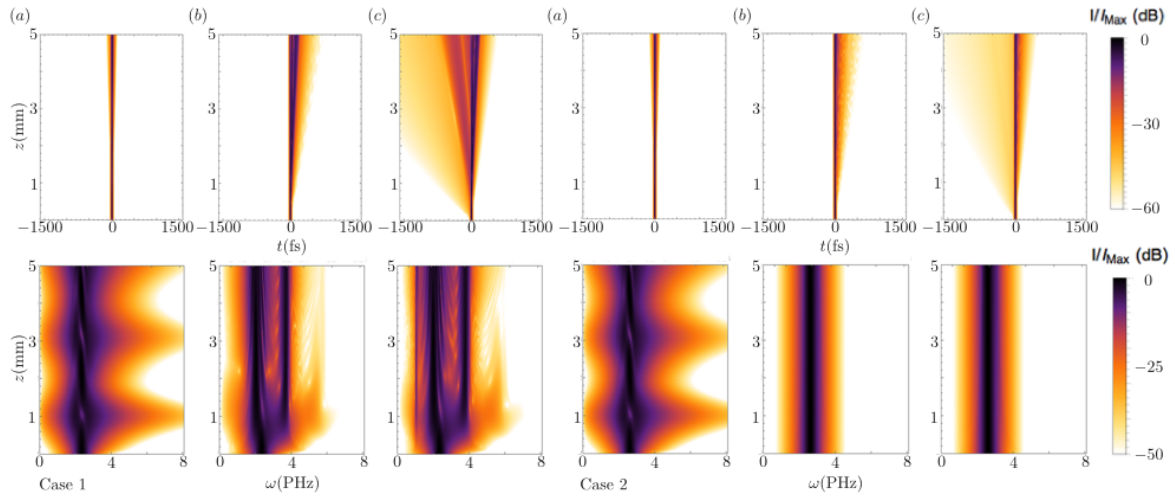
$$\mathcal{E}_{\text{FS}} = \sqrt{\frac{1}{\gamma L_D}}, \quad N^2 = \frac{L_D}{L_{\text{NL}}}, \quad L_D = \frac{\tau_0}{|\beta_2|}, \quad L_{\text{NL}} = \frac{1}{\gamma P_0}, \quad \tau_0 = \frac{\tau_{\text{FWHM}}}{2 \log(1 + \sqrt{2})}. \quad (11)$$

$L_D$  and  $L_{\text{NL}}$  are the dispersion and nonlinear lengths and  $\tau_{\text{FWHM}}$  is the time for full-width at half-maximum (FWHM), the most common way to measure time duration of optical pulses.

For  $N = 1$  we have the fundamental soliton, whose shape does not change over propagation if we approximate dispersion given only by  $\beta_2$  factor. Eq. (10) for  $N = 1$  fixes a relation between duration  $\tau_0$  and amplitude  $\sqrt{P_0}$ . Keeping  $\tau_0$  fixed and increasing  $\sqrt{P_0}$  leads to  $N > 1$ , which are the higher-order solitons, whose propagation is not constant but periodic. Usually, the integer part of  $N$  takes a relevant part on the actual form of its propagation, i.e.,  $\tilde{N}$  if we take  $N = \tilde{N} + \epsilon$ , with  $\epsilon < 1/2$ .

#### 5. Ultra-short pulse propagation

Now we will use the model equations and the dispersion relation we have developed to explain through numerical simulations the dynamics of ultra-short higher-order solitons in a PCF. All



**Figure 3.** (Color online.) Left. Case 1, or base case, with  $\lambda = 800$  nm,  $N = 2$  and  $\tau_{FWHM} = 7$  fs. Right. Case 2, same as Case 1, except that now  $\lambda = 720$  nm, i.e., very close to the ZDW of the dispersion.

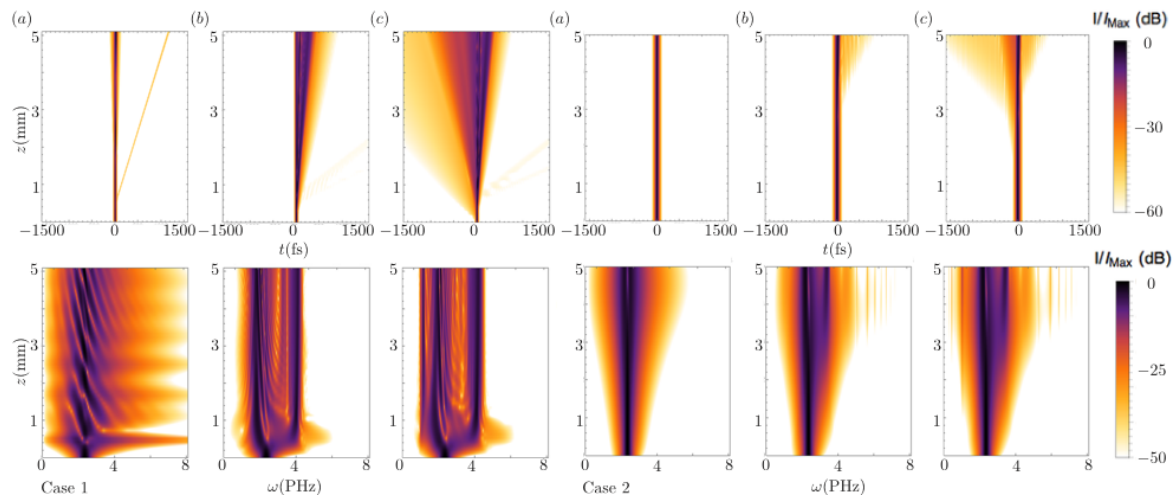
the simulations are done with a Dormand-Prince method, an adaptive version of the fourth-order Runge-Kutta method that contains an error monitoring function [16]. In addition, we choose the propagation step  $dz$  by checking norm conservation.

In our first studies, we determined that the SRS does not change the temporal shape, but for completeness we include it in our simulations. Furthermore, we found that the most important effects are given by higher-order terms in the dispersion. To study this, we will use three degrees of approximation in the dispersion: (a) second-order, (b) third-order and (c) rational approximation. For each of them, we will simulate four different pulses:

- Case 1. It is the base case, a higher-order soliton with  $\lambda = 800$  nm,  $N = 2$  and  $\tau_{FWHM} = 7$  fs, i.e., well inside the anomalous GVD of the dispersion. With second-order dispersion we see the usual dynamics from higher-order solitons and the dispersive effects do not enter. Already for the third-order dispersion some dispersive waves are shed around 500 nm. For the higher-order dispersion the pulse breaks further in time. See Fig. 3.
- Case 2. Similar to the base case except that  $\lambda = 720$  nm. Because we are very close to one ZDW, for the third-order and the full-dispersion there is no much dynamics except the usual broadening in time. Also, the effect of the higher-order dispersion is diminished and it does not allow steeper pulses. See Fig. 3.
- Case 3. Like the base case but with  $N = 3$ , the dynamics is much faster, as it is shown in shorter periods for the second-order dispersion and the faster break-up and shed of dispersive waves with the higher-order dispersions. See Fig. 4.
- Case 4. As the base case with  $\tau_{FWHM} = 15$  fs, here the dynamics is slowed down, we have a slow broadening of the pulse. All of the dispersions give similar results except of few dispersive waves from the third- and higher-order dispersions. See Fig. 4.

## 6. Conclusions

In this work we studied the pulse dynamics of ultra-short higher-order solitons traveling through a PCF. We obtained that the third-order dispersion is essential to obtain the correct qualitative result in the spectrum. The inclusion of further terms modifies the pulse in time and accelerates its broadening. In most cases we found an initial steepening in one side of the soliton followed by the known soliton fission dynamics. On the case of higher soliton number, the pulses can



**Figure 4.** (Color online.) Left. Case 3, same as Case 1, but now the initial pulse has  $N = 3$ , i.e., larger energy. Right. Case 4, same as Case 1 with the exception of that  $\tau_{\text{FWHM}} = 15$  fs.

be steeper but for a shorter propagation distance. The case of longer pulses starts with a disadvantage and even though there is some steepening, it does not surpass other cases. Finally, we obtained pulses steeper than the initial ones in very short propagation distances, which can be useful in the search of an analogue of Hawking radiation [10, 11].

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