Approximate yet Confident Solution for a Parametric Oscillator in a Kerr Medium

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Abstract. We study the temporal evolution of a coherent state under the action of a parametric oscillator immersed in a nonlinear Kerr-like medium. Applying a self consistent method we obtain an approximate time evolution operator. This operator behaves like a squeezing operator due to the temporal dependence of the oscillator's frequency. We analyze Mandel's parameter, the presence of squeezing in the field quadratures and the generation of photons from the vacuum state.

1. Introduction

Coherent states were introduced by Schrödinger in 1926 since the early stages of quantum mechanics [1] in connection with the classical states of the quantum harmonic oscillator. These quantum states are characterized by the fact that the trajectory of the center of the coherent wave packet evolves in time in the same way as a classical harmonic oscillator and its dispersion takes the minimum value allowed by Heisenberg's principle. Much later, in 1963, Glauber introduced the *field* coherent states, that is, coherent states for the electromagnetic field, as right eigenstates of the annihilation operator. These states have played an important role in quantum optics [2]. The field coherent states can be obtained from any one of the three mathematical definitions: (i) as the right hand eigenstates of the boson annihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with α a complex number, (ii) as those states obtained by application of the displacement operator upon the vacuum state of the harmonic oscillator $D(\alpha)|0\rangle = |\alpha\rangle$ with $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, and (iii) as the quantum states with a minimum uncertainty product $(\Delta p)(\Delta q) = 1/2$ with $\Delta q = \Delta p$. The coherent states obtained from any one of these definitions are identical when one makes use of the harmonic oscillator algebra. Subject to a linear interaction a coherent state evolves into a new coherent state, that is, they show temporal stability [3, 4].

In the presence of a nonlinear interaction, field coherent states evolve into non classical states. This can be achieved experimentally by passing a coherent state through a Kerr medium resulting in the appearance of distinguishable macroscopic superpositions of coherent states, the so called cat states [5, 6].

The parametric harmonic oscillator, namely a harmonic oscillator with a time dependent frequency, has been studied from several points of view: using the method of adiabatic invariants [7–9], super symmetric quantum mechanics [10], algebraic methods [11], and different

approximation methods [12] among others. A particularly relevant realization of the parametric oscillator is cavity quantum electrodynamics (CQED) where the frequency of a given field mode in the cavity can change in time due to the motion of the cavity walls or to changes in the dielectric function of the medium [13]. For instance, Wineland et. al. [14] analyzed both theoretically and experimentally the loss of coherence caused by fluctuations in the trap parameters and in the amplitude and frequency of the laser beams, heating due to collisions with background gas, internal state decoherence due to radiative decay, and coupling to spectator levels.

In this work we consider a nonlinear system corresponding to a single mode field propagating in a Kerr-like medium immersed in a cavity with a time dependent frequency. In Section 2 we write the Hamiltonian and construct its time evolution operator. In section 3 we follow the evolution of coherent states under the nonlinear Hamiltonian and analyze some of their statistical properties like the Mandel parameter, the average value of the number operator and the dispersion of the quadratures.

2. Parametric oscillator in a Kerr medium

Consider a parametric harmonic oscillator immersed in a Kerr-like medium. Its Hamiltonian is given by:

$$H(t) = \frac{1}{2}[p^2 + \Omega^2(t)q^2] + H_{Kerr} \tag{1}$$

where $\Omega(t)$ is an explicit time dependent frequency and H_{Kerr} has to do with the Kerr-like medium. We can define the usual annihilation, creation and number operators as:

$$\hat{a} = \frac{1}{\sqrt{2\Omega_0}}(\Omega_0 q + ip), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\Omega_0}}(\Omega_0 q - ip), \quad \hat{n} = \hat{a}^\dagger \hat{a}. \tag{2}$$

where we have set $\hbar = 1$ and we can write the Kerr medium [15] as $H_{Kerr} = \chi \hat{n}^2$, where χ is proportional to a third-order nonlinear susceptibility $\chi^{(3)}$. To be specific, in what follows we will choose $\Omega(t) = \Omega_0[1 + 2\kappa \cos(2\Omega_0 t)]$ [16] with κ and χ small parameters. The Hamiltonian can be written in terms of $\hat{a}^\dagger$, $\hat{a}$ and $\hat{n}$ as [17]:

$$H(t) = \Omega_0(\hat{n} + 1/2) + \chi \hat{n}^2 + g(t)(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{n} + 1) \tag{3}$$

and $g(t) = \Omega_0 \kappa \cos(2\Omega_0 t)(1 + \kappa \cos(2\Omega_0 t))$.

The time evolution operator corresponding to the non linear time independent part of the Hamiltonian is given by:

$$U_0 = \exp\left(-i\Omega_0 t(\hat{n} + 1/2) - it\chi \hat{n}^2\right) \tag{4}$$

and we can write the time dependent Hamiltonian in the interaction picture as

$$H_I(t) = g(t)\left(e^{-2i\Omega(\hat{n})t}\hat{a}^2 + \hat{a}^{\dagger 2}e^{2i\Omega(\hat{n})t} + 2\hat{n} + 1\right) \tag{5}$$

where $\Omega(\hat{n}) = \Omega_0 + 2\chi(1 + \hat{n})$ is a function of the number operator.

The set $\{\hat{a}^{\dagger 2}, \hat{n}, \hat{a}^2, 1\}$ forms the basis of a Lie algebra closed under commutation. However, the Hamiltonian $H_I(t)$ includes terms where the number operator appears in the exponent. Invoking the mean field approximation [18] we make the replacement $\exp[\pm 2i\Omega(\hat{n})t]$ by $\langle \exp[\pm 2i\Omega(\hat{n})t]\rangle$ obtaining:

$$H_I(t) = g(t)\left(\hat{a}^2\langle e^{-2i\Omega(\hat{n})t}\rangle + \hat{a}^{\dagger 2}\langle e^{2i\Omega(\hat{n})t}\rangle + 2\hat{n} + 1\right) \tag{6}$$

where the expectation value is taken with respect to the coherent state in the interaction picture representation $|\alpha_0\rangle_I = U_I|\alpha_0\rangle$. The resulting approximate Hamiltonian is similar to that of a

degenerate parametric amplifier where a non linear medium is pumped by a strong laser inducing the emission and absorption of photon pairs [19].

With this simplification, the Hamiltonian in the interaction picture becomes an element of the Lie algebra with time dependent coefficients and the corresponding time evolution operator may be written in the product form [20]

$$H_I(t) = \sum_{n=1}^{4} f_n(t) X_n, \quad U_I(t) = \prod_{n=1}^{4} e^{\alpha_n(t) X_n}. \tag{7}$$

with initial conditions $\alpha_n(t_0) = 0$, and we have chosen the ordering $X_1 = \hat{a}^{\dagger 2}$, $X_2 = \hat{n}$, $X_3 = \hat{a}^2$ and $X_4 = 1$.

The average takes the form:

$$\langle e^{\pm 2i\Omega(\hat{n})t} \rangle = e^{\pm 2i(\Omega_0 + 2\chi)t} \langle \alpha_0 | U_I^{\dagger} e^{\pm 4i\chi t\hat{n}} U_I | \alpha_0 \rangle. \tag{8}$$

Using

$$\begin{aligned} U_I^{\dagger} \hat{n} U_I &= [2\alpha_1 e^{-2\alpha_2}]\hat{a}^{\dagger 2} + [1 - 8\alpha_1\alpha_3 e^{-2\alpha_2}]\hat{n} + [2\alpha_3(4\alpha_1\alpha_3 e^{-2\alpha_2} - 1)]\hat{a}^2 - [4\alpha_1\alpha_3 e^{-2\alpha_2}] \\ &\equiv a_1 \hat{a}^{\dagger 2} + a_2 \hat{n} + a_3 \hat{a}^2 + a_4 \end{aligned} \tag{9}$$

we obtain the following expression for the average:

$$\langle e^{\pm 2i\Omega(\hat{n})t} \rangle = e^{\pm 2i(\Omega_0 + 2\chi)t} \langle \alpha_0 | e^{\pm 4i\chi t(a_1 \hat{a}^{\dagger 2} + a_2 \hat{n} + a_3 \hat{a}^2 + a_4)} | \alpha_0 \rangle. \tag{10}$$

Here the functions a_i are complex functions of time, and we now face the problem of writing the exponential in a product form.

If $\kappa \ll 1$ then $a_2 \to 1$ and $a_i \to 0$ for $i = 1, 3, 4$. In that case the average is given by:

$$e^{\pm 2i(\Omega_0 + 2\chi)t} \langle \alpha_0 | e^{\pm 4i\chi t\hat{n}} | \alpha_0 \rangle = e^{\pm 2i(\Omega_0 + 2\chi)t} \exp[|\alpha_0|^2 (e^{\pm 4i\chi t} - 1)]. \tag{11}$$

If we take the average between initial coherent states $|\alpha_0\rangle$ instead of the evolved coherent states $|\alpha_0\rangle_I$ we get the same result as that given by Eq. 11 (see [17]). For the general case we make use of the fact that the set $\{\hat{a}^{\dagger 2}, \hat{a}^2, \hat{n}, 1\}$ is closed under commutation and we write [21, 22]:

$$\exp\left[\lambda \sum_{n=1}^{N} \alpha_n \hat{O}_n\right] = \exp[\phi_1(\lambda)\hat{O}_1] \cdots \exp[\phi_N(\lambda)\hat{O}_N] \tag{12}$$

where the set of functions $\{\phi_i(\lambda)\}$ must be determined under the constraint $\phi_i(\lambda = 0) = 0$. Consider first the case with a positive sign in the exponent. Then we write:

$$\langle e^{2i\Omega(\hat{n})t} \rangle = e^{2i(\Omega_0 + 2\chi)t} \langle \alpha_0 | e^{4i\chi t(a_1 \hat{a}^{\dagger 2} + a_2 \hat{n} + a_3 \hat{a}^2 + a_4)} | \alpha_0 \rangle \tag{13}$$

using Eq. 12 we write

$$e^{\lambda[\beta_1 \hat{a}^{\dagger 2} + \beta_2 \hat{n} + \beta_3 \hat{a}^2 + \beta_4]} = e^{\phi_1(\lambda)\hat{a}^{\dagger 2}} e^{\phi_2(\lambda)\hat{n}} e^{\phi_3(\lambda)\hat{a}^2} e^{\phi_4(\lambda)} \tag{14}$$

where we have renamed $\beta_i = 4i\chi t a_i$. From Eq. 14 we obtain the following set of ordinary differential equations:

$$\frac{d\phi_1}{d\lambda} = \beta_1 + 2\beta_2\phi_1 + 4\beta_3\phi_1^2, \qquad \frac{d\phi_2}{d\lambda} = \beta_2 + 4\beta_3\phi_1$$

$$\frac{d\phi_3}{d\lambda} = \beta_3 e^{2\phi_2}, \qquad \frac{d\phi_4}{d\lambda} = \beta_4 + 2\beta_3\phi_1.$$

This set of equations can be solved analytically (see Appendix A) and as a result we get:

$$l\phi_1(\lambda) = \frac{\beta_1 \sinh(2\Sigma\lambda)}{2\Sigma \cosh(2\Sigma\lambda) - \beta_2 \sinh(2\Sigma\lambda)},$$

$$\phi_2(\lambda) = -\ln[\cosh(2\Sigma\lambda) - \frac{\beta_2}{2\Sigma} \sinh(2\Sigma\lambda)]$$

$$\phi_3(\lambda) = \frac{\beta_3 \sinh(2\Sigma\lambda)}{2\Sigma \cosh(2\Sigma\lambda) - \beta_2 \sinh(2\Sigma\lambda)},$$

$$\phi_4(\lambda) = (\beta_4 - \beta_2/2)\lambda - \frac{1}{2} \ln[\cosh(2\Sigma\lambda) - \frac{\beta_2}{2\Sigma} \sinh(2\Sigma\lambda)]$$

where $\Sigma = \sqrt{\beta_2^2/4 - \beta_1\beta_3}$. Once we have found the functions $\phi_i(\lambda)$ we can take the average given in Eq. 10, getting:

$$\langle e^{2i\Omega(\hat{n})t} \rangle = e^{2i(\Omega_0 + 2\chi)t} e^{\phi_4 + \phi_3\alpha_0^2 + \phi_1\alpha_0^{*2}} e^{|\alpha_0|^2(e^{\phi_2} - 1)} \tag{15}$$

and

$$\langle e^{-2i\Omega(\hat{n})t} \rangle = e^{-2i(\Omega_0 + 2\chi)t} e^{\phi_4^* + \phi_3^*\alpha_0^{*2} + \phi_1^*\alpha_0^2} e^{|\alpha_0|^2(e^{\phi_2^*} - 1)} \tag{16}$$

where the functions $\phi_i(\lambda)$ must be evaluated at $\lambda = 1$.

The complex, time dependent functions $\alpha_n(t)$ needed to construct the time evolution operator in the interaction picture are likewise obtained from the following set of coupled, nonlinear, ordinary differential equations obtained after substitution of Eq. 7 into Schrödinger's equation

$$\frac{d\alpha_1}{dt} = -i(f_1 + 2\alpha_1 f_2 + 4\alpha_1^2 f_3), \qquad \frac{d\alpha_2}{dt} = -i(f_2 + 4\alpha_1 f_3)$$

$$\frac{d\alpha_3}{dt} = -if_3 e^{2\alpha_2}, \qquad \frac{d\alpha_4}{dt} = -i(f_4 + 2\alpha_1 f_3)$$

where the functions f_n correspond to the coefficients in the mean field Hamiltonian, namely:

$$f_1 = g(t)\langle e^{2i\Omega(\hat{n})t} \rangle, \quad f_2 = 2g(t), \quad f_3 = g(t)\langle e^{-2i\Omega(\hat{n})t} \rangle, \quad f_4 = g(t).$$

At the initial time t_0, the complex functions $\alpha_n(t_0) = 0$ so that we can construct the functions $f_n(t_0)$ corresponding to the mean field Hamiltonian. With these we integrate for a time Δt and obtain new values for the functions $\alpha_n(t = \Delta t)$. At each step we produce a new value for the functions $f_n(t)$ which is then used to compute the new $\alpha_n(t)$. In this way we couple the differential equations with the expectation value of the exponential of the number operator in a self consistent way [18].

If we take the average value of the number operator (see Eq. 9) between number states $|n_i\rangle$ we obtain:

$$\langle n_i|\hat{n}(t)|n_i\rangle = [1 - 8\alpha_1\alpha_3 e^{-2\alpha_2}]n_i - [4\alpha_1\alpha_3 e^{-2\alpha_2}]. \tag{17}$$

If instead, we take the average value using coherent states $|\alpha\rangle$, then the corresponding expectation value of the number operator is

$$\langle \alpha|\hat{n}(t)|\alpha\rangle = [1 - 8\alpha_1\alpha_3 e^{-2\alpha_2}]|\alpha|^2 - [4\alpha_1\alpha_3 e^{-2\alpha_2}] + [2\alpha_3(4\alpha_1\alpha_3 e^{-2\alpha_2} - 1)]\alpha^2 + [2\alpha_1 e^{-2\alpha_2}]\alpha^{*2}. \tag{18}$$

Notice that the behavior of $\langle \hat{n}(t) \rangle$ is different depending upon the kind of averaging that is chosen. In what follows we will use the averaging with respect to coherent states.

3. Statistical properties

3.1. Mandel parameter

The Mandel parameter Q gives information about the nature of the photon statistics of any state. It is defined as:

$$Q = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle}. \tag{19}$$

For a state with Q in the range $0 \leq Q < 1$ the statistics are sub-Poissonian and for $Q > 1$, super-Poissonian. A field coherent state has $Q = 1$. States with sub-Poissonian distribution are non classical states [23]. When $\kappa = 0$ the time evolution operator is that given by Eq. 4, that is: $U = U_0$ and $\langle \alpha | U_0^\dagger \hat{n} U_0 | \alpha \rangle = \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2$ and $\langle \alpha | U_0^\dagger \hat{n}^2 U_0 | \alpha \rangle = |\alpha|^4 + |\alpha|^2$ so that the Mandel parameter is equal to one regardless of the value of χ.

In figure 1 we plot the temporal evolution of the Mandel parameter for fixed α with average photon number $\langle \hat{n} \rangle = 18$ and potential parameters $\kappa = 0.05$, $\chi = 0.0$ and $\Omega_0 = 1$ corresponding to a parametric oscillator. In this case the Mandel parameter starts at one as corresponds to a usual coherent state. When $\alpha = (3, 3)$ (green line) the Mandel parameter takes values smaller than one at the beginning of the evolution corresponding to a non classical state and remains sub-Poissonian for some time until it becomes larger than one and maintains an increasing behavior as a function of time. The results for real $\alpha = (\sqrt{18}, 0)$ are shown in purple. As in the previous case the Mandel parameter starts at one as corresponds to a usual coherent state but in contrast with that case here it is always an increasing function of time taking values corresponding to a classical state. The difference is due to the fact that for the evaluation of the Mandel parameter we have taken the averages from Eq. 18 where the importance of the nature of α (it being real or complex) is evident. The undulations are due to the temporal dependence of the oscillator's frequency. Also shown in the figure are the cases with $\kappa = 0.7$ and $\chi = 0.25$ corresponding to a nonlinear parametric oscillator. We present, in blue, the case for $\alpha = (\sqrt{18}, 0)$ and in red the case for $\alpha = (3, 3)$. The general conduct is similar to that of the previous cases, however the undulations we mentioned above are washed out and we see instead the appearance of strong oscillations between intervals where the Mandel parameter remains practically constant. It is precisely when these oscillations appear that the Mandel parameter changes. Notice that for non negligible values of χ, the nonlinear term H_{Kerr} in the Hamiltonian becomes important in the evolution of the system and is responsible for a slower change in the average value of the number operator.
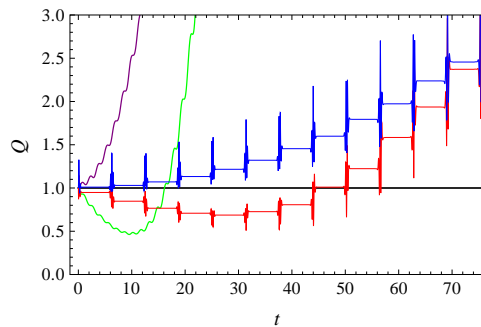

Figure 1. Temporal evolution of the Mandel parameter for $\alpha = \sqrt{18}$ (purple and blue), $\alpha = (3, 3)$ (green and red), corresponding to an initial average number $\langle \hat{n} \rangle = 18$ and parameters $\kappa = 0.05$, $\chi = 0$ (purple and green), $\kappa = 0.7$, $\chi = 0.25$ (blue and red). In all cases $\Omega_0 = 1$ and the initial condition used was $\alpha_0 = \sqrt{18}$.

3.2. Average value of the number operator for the vacuum

If we evaluate the average number of photons with respect to the vacuum state we find an exponential grow when $\chi = 0$ and some undulations due to the temporal dependence of the frequency (see figure 2, purple); this effect is related to the dynamical Casimir effect [24] namely the generation of real photons starting from the vacuum. When the nonlinear term $\chi \neq 0$ the average number of photons with respect to the vacuum state decreases rapidly (see figure 2, black, green and blue) with increasing oscillations as the value of χ increases. This conduct was reported in [25] where they studied the interaction between two two-level atoms in a cavity with time dependent frequency. The similarity with their results is due to the fact that the atoms in the cavity emulate an intensity dependent refraction index.
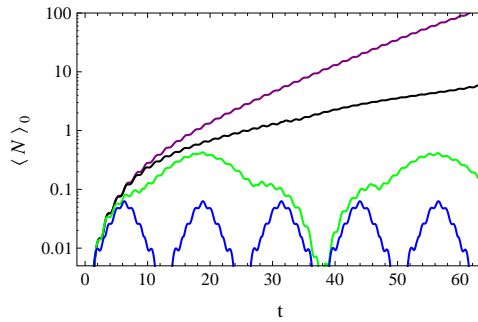

Figure 2. Temporal evolution of the average value of the number operator with respect to the vacuum state. Parameters used: $\chi = 0$ (purple), $\chi = 0.025$ (black), $\chi = 0.085$ (green), $\chi = 0.25$ (blue). In all cases $\kappa = 0.05$ and $\Omega_0 = 1$.

3.3. Squeezing and dispersions

In order to analyze the temporal behavior of different observables let us consider the temporal evolution of the creation-annihilation operators. The creation operator in the Heisenberg picture is given by

$$\hat{a}^\dagger(t) = U_I^\dagger U_0^\dagger \hat{a}^\dagger U_0 U_I \tag{20}$$

Applying the first transformation we get:

$$U_0^\dagger \hat{a}^\dagger U_0 = e^{i\chi t\hat{n}^2} e^{i\Omega_0 t\hat{n}} \hat{a}^\dagger e^{-i\Omega_0 t\hat{n}} e^{-i\chi t\hat{n}^2} = \hat{a}^\dagger e^{i\chi t(2\hat{n}+1)} e^{i\Omega_0 t} \tag{21}$$

The number operator appears in the exponential again. To be consistent with the approximation used to get the interaction picture Hamiltonian (Eq. 6) we replace the exponential by its average value, and apply the second transformation (only to the operator $\hat{a}^\dagger$) obtaining:

$$\hat{a}^\dagger(t) = U^\dagger \hat{a}^\dagger U = e^{i(\Omega_0+\chi)t} \langle e^{2i\chi t\hat{n}} \rangle U_I^\dagger \hat{a}^\dagger U_I = e^{i(\Omega_0+\chi)t} \langle e^{2i\chi t\hat{n}} \rangle e^{-\alpha_2} \left[\hat{a}^\dagger - 2\alpha_3 \hat{a} \right]. \tag{22}$$

Proceeding in a similar form for the annihilation operator we get:

$$\hat{a}(t) = \left[\hat{a}(e^{\alpha_2} - 4\alpha_1\alpha_3 e^{-\alpha_2}) + 2\alpha_1 e^{-\alpha_2} \hat{a}^\dagger \right] e^{-i(\Omega_0+\chi)t} \langle e^{-2i\chi t\hat{n}} \rangle. \tag{23}$$

The above expressions can be written as:

$$\begin{pmatrix} \hat{a}^\dagger(t) \\ \hat{a}(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} \hat{a}^\dagger \\ \hat{a} \end{pmatrix}. \tag{24}$$

The complex, time dependent coefficients $a_{ij}(t)$ fulfill the conditions $a_{11}(t) = a_{22}^*(t)$ and $a_{12}(t) = a_{21}^*(t)$ (these have been checked numerically).

The quadratures q and p at time t are given by:

$$q(t) = \frac{1}{\sqrt{2}}\left[\hat{a}(t) + \hat{a}^\dagger(t)\right], \quad p(t) = \frac{\mathrm{i}}{\sqrt{2}}\left[\hat{a}^\dagger(t) - \hat{a}(t)\right] \tag{25}$$

Using the expressions given above we obtain

$$\langle\alpha|q(t)|\alpha\rangle = \frac{1}{\sqrt{2}}\left[\alpha^*(a_{11}(t) + a_{21}(t)) + \alpha(a_{12}(t) + a_{22}(t))\right] \tag{26}$$

and

$$\langle\alpha|p(t)|\alpha\rangle = \frac{\mathrm{i}}{\sqrt{2}}\left[\alpha^*(a_{11}(t) - a_{21}(t)) + \alpha(a_{12}(t) - a_{22}(t))\right]. \tag{27}$$

And for $q(t)^2$, $p(t)^2$ we get:

$$\langle\alpha|q(t)^2|\alpha\rangle = \frac{1}{2}[(a_{11}(t) + a_{21}(t))^2\alpha^{*2} + (a_{12}(t) + a_{22}(t))^2\alpha^2 +$$
$$2(a_{11}(t) + a_{21}(t))(a_{12}(t) + a_{22}(t))|\alpha|^2 + (a_{11}(t) + a_{21}(t))a_{12}(t)$$
$$+(a_{12}(t) + a_{22}(t))a_{21}(t) + 1] \tag{28}$$

$$\langle\alpha|p(t)^2|\alpha\rangle = -\frac{1}{2}[(a_{11}(t) - a_{21}(t))^2\alpha^{*2} + (a_{12}(t) - a_{22}(t))^2\alpha^2 +$$
$$2(a_{11}(t) - a_{21}(t))(a_{12}(t) - a_{22}(t))|\alpha|^2 + (a_{11}(t) - a_{21}(t))a_{12}(t)$$
$$+(a_{22}(t) - a_{12}(t))a_{21}(t) - 1] \tag{29}$$

with these expressions we can obtain the dispersions $\Delta q(t)$, $\Delta p(t)$.

In figure 3 we show the dispersions $\Delta q(t)$ (purple) and Δp (green) for Hamiltonian parameters $\kappa = 0.05$, $\chi = 0.0$, $\Omega_0 = 1$ corresponding to a parametric oscillator. Notice that at the initial time the state is a minimum uncertainty state with $\Delta q(t = 0) = \Delta p(t = 0) = 1/\sqrt{2}$ and as time evolves the dispersions oscillate and are such that when one increases the other decreases and there are regions where either Δq or Δp take values corresponding to a squeezed state. The amplitude of the oscillations in the dispersions is an increasing function of time. However, the presence of squeezing is maintained. In black we show the product $\Delta q(t)\Delta p(t)$. It can be seen that it oscillates and at those times when the squeezing is present the product corresponds to that of a minimum uncertainty state. For most of the time, the product $\Delta q\Delta p$ is larger and we can say that these states are not minimum uncertainty states but for a limited set of instants of time. A coherent state corresponding to a parametric oscillator does not evolve into a coherent state.

In figure 4 we show the dispersion $\Delta q(t)$ (red) and the product $\Delta q\Delta p$ (black) for Hamiltonian parameters $\kappa = 0.7$, $\chi = 0.25$, $\Omega_0 = 1$, $\alpha = 3 + \mathrm{i}3$ and $\alpha_0 = \sqrt{18}$ corresponding to a nonlinear parametric oscillator. At the initial time $t = 0$ the state is a minimum uncertainty state. At the initial stages of the evolution there is squeezing present, this however does not last long and the dispersion returns to that corresponding to a coherent state until after about $t = 4\pi$ it oscillates for a short time attaining values larger and smaller than $1/\sqrt{2}$ and returning to $1/\sqrt{2}$ until about $t = 8\pi$. This conduct is repeated periodically. In black we show the product $\Delta q(t)\Delta p(t)$ for the same set of potential parameters. It can be seen that the product corresponds practically to a minimum uncertainty state with deviations at those times where squeezing is present.
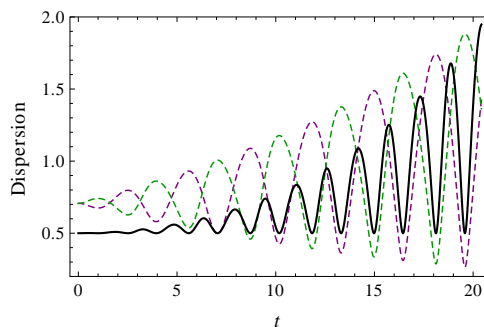
Figure 3. Dispersions Δq (purple) and Δp (green) and their product $\Delta q \Delta p$ (black) with parameters $\chi = 0$, $\kappa = 0.05$, $\alpha = 3 + \mathrm{i}3$, $\alpha_0 = \sqrt{18}$ and $\Omega_0 = 1$.
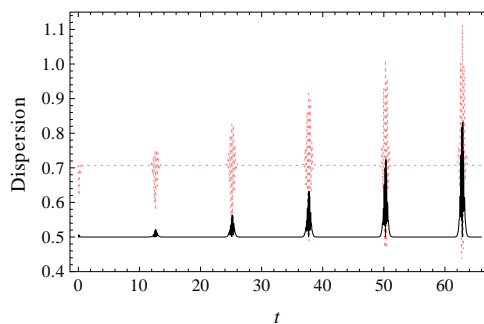

Figure 4. Dispersions Δq (red) and the product $\Delta q \Delta p$ (black) with parameters $\chi = 0.25$, $\kappa = 0.7$, $\alpha = 3 + \mathrm{i}3$, $\alpha_0 = \sqrt{18}$ and $\Omega_0 = 1$.

4. Conclusions

In this work we have built an approximate time evolution operator for a system composed of a parametric oscillator in a nonlinear Kerr-like medium. The Hamiltonian is transformed into the interaction picture and as a result we obtained a time dependent Hamiltonian that contains the number operator in an exponential. In order to solve this problem we approximate the exponential by its average value taken between time dependent coherent states. With this simplification we can write the Hamiltonian in the interaction picture as an element of a finite Lie algebra. The time dependent coefficients of this linear combination are obtained in a self consistent form. The time evolution operator can then be expressed as a product of exponentials. We calculated several statistical properties like the Mandel parameter the dispersions in the quadratures and the generation of photons from the vacuum state. We found squeezing in the quadratures due to the presence of quadratic creation and annihilation operators in the Hamiltonian. The Mandel parameter yields sub-Poissonian statistics at the initial stages of the evolution when α is complex and super-Poissonian statistics when α is real. When α is complex the Mandel parameter is sub-Poissonian at the start and becomes super-Poissonian after some time. For real α the Mandel parameter is always super-Poissonian.

Appendix A

From the set of coupled, nonlinear, ordinary differential equations we got for the functions $\phi_i(t)$ we see that equation 15 has the form of a Riccati equation:

$$y'(\lambda) = q_0(\lambda) + q_1(\lambda)y(\lambda) + q_2(\lambda)y^2(\lambda) \tag{A.1}$$

with:

$$q_0(\lambda) = \beta_1(t), \quad q_1(\lambda) = 2\beta_2(t), \quad q_2(\lambda) = 4\beta_3(t) \tag{A.2}$$

so that,

$$q_2'(\lambda) = 0, \rightarrow \frac{q_2'(\lambda)}{q_2(\lambda)} = 0, \quad q_2(\lambda) \neq 0. \tag{A.3}$$

Introducing the variable $z(\lambda)$ as:

$$\phi_1(\lambda)q_2(\lambda) = -\frac{z'(\lambda)}{z(\lambda)} \tag{A.4}$$

we obtain the following linear, second order equation with constant coefficients

$$z''(\lambda) - 2\beta_2(t)z'(\lambda) + 4\beta_1(t)\beta_2(t)z(\lambda) = 0 \tag{A.5}$$

proposing a solution of the form $z(\lambda) = \exp(\theta\lambda)$ we get the characteristic polynomial

$$\theta^2 - 2\beta_2(t)\theta + 4\beta_1(t)\beta_3(t) = 0 \tag{A.6}$$

with roots:

$$\theta = \beta_2(t) \pm 2\Sigma(t), \quad \Sigma^2(t) = \frac{\beta_2^2(t)}{4} - \beta_1(t)\beta_3(t) \tag{A.7}$$

so the function $z(\lambda)$ can be written as:

$$z(\lambda) = A\exp[(\beta_2(t) + 2\Sigma(t))\lambda] + B\exp[(\beta_2(t) - 2\Sigma(t))\lambda] \tag{A.8}$$

The constants A, B are fixed by the initial condition $\phi_i(t_0) = 0$ and we obtain finally:

$$\phi_1(\lambda) = \frac{\beta_1 \sinh(2\Sigma\lambda)}{2\Sigma \cosh(2\Sigma\lambda) - \beta_2 \sinh(2\Sigma\lambda)} \tag{A.9}$$

substitution into the differential equation for $\phi_2(\lambda)$ yields

$$\frac{d\phi_2(\lambda)}{d\lambda} = \frac{\beta_2(t)\cosh(2\Sigma(t)\lambda) - 2\Sigma(t)\sinh(2\Sigma(t)\lambda)}{\cosh(2\Sigma(t)\lambda) - \frac{\beta_2(t)}{2\Sigma(t)}\sinh(2\Sigma(t)\lambda)} \tag{A.10}$$

integrating we get

$$\phi_2(\lambda) = -\ln\left[\cosh(2\Sigma(t)\lambda) - (\beta_2(t)/2\Sigma(t))\sinh(2\Sigma(t)\lambda)\right] \tag{A.11}$$

substitution in the differential equation for $\phi_3(\lambda)$ yields

$$\frac{d\phi_3(\lambda)}{d\lambda} = \frac{4\Sigma^2(t)\beta_3(t)}{[2\Sigma(t)\cosh(2\Sigma(t)\lambda) - \beta_2(t)\sinh(2\Sigma(t)\lambda)]^2} \tag{A.12}$$

integrating we get:

$$\phi_3(\lambda) = \frac{\beta_3(t)\sinh(2\Sigma(t)\lambda)}{2\Sigma(t)\cosh(2\Sigma(t)\lambda) - \beta_2(t)\sinh(2\Sigma(t)\lambda)} \tag{A.13}$$

Finally from the equation for $\phi_4(\lambda)$ we get:

$$\frac{d\phi_4(\lambda)}{d\lambda} = \beta_4(t) + \frac{2\beta_1(t)\beta_3(t)\sinh(2\Sigma(t)\lambda)}{2\Sigma(t)\cosh(2\Sigma(t)\lambda) - \beta_2(t)\sinh(2\Sigma(t)\lambda)} \tag{A.14}$$

integrating we obtain:

$$\phi_4(\lambda) = (\beta_4(t) - \beta_2(t)/2)\lambda - \frac{1}{2}\ln\left[\cosh(2\Sigma(t)\lambda) - \beta_2(t)/(2\Sigma(t))\sinh(2\Sigma(t)\lambda)\right] \tag{A.15}$$

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