First-Order Polynomial Heisenberg Algebras and Coherent States

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Abstract. The polynomial Heisenberg algebras (PHA) are deformations of the Heisenberg-Weyl algebra characterizing the underlying symmetry of the supersymmetric partners of the Harmonic oscillator. When looking for the simplest system ruled by PHA, however, we end up with the harmonic oscillator. In this paper we are going to realize the first-order PHA through the harmonic oscillator. The associated coherent states will be also constructed, which turn out to be the well known even and odd coherent states.

1. Introduction

The polynomial deformations of the Heisenberg-Weyl algebra, polynomial Heisenberg algebras (PHA) by short, start to be important in physics [1–6]. For example, the underlying algebraic symmetry for the supersymmetric (SUSY) partners of the harmonic oscillator is precisely described by a PHA. In addition, when looking for the general one-dimensional Schrödinger Hamiltonians ruled by second (third) order PHA, the key turns out to be the determination of a function which must satisfy the Painlevé IV (Painlevé V) equation [5]. As a consequence, through this link it has been possible to design a simple method to generate solutions to these non-linear second-order ordinary differential equations [6].

It is important to note that the spectrum for systems ruled by PHA can be obtained by identifying the physical extremal states, those eigenstates of the Hamiltonian which belong also to the kernel of the annihilation operator. In principle, the number of physical extremal states could be equal to the order of the annihilation operator. However, we have observed in most of the works that the number of extremal states is less than the order of the annihilation operator.

In this paper we would like to work with the simplest systems realizing the PHA. Moreover, we want to explore the possibility that every extremal state, if possible, would have an eigenvalue which is in the spectrum of the Hamiltonian. Once this has been done, we will construct the associated coherent states and we will study their properties.

2. Polynomial Heisenberg algebras

In the standard harmonic oscillator (Heisenberg-Weyl) algebra there are three generators H, a, a^+ satisfying the following commutation relations:

$$[H, a^+] = a^+, \quad [H, a] = -a, \quad [a, a^+] = 1, \tag{1}$$

where there is a linear dependence between the number operator N and the Hamiltonian H:

$$N = a^+ a = H - \tfrac{1}{2}. \tag{2}$$

On the other hand, the $(m-1)$-th order polynomial Heisenberg algebras are deformations of the oscillator algebra of kind [5]:

$$[H, \mathcal{L}_m^+] = \omega\, \mathcal{L}_m^+, \tag{3}$$

$$[H, \mathcal{L}_m^-] = -\omega\, \mathcal{L}_m^-, \tag{4}$$

$$[\mathcal{L}_m^-, \mathcal{L}_m^+] \equiv N_m(H+\omega) - N_m(H) \equiv P_{m-1}(H), \tag{5}$$

where the analogue of the number operator is now an m-th degree polynomial in H:

$$N_m(H) \equiv \mathcal{L}_m^+ \mathcal{L}_m^- = \prod_{i=1}^{m} (H - \mathcal{E}_i)\,. \tag{6}$$

The most common realization of a PHA is a differential one, where H has the standard one-dimensional Schrödinger form,

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \tag{7}$$

while $\mathcal{L}_m^{\pm}$ are differential ladder operators of order m-th.

The spectrum of H, $\mathrm{Sp}(H)$, can be obtained from the analysis of the Kernel $K_{\mathcal{L}_m^-}$ of $\mathcal{L}_m^-$. In fact, it turns out that

$$\mathcal{L}_m^- \psi = 0 \quad \Rightarrow \quad \mathcal{L}_m^+ \mathcal{L}_m^- \psi = \prod_{i=1}^{m} (H - \mathcal{E}_i)\, \psi = 0. \tag{8}$$

Since $K_{\mathcal{L}_m^-}$ in invariant under H,

$$\mathcal{L}_m^-(H\psi) = (H+\omega)\mathcal{L}_m^- \psi = 0 \quad \forall \quad \psi \in K_{\mathcal{L}_m^-}, \tag{9}$$

then a basis in $K_{\mathcal{L}_m^-}$ can be chosen such that,

$$H\psi_{\mathcal{E}_i} = \mathcal{E}_i \psi_{\mathcal{E}_i}. \tag{10}$$

The states $\psi_{\mathcal{E}_i}$ are the so-called extremal states; departing from them we can build several energy ladders with spacing $\Delta E = \omega$.

Depending on how many extremal states $\psi_{\mathcal{E}_i}$ are eigenstates of H (satisfying also the boundary conditions), two possibilities appear:

(a) If s extremal states are eigenstates of H, $\{\psi_{\mathcal{E}_i}, i = 1, \ldots, s\}$, there will be s energy ladders arising from the iterated action of $\mathcal{L}_m^+$ onto $\psi_{\mathcal{E}_i}$.

(b) If for the j-th ladder it turns out that

$$\left(\mathcal{L}_m^+\right)^{n-1} \psi_{\mathcal{E}_j} \neq 0, \qquad \left(\mathcal{L}_m^+\right)^{n} \psi_{\mathcal{E}_j} = 0, \tag{11}$$

then we will have

$$\mathcal{L}_m^-(\mathcal{L}_m^+)^n \psi_{\mathcal{E}_j} = \mathcal{L}_m^- \mathcal{L}_m^+ (\mathcal{L}_m^+)^{n-1} \psi_{\mathcal{E}_j} = \prod_{i=1}^{m} (\mathcal{E}_j + n\omega - \mathcal{E}_i)\, (\mathcal{L}_m^+)^{n-1} \psi_{\mathcal{E}_j} = 0. \tag{12}$$

Therefore $\mathcal{E}_k = \mathcal{E}_j + n\omega$ for some $k \in \{s+1, \ldots, m\}$, $j \in \{1, \ldots, s\}$; thus, $\mathrm{Sp}(H)$ will consist of $s-1$ infinite ladders and a finite one of lenght $(n-1)\omega$ which starts from $\mathcal{E}_j$ and ends at $\mathcal{E}_j + (n-1)\omega$. We can conclude that $\mathrm{Sp}(H)$ can have up to m infinite ladders with spacing $\Delta E = \omega$ between steps.

3. Harmonic oscillator and polynomial Heisenberg algebras

It is possible to realize the PHA through the harmonic oscillator Hamiltonian H and deformed versions a_g, a_g^+ of the standard annihilation and creation operators a, a^+. One possibility is to take $a_g = P_{k-1}(H)a,\ a_g^+ = a^+P_{k-1}(H)$, where $P_{k-1}(x)$ is a polynomial of degree $k-1$ with real roots; then, the set of operators $\{H, a_g, a_g^+\}$ satisfy the commutation relations of Eqs. (3-5). However, in these deformations it is typical that some extremal states have formal eigenvalues which do not belong to $\mathrm{Sp}(H)$.

We are looking for deformations of the annihilation and creation operators such that all extremal states become physical and, consequently, from them we will generate infinite ladders of eigenfunctions and eigenvalues of H. For that reason, let us take now a different deformation,

$$a_g = a^2, \qquad a_g^+ = (a^+)^2. \tag{13}$$

The operator set $\{H, a_g, a_g^+\}$ gives place to a first-order PHA with $\omega = 2$ since,

$$[H, a_g] = -2a_g, \tag{14}$$
$$[H, a_g^+] = 2a_g^+, \tag{15}$$
$$[a_g, a_g^+] = N(H+2) - N(H), \tag{16}$$

where

$$N(H) = \left(H - \tfrac{1}{2}\right)\left(H - \tfrac{3}{2}\right). \tag{17}$$

There are two extremal state energies,

$$\mathcal{E}_1 = E_0 = \tfrac{1}{2}, \quad \mathcal{E}_2 = E_1 = \tfrac{3}{2}. \tag{18}$$

Thus, the eigenvalues associated to the j-th ladder are

$$\mathcal{E}_n^j = \mathcal{E}_j + 2n \quad n = 0, 1, \dots, \quad j = 1, 2. \tag{19}$$

The corresponding eigenstates turn out to be

$$|\psi_n^j\rangle = |2n + j - 1\rangle = \sqrt{\frac{(j-1)!}{(2n+j-1)!}}(a_g^+)^n|j-1\rangle. \tag{20}$$

The spectrum of H becomes,

$$\mathrm{Sp}(H) = \{\mathcal{E}_0^1, \mathcal{E}_1^1, \dots\} \cup \{\mathcal{E}_0^2, \mathcal{E}_1^2, \dots\}, \tag{21}$$

which coincides with the harmonic oscillator spectrum, but seen from a different point of view: the Hilbert space $\mathcal{H}$ is now decomposed as the direct sum of two orthogonal supplementary subspaces, namely, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ (a representation of both ladders can be seen in Figure 1). Note that states in each subspace have definite parity: the ones in $\mathcal{H}_0$ have positive parity and they are called even states, while those in $\mathcal{H}_1$ have negative parity, being called odd states.

4. Coherent states

Once we have identified the algebraic structure underlying the system under study, let us derive the corresponding coherent states. We will look for them as eigenstates of the deformed annihilation operator:

$$a_g|\alpha\rangle_j = \alpha|\alpha\rangle_j, \quad j = 0, 1, \tag{22}$$

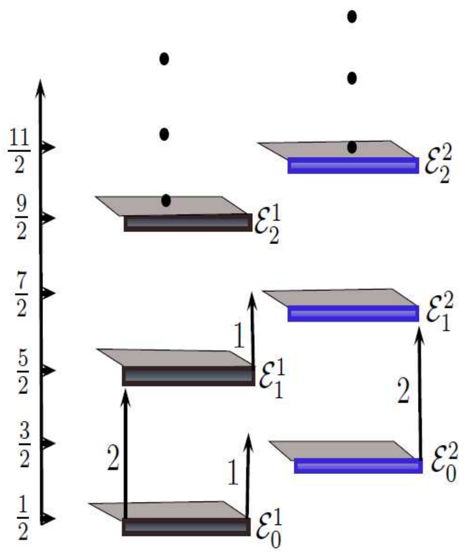

Figure 1: The two independent energy ladders (with level spacing $\Delta E = 2$) associated to the first-order polynomial Heisenberg algebra of Eqs. (14-16). Globally they give place to the harmonic oscillator spectrum with the standard level spacing $\Delta E = 1$.

where

$$|\alpha\rangle_j = \sum_{n=0}^{\infty} C_n |2n+j\rangle. \tag{23}$$

We thus get the following recurrence relationship:

$$C_{n+1} = \frac{\alpha\, C_n}{\sqrt{(2n+j+2)(2n+j+1)}}, \tag{24}$$

which after iteration leads to:

$$C_n = \sqrt{\frac{j!}{(2n+j)!}}\alpha^n C_0. \tag{25}$$

Using C_0 as a normalization constant, we finally obtain the coherent states we were looking for:

$$|\alpha\rangle_j = \frac{1}{\sqrt{\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{(2n+j)!}}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(2n+j)!}} |2n+j\rangle. \tag{26}$$

Some quantities useful to analyze the Heisenberg uncertainty relation for these coherent states are the following:

$$\langle x\rangle_j = \langle p\rangle_j = 0, \tag{27}$$

$$\langle x^2\rangle_j = |a|\alpha\rangle_j|^2 + \tfrac{1}{2} + \mathrm{Re}(\alpha), \tag{28}$$

$$\langle p^2\rangle_j = |a|\alpha\rangle_j|^2 + \tfrac{1}{2} - \mathrm{Re}(\alpha), \tag{29}$$

where

$$|a|\alpha\rangle_j|^2 = \begin{cases} \frac{1}{\sum\limits_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(2r)!}} \sum\limits_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{(2n+1)!} = |\alpha| \tanh|\alpha| & \text{for } j=0, \\ \frac{1}{\sum\limits_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(2r+1)!}} \sum\limits_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(2n)!} = |\alpha| \coth|\alpha| & \text{for } j=1. \end{cases} \tag{30}$$

Therefore:

$$(\Delta x)_j^2(\Delta p)_j^2 = \begin{cases} \left(|\alpha|\tanh|\alpha| + \frac{1}{2}\right)^2 - [\mathrm{Re}(\alpha)]^2 \geq \frac{1}{4} & \text{for } j=0, \\ \left(|\alpha|\coth|\alpha| + \frac{1}{2}\right)^2 - [\mathrm{Re}(\alpha)]^2 \geq \frac{9}{4} & \text{for } j=1. \end{cases} \tag{31}$$

Plots of these uncertainty relations are shown in Figure 2 for $j=0$ and Figure 3 for $j=1$.

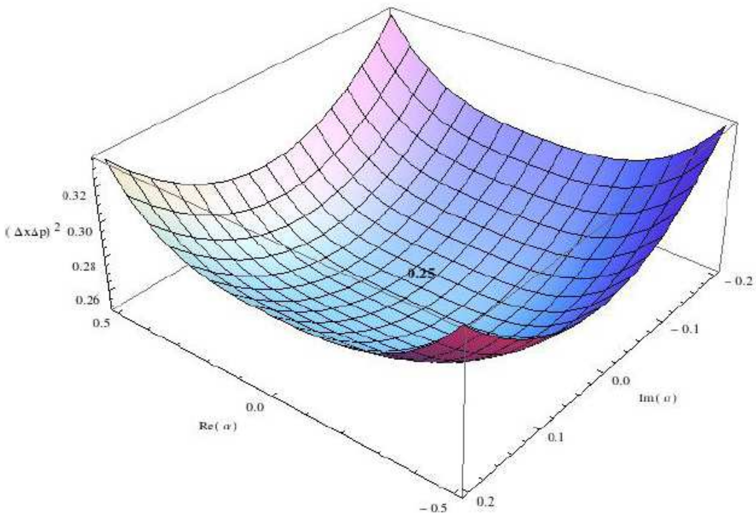

Figure 2: Heisenberg uncertainty relation $(\Delta x)_0^2(\Delta p)_0^2$ of Eq. (31) for $j=0$.

On the other hand, the mean energy value for a system being in a coherent state becomes:

$$\langle H\rangle_j = |a|\alpha\rangle_j|^2 + \frac{1}{2} = \begin{cases} |\alpha|\tanh|\alpha| + \frac{1}{2} & \text{for } j=0, \\ |\alpha|\coth|\alpha| + \frac{1}{2} & \text{for } j=1. \end{cases} \tag{32}$$

It is also important to guarantee the completeness relation in the subspace $\mathcal{H}_j$ (remember that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$), namely:

$$\int |\alpha\rangle_j\, {}_j\langle\alpha| d\mu_j(\alpha) = I_j, \quad j=0,1, \tag{33}$$

where

$$d\mu_j(\alpha) = \frac{1}{\pi|\alpha|}\left(\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(2r+j)!}\right) f_j(|\alpha|^2) d|\alpha| d\varphi. \tag{34}$$

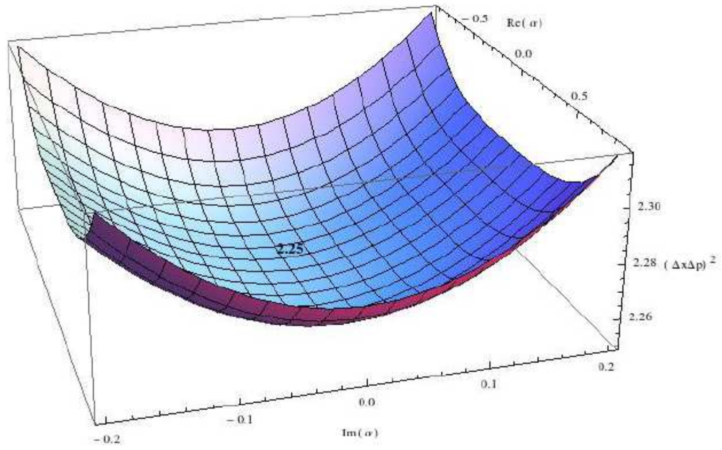

Figure 3: Heisenberg uncertainty relation $(\Delta x)_1^2(\Delta p)_1^2$ of Eq. (31) for $j=1$.

It turns out that the function $f_j(x)$ must satisfy:

$$\int_0^\infty x^{n-1} f_j(x)dx = \Gamma(2n+j+1). \tag{35}$$

Thus, provided that Eq. (35) is satisfied, it is true that any state vector can be decomposed in terms of our coherent states.

Finally, the time evolution of a coherent state is quite simple:

$$U(t)|\alpha\rangle_j = e^{-i(j+\frac{1}{2})t}|\alpha(t)\rangle_j, \quad \alpha(t) = \alpha\, e^{-ikt}. \tag{36}$$

Let us consider next the following non-normalized coherent states:

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle, \tag{37}$$

$$|z\rangle_j = \sum_{n=0}^{\infty} \frac{z^{2n+j}}{\sqrt{(2n+j)!}}|2n+j\rangle, \quad \alpha = z^2. \tag{38}$$

Up to normalization, the states of Eq. (37) are the standard coherent states and those of Eq. (38) are the even ($j=0$) and odd ($j=1$) coherent states [7–14] (see also the discussion at the end of Section 3). The completeness relationship of Eq. (33) guarantees that $|z\rangle$ and $|e^{i\pi}z\rangle$ can be decomposed in terms of $|z\rangle_0$ and $|z\rangle_1$ as:

$$|z\rangle = |z\rangle_0 + |z\rangle_1, \quad |e^{i\pi}z\rangle = |z\rangle_0 - |z\rangle_1. \tag{39}$$

On the other hand, from these equations we can solve $|z\rangle_0$ and $|z\rangle_1$ in terms of $|z\rangle$ and $|e^{i\pi}z\rangle$. After normalization we get that:

$$|z\rangle_0 = \frac{e^{-|z|^2/2}}{\sqrt{2(1+e^{-2|z|^2})}}[|z\rangle + |e^{i\pi}z\rangle], \tag{40}$$

$$|z\rangle_1 = \frac{e^{-|z|^2/2}}{\sqrt{2(1-e^{-2|z|^2})}}[|z\rangle - |e^{i\pi}z\rangle], \tag{41}$$

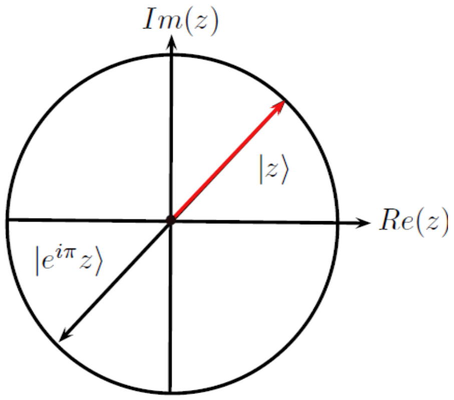

Figure 4: Diagram indicating the position in the complex plane of the standard coherent states which give place to the even and odd coherent states $|z\rangle_0$ and $|z\rangle_1$.

which means that the even and odd coherent states are linear combinations of two standard coherent states with *opposite directions* in the z complex plane, as it is represented in Figure 4. The expressions (40-41) can be used now to identify in a simple way the wave packets for $U(t)|z\rangle_0$ and $U(t)|z\rangle_1$. In fact, the wavefunction for a normalized standard coherent state is [15]:

$$\psi_z(x) = \langle x|z\rangle = \left(\frac{1}{\pi}\right)^{1/4} e^{-\frac{1}{2}(x-\langle x\rangle)^2 + ix\langle p\rangle}, \tag{42}$$

with $\langle x\rangle = \sqrt{2}\,\mathrm{Re}(z)$ and $\langle p\rangle = \sqrt{2}\,\mathrm{Im}(z)$. Thus, the wavefunctions associated to $|z\rangle_0$ and $|z\rangle_1$ in Eqs. (40-41) are:

$$\psi_z^0(x) = \langle x|z\rangle_0 = N_0\left[e^{-\frac{1}{2}(x-\langle x\rangle)^2 + ix\langle p\rangle} + e^{-\frac{1}{2}(x+\langle x\rangle)^2 - ix\langle p\rangle}\right], \tag{43}$$

$$\psi_z^1(x) = \langle x|z\rangle_1 = N_1\left[e^{-\frac{1}{2}(x-\langle x\rangle)^2 + ix\langle p\rangle} - e^{-\frac{1}{2}(x+\langle x\rangle)^2 - ix\langle p\rangle}\right], \tag{44}$$

where

$$N_0 = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2(1+e^{-\langle x\rangle^2 - \langle p\rangle^2})}}, \quad N_1 = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2(1-e^{-\langle x\rangle^2 - \langle p\rangle^2})}}.$$

Finally, up to global phase factors the wavefunctions $\psi_z^j(x,t) = \langle x|U(t)|z\rangle_j$, $j = 0,1$ arise from the right hand side of Eqs. (43-44) substituting $\langle x\rangle$ by $\langle x\rangle\cos t + \langle p\rangle\sin t$ and $\langle p\rangle$ by $\langle p\rangle\cos t - \langle x\rangle\sin t$. The corresponding probability densities are shown in Figure 5 for $j = 0$ and in Figure 6 for $j = 1$.

Note that for certain times the probability density is maximum at $x = 0$ for the even coherent states, while this never happens for the odd ones. It is interesting to observe also that the coherent states $U(t)|z\rangle_0$ and $U(t)|z\rangle_1$ are cyclic, with a period $\tau = \pi$ which is half the period of the oscillator (2π). This property reflects clearly the very quantum nature of the even and odd coherent states compared with the somehow classical behavior of the standard ones, which always have the period of the oscillator.

5. Conclusions

In this paper we have explored an interesting realization of the first-order polynomial Heisenberg algebra. The generators of such an algebra are the harmonic oscillator Hamiltonian H and the

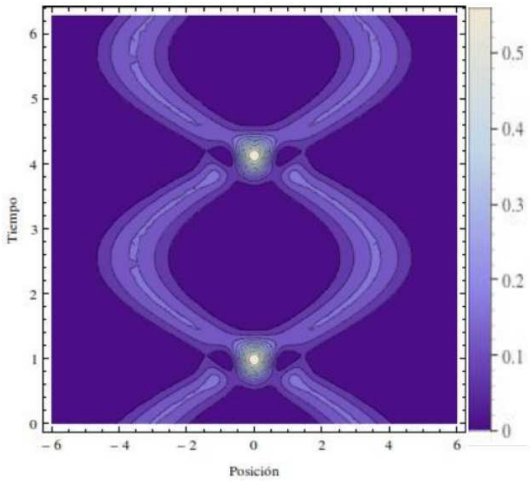

Figure 5: Probability density $|\psi_z^0(x,t)|^2$ for the even coherent states.

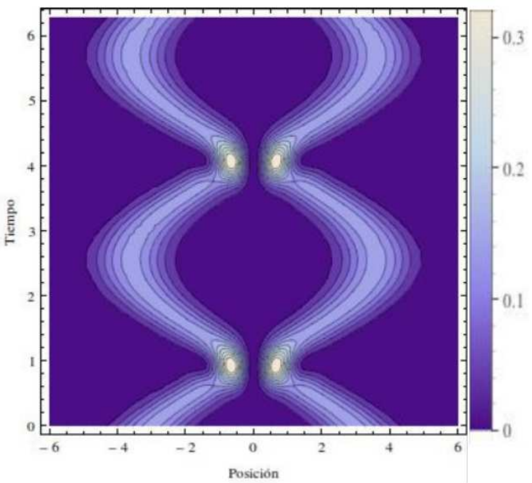

Figure 6: Probability density $|\psi_z^1(x,t)|^2$ for the odd coherent states.

deformed annihilation and creation operators $a_g = a^2$, $a_g^+ = (a^+)^2$. We have shown that the two extremal states involved in our treatment are physical eigenstates of the Hamiltonian. We have seen also that the two ladders generated from these extremal states are of infinite length, and the associated coherent states become the so-called even and odd. We have realized that the period of the even and odd coherent states is a fraction (1/2) of the original period (2π) for the oscillator. This property indicates the very quantum nature of such coherent states. This statement should be reinforced by calculating other quantities indicating how quantum a given state is. We hope to perform this analysis in the near future.

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