

# Coupling $n$ -level Atoms with $\ell$ -modes of Quantised Light in a Resonator

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**Abstract.** We study the quantum phase transitions associated to the Hamiltonian of a system of  $n$ -level atoms interacting with  $\ell$  modes of electromagnetic radiation in a resonator. The quantum phase diagrams are determined in analytic form by means of a variational procedure where the test function is constructed in terms of a tensorial product of coherent states describing the matter and the radiation field. We demonstrate that the system can be reduced to a set of Dicke models.

## 1. Introduction

The quantum electrodynamics (QED) in cavities has its origin in the works of Einstein on the statistical theory of light and its interaction with matter. The photoelectric effect and the ultraviolet photoionization are considered the first applications of QED, where the quantisation of the electromagnetic field is fundamental [1]. The quantized coupling between matter and radiation was firstly proposed in the Jaynes-Cummings model (JCM) [2]. However, the JCM model was regarded as a toy model because an excited atom in free space decays in an irreversible form. This situation changes dramatically when the atom is inside a resonator because, depending on its quality factor, the half-life of a photon in the cavity can be up to one second. This means that the atom can now emit and absorb the photon in a reversible manner [3, 4].

A schematic model of a resonator is formed by two mirrors in whose interior is an atom at rest (or passing through the cavity). There are 4 parameters appearing in the physical system: the atom-field coupling,  $\mu$ , the dipolar atomic radiation of the atom to modes of the electromagnetic field different from the mode of the cavity,  $\gamma$ , the rate of photon decay in the cavity  $\kappa$  (related to its quality factor), and, for atoms passing through, the transit time through the cavity  $T$ . The strong coupling regime considers that  $\mu \gg \{\gamma, \kappa, T^{-1}\}$ . This regime has been used to study entanglement properties between matter and radiation; in particular Haroche and his collaborators have generated Schrödinger cat states using the entanglement properties as a source of information [5]. There are other procedures that use the entanglement properties to generate Schrödinger cat states, for example Wineland and his group used the entanglement between the vibrational states of the molecules with their corresponding hyperfine states [6].

Although the JCM model captures the essence of the interaction dynamics between the matter and the radiation field, there are dissipative processes that appear naturally in these systems of cavities and atoms. Recently effects on the dissipation rates of the uncertainty in the



position of the atoms when they are interacting with a reservoir has been considered [7]. The dissipative processes are related to the parameters mentioned above  $\gamma$ ,  $\kappa$ , and  $T$ ; additionally one has to consider the position of the atoms. The position dependence can be made negligible by considering Fabry-Perot optical cavities [3]. By means of these physical systems one may transmit information from one atom in a cavity to another atom in a second cavity [8].

In this work we determine the quantum phase transitions for  $n$ -level atoms interacting with  $\ell$  modes of electromagnetic radiation in a cavity [9]. We establish analytic expressions for the energies associated to the ground state for arbitrary values of the dipolar strengths, together with their corresponding eigenstates. Additionally we demonstrate that the phase diagram associated to the transition from the normal to the superradiant regime is reduced to a set of phase sub-diagrams, each of them according as the quantum phase transition found by Hepp and Lieb for the Dicke model [10,11]. For the transitions between superradiant regions one finds always first order phase transitions and the different boundaries correspond to Maxwell sets.

## 2. Model Hamiltonian

The matter and the electromagnetic field are described by the Hamiltonian

$$\mathbf{H} = \sum_{j=1}^n \hbar \omega_j \mathbf{A}_{jj} + \sum_{j < k}^n \hbar \Omega_{jk} \boldsymbol{\nu}_{jk}, \quad (1)$$

where one introduces the convention  $\omega_1 \leq \omega_2 \leq \dots \leq \omega_n$ . The operators  $\mathbf{A}_{jj}$  and  $\boldsymbol{\nu}_{jk} \equiv \mathbf{a}_{jk}^\dagger \mathbf{a}_{jk}$  define the number operator of particles in level  $j$  and the number of photons in the mode  $\Omega_{jk}$ , respectively. The detuning parameter of the transition  $j \rightarrow k$  is denoted by  $\Delta_{jk} = \Omega_{jk} - (\omega_k - \omega_j)$ , with  $\omega_k > \omega_j$  [9]. A schematic diagram of the system is shown in Fig. 1.

The interaction between the matter and the radiation field, under the long wavelength and dipolar approximations, is written in two parts as follows:

$$\mathbf{H}_{FM} = \mathbf{O}_R + \mathbf{O}_{CR}.$$

They are called the rotating and the counter-rotating terms, and are given by the expressions

$$\mathbf{O}_R = -\frac{1}{\sqrt{N_a}} \sum_{j=1}^{n-1} \sum_{k=j+1}^n \mu_{jk} (\mathbf{A}_{jk} \mathbf{a}_{jk}^\dagger + \mathbf{A}_{kj} \mathbf{a}_{jk}), \quad (2)$$

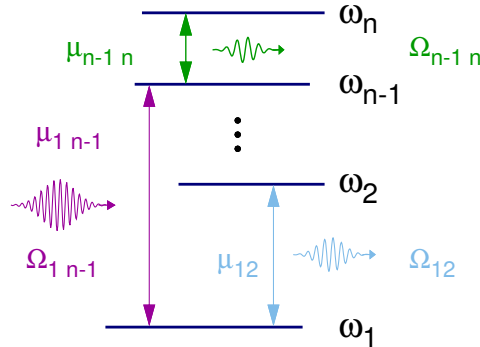
$$\mathbf{O}_{CR} = -\frac{1}{\sqrt{N_a}} \sum_{j=1}^{n-1} \sum_{k=j+1}^n \mu_{jk} (\mathbf{A}_{jk} \mathbf{a}_{jk} + \mathbf{A}_{kj} \mathbf{a}_{jk}^\dagger), \quad (3)$$

where the coupling strength of the matter-field interaction  $\mu_{jk}$  is

$$\mu_{jk} = \hbar (\omega_j - \omega_k) d_{jk} \sqrt{\frac{2\pi \rho_m}{\hbar \Omega_{jk}}}.$$

The symbol  $d_{jk}$  is the matrix element of the dipolar operator  $\vec{d} = e \vec{r}$ ,  $\Omega_{jk}$  is a photonic mode of the cavity, and  $\rho_m$  indicates the number of atoms in the quantization volume. The matter operators can be realized in terms of bosonic operators  $\mathbf{A}_{jk} = \mathbf{b}_j^\dagger \mathbf{b}_k$ , and they thus satisfy the commutation relations of a unitary algebra in  $n$  dimensions, i.e.,

$$[\mathbf{A}_{jk}, \mathbf{A}_{mn}] = \mathbf{A}_{jn} \delta_{mk} - \mathbf{A}_{mk} \delta_{jn}, \quad (4)$$



**Figure 1.** Schematic diagram of an  $n$ -level atom interacting with  $\ell$  modes of electromagnetic field, where transitions between arbitrary pair of atomic levels are promoted by one mode of the electromagnetic field.

while the creation and annihilation photon operators  $\mathbf{a}_{jk}^\dagger, \mathbf{a}_{jk}$ , satisfy the commutation relations of a Heisenberg-Weyl algebra

$$[\mathbf{a}_{jk}, \mathbf{a}_{mn}^\dagger] = \delta_{jm} \delta_{kn}, \quad [\mathbf{a}_{jk}, \mathbf{a}_{mn}] = [\mathbf{a}_{jk}^\dagger, \mathbf{a}_{mn}^\dagger] = 0. \quad (5)$$

In the rotating wave approximation (RWA), the Hamiltonian system has the following constants of motion

$$\mathbf{K}_j = \mathbf{A}_{jj} + \sum_{k=1}^{j-1} \nu_{kj} - \sum_{k=j+1}^n \nu_{jk}, \quad (6)$$

because it is straightforward to calculate that  $[\mathbf{K}_j, \mathbf{O}_R] = 0$  for all  $j = 1, 2, \dots, n$ . Therefore, for the RWA approximation one has  $n$  constants of the motion (notice that they include the total number of particles  $N_a = \sum_j A_{jj}$ ). From here on, we shall call the Hamiltonian in this approximation the *generalised Tavis-Cummings model* [12].

For the counter-rotating term one finds  $[\mathbf{K}_j, \mathbf{O}_{CR}] \neq 0$ . However, it is easy to prove that the parity operators of the mentioned constants of motion are invariants, that is,

$$e^{i\pi \mathbf{K}_j} \mathbf{O}_{CR} e^{-i\pi \mathbf{K}_j} = \mathbf{O}_{CR} \quad (7)$$

because  $e^{i\pi \mathbf{K}_j} \mathbf{a}_{jk} e^{-i\pi \mathbf{K}_j} = -\mathbf{a}_{jk}$  and  $e^{i\pi \mathbf{K}_j} \mathbf{A}_{jk} e^{-i\pi \mathbf{K}_j} = -\mathbf{A}_{jk}$ , for all  $j, k$ .

The basis states of the Hilbert space can be defined in terms of Fock states as

$$\left\{ \bigotimes_{j < k} |\nu_{jk}\rangle \right\} \otimes \left\{ \bigotimes_{m=1}^n |n_m\rangle \right\} \equiv |\nu_{12} \cdots \nu_{1n} \nu_{23} \cdots \nu_{2n} \cdots \nu_{(n-1)n}; n_1, \dots, n_n\rangle, \quad (8)$$

with  $(j, k)$  indicating the mode of the cavity, and  $m$  the atomic quantum level.

In the RWA approximation we have to substitute the quantum labels of the Fock states of the electromagnetic field in terms of  $n - 1$  constants of motion and one of the labels of the matter Fock states in terms of the number of atoms. In contrast, for the Hamiltonian system without the RWA one has to ask that the basis states have the defined parity of  $n - 1$  of the  $n$  constants of motion, not including the number of atoms. Thus the Hilbert space of the complete Hamiltonian, called the *generalised Dicke model*, is divided into  $2^{n-1}$  subspaces.

### 3. Quantum phase diagrams

In order to determine the ground state of the Hamiltonian system, one can consider the fidelity susceptibility approach to find the values of the interaction parameters where the ground state changes suddenly [13–16]. However, the size of the matrices increases very much for large values of the coupling parameters of the order of  $\mu_{jk} \approx 1$ . Instead we use the catastrophe formalism which considers the following procedure [15, 17]:

- Propose a variational state, living in a large enough Hilbert space; we have considered the associated coherent states to the matter and field parts.
- Calculate the expectation value of the Hamiltonian, which defines the energy surface. The energy surface is constructed as an intensive function that depends on variables and parameters, i.e.,

$$\{E(x, c) | x = x_1, x_2, \dots; c = c_1, c_2, \dots\},$$

where the  $x$ 's are order parameters and the  $c$ 's control parameters. This energy surface defines also a Hamiltonian function in classical mechanics, and can be used to study the corresponding semiclassical dynamics [18]

- The catastrophe term was invented by E. C. Zeeman. Catastrophes are abrupt changes in a state equation as a sudden response to a smooth change in the external conditions [19]. The catastrophe formalism is used to determine the quantum phase diagrams of the system through the expressions

$$\frac{\partial E}{\partial x_k} = 0, \quad x = x^p(c), \quad E_{ij} = \frac{\partial^2 E}{\partial x_i \partial x_j} \Big|_{x^p(c)},$$

which determine the state equation, the critical points, and the stability matrix, respectively.

- When  $E^p = E^{p+1}$  for two different critical points denoted by  $p$  and  $p + 1$ ,

$$\left( \frac{\partial E^p}{\partial c_\alpha} - \frac{\partial E^{p+1}}{\partial c_\alpha} \right) \delta c_\alpha = 0,$$

define the Maxwell sets.

A quantum phase transition occurs when the system jumps from one critical point to another or there are degenerate critical points.

The order of the phase transition can be determined following the Ehrenfest classification. A phase transition takes place between the  $p$  and  $q$  branches of critical points, is of  $n$ th-order, if

$$\lim_{\delta \rightarrow 0} \frac{\partial^i E^{(p)}(s)}{\partial s^i} \Big|_{s_0 - \delta} = \lim_{\delta \rightarrow 0} \frac{\partial^i E^{(q)}(s)}{\partial s^i} \Big|_{s_0 + \delta}, \quad (9)$$

for  $i = 0, 1, 2, \dots, n - 1$ , but fails for  $i = n$  [15].

### 4. Variational function and the energy surface

Glauber [20] developed the theory of quantum coherence and indicated its importance in optics problems. Sudarshan [21] showed the equivalence of semiclassical and quantum descriptions of beam lights, while Klauder [22] established their connections with the continuous representations theory. Therefore, the variational state for the electromagnetic field is determined by the tensorial product of the coherent states introduced in the sixties

$$|\vec{\alpha}\rangle = \bigotimes_{j < k}^n \left\{ e^{-\frac{|\alpha_{jk}|^2}{2}} \sum_{\ell_{jk}} \frac{(\alpha_{jk})^{\ell_{jk}}}{\sqrt{\ell_{jk}!}} |\ell_{jk}\rangle \right\}, \quad (10)$$

while the coherent states of the unitary group in  $n$  dimensions  $U(n)$  are used for the matter part [23–25]

$$|\vec{\gamma}\rangle = \frac{1}{\sqrt{N_a!}} \left\{ \frac{1}{\sqrt{\sum |\gamma_k|^2}} \sum_k \gamma_k \mathbf{b}_k^\dagger \right\}^{N_a} |0\rangle. \quad (11)$$

with  $\mathbf{b}_k|0\rangle = 0$  for  $k = 1, 2, \dots, n$ . The condensate or the coherent state for the completely symmetric representation of  $U(n)$  has a redundant parameter which without loss of generality one takes that  $\gamma_1 = 1$  [25]. Therefore the test function is given by  $|\vec{\alpha}\rangle \otimes |\vec{\gamma}\rangle$ , which depends on  $2(n + \ell - 1)$  real parameters.

Then the energy surface, defined as the expectation value of the Hamiltonian with respect to the test function, takes the form

$$E = \sum_{j < k}^n \Omega_{jk} r_{jk}^2 + \sum_j^n \omega_j \frac{\rho_j^2}{1 + R_2} - 4 \sum_{j < k}^n \mu_{jk} r_{jk} \frac{\rho_j \rho_k \cos \theta_{jk} \cos(\phi_k - \phi_j)}{1 + R_2}, \quad (12)$$

where we have defined

$$R_2 = \sum_{k=2}^n \rho_k^2, \quad \alpha_{jk} = \sqrt{N_a} r_{jk} \quad e^{i\theta_{jk}} \gamma_k = \rho_k e^{i\phi_k},$$

with  $j, k = 1, 2, \dots, n$ . Notice that the energy surface is an intensive quantity, i.e., it is independent of the number of atoms.

The critical points are obtained by the following set of derivatives

$$\left( \frac{\partial E}{\partial \rho_2}, \dots, \frac{\partial E}{\partial \rho_n}, \frac{\partial E}{\partial r_{12}}, \dots, \frac{\partial E}{\partial r_{n-1,n}}, \frac{\partial E}{\partial \theta_{12}}, \dots, \frac{\partial E}{\partial \theta_{n-1,n}}, \frac{\partial E}{\partial \phi_2}, \dots, \frac{\partial E}{\partial \phi_n} \right) = 0, \quad (13)$$

which imply the conditions:

$$\theta_{jk}^c = 0, \pi, \quad \phi_k^c - \phi_j^c = 0, \pi, \quad r_{jk}^c = 2 \frac{\mu_{jk} \rho_j^c \rho_k^c}{\Omega_{jk}(1 + R_2)}, \quad (14)$$

with  $j < k$ , and  $k = 2, \dots, n$ . We have taken the critical angles equal to zero because in this work we are only considering positive values for the dipolar interactions  $\mu_{ij}$ . By substituting these conditions into the expression for the energy surface, one gets

$$E = \sum_{j=2}^n \frac{\omega_j \rho_j^c}{1 + R_2} - 4 \sum_{j < k}^n \frac{\mu_{jk}^2}{\Omega_{jk}} \left( \frac{\rho_j^c \rho_k^c}{1 + R_2} \right)^2. \quad (15)$$

The energy surface for the RWA approximation is obtained by replacing  $\mu_{jk} \rightarrow \mu_{jk}^{\text{RWA}}/2$  [9], implying a rescaling of the phase diagram. This must be also done for any expectation values of matter-field observables.

The critical values for the variables  $\rho_j^c$  are obtained from the set of algebraic equations

$$\frac{\rho_j}{1 + R_2} \left( \omega_j - \sum_{k=1}^n \frac{\omega_k \rho_k^2}{1 + R_2} - 4 \sum_{k=1, k \neq j}^n \frac{\mu_{jk}^2 \rho_k^2}{\Omega_{jk} (1 + R_2)} + 8 \sum_{j' \leq k'}^n \frac{\mu_{j'k'}^2 \rho_{j'}^2 \rho_{k'}^2}{\Omega_{j'k'} (1 + R_2)^2} \right) \Big|_{\{\rho^c\}} = 0.$$

The critical values of  $\rho_k$  corresponding to the minima together with the conditions for their existence are given in Table 1. For  $\rho_j^c = 0$ , with  $j = 2, \dots, n$  one gets  $E_N = 0$ , because we took  $\omega_1 = 0$ . Considering  $\rho_s^c = 0$  for some value of  $s$  all  $(n - 1)$   $\mu_{jk}$  related to that level can be taken

Regime	Energy	$\rho_k^c$	Condition
N	0	0	$k = 1, 2, \dots, n$
$S_{1k}$	$-\frac{(\omega_k \Omega_{1k} - 4\mu_{1k}^2)^2}{16\Omega_{1k}\mu_{1k}^2}$	$\sqrt{\frac{4\mu_{1k}^2 - \omega_k \Omega_{1k}}{4\mu_{1k}^2 + \omega_k \Omega_{1k}}}$	$4\mu_{1k}^2 - \omega_k \Omega_{1k} \geq 0, k = 2, \dots, n$
$S_{jk}$	$\omega_j - \frac{((\omega_k - \omega_j)\Omega_{jk} - 4\mu_{jk}^2)^2}{16\Omega_{jk}\mu_{jk}^2}$	$\sqrt{\frac{4\mu_{jk}^2 - (\omega_k - \omega_j)\Omega_{jk}}{4\mu_{jk}^2 + (\omega_k - \omega_j)\Omega_{jk}}}$	$4\mu_{jk}^2 - (\omega_k - \omega_j)\Omega_{jk} \geq 0$

**Table 1.** Minima critical values for  $\rho_k$ . In the third row we give the expression for  $\eta_k^c$  instead of  $\rho_k^c$  (see text), and we take values of  $j > 1$  and  $k > j$ .

equal to zero; thus the expression of the energy surface effectively reduces to that of a system with  $n - 1$  levels. For each variable  $\rho_s^c = 0$  with  $s = 2, \dots, n$  our physical system is reduced to the case of atoms with  $n - 1$  levels and from them one selects the one with lowest energy, i.e.,  $E = \text{Min}\{E_2, E_3, \dots, E_n\}$ . For  $\rho_j^c \rightarrow \infty$  with  $\rho_{k \neq j}^c$  finite, one has  $E \rightarrow \omega_j$  which is greater than zero, and so it is not a minimum value of the energy.

The interaction terms, which give a negative contribution to the energy surface, are different from zero if and only if we assume that the critical points satisfy

$$\varrho_2 \rightarrow \infty, \quad \varrho_j \rightarrow \eta_j \varrho_2, \quad (16)$$

for  $j = 3, \dots, n$ . From the expression of the energy surface one gets

$$E_\infty = \sum_{j=2}^n \omega_j \frac{\eta_j^2}{1 + \sum_{k=3}^n \eta_k^2} - 4 \sum_{2 \leq j < k} \frac{\mu_{jk}^2}{\Omega_{jk}} \left( \frac{\eta_j \eta_k}{1 + \sum_{\ell=3}^n \eta_\ell^2} \right)^2, \quad (17)$$

where  $\eta_2 \equiv 1$  and we have used the relations

$$r_{ij} \rightarrow 2 \frac{\mu_{ij}}{\Omega_{ij}} \frac{\eta_i \eta_j}{1 + \sum_{k=3}^n \eta_k^2}, \quad \frac{\varrho_i \varrho_j}{1 + \sum_{k=2}^n \varrho_k^2} \rightarrow \frac{\eta_i \eta_j}{1 + \sum_{k=3}^n \eta_k^2}. \quad (18)$$

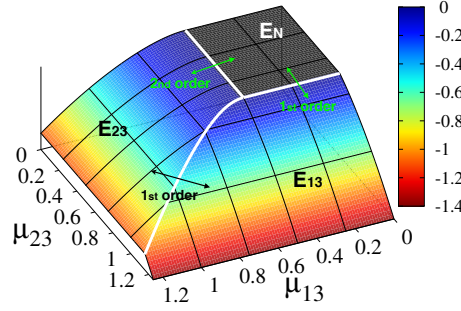
Comparing (17) with (15) it is clear that we have obtained an equivalent system with  $n - 1$  levels where the atomic variables  $\varrho_j$  are replaced by new variables  $\eta_j$ , which upon finding its critical points leads to an equivalent algebraic system of  $n - 2$  levels and so on until we reach a two-level system with one radiation mode whose properties, including the complete structure of the phase diagram, have been studied extensively [26–29].

From Table 1 we can see that the minimum energy surface can be divided into two regimes, the normal (N) and the superradiant. This last regime in turn is divided into two sets of monochromatic regions denoted by  $S_{1k}$  with  $k = 2, \dots, n$  and  $S_{jk}$  with  $j > 1$  and  $k > j$ . In the normal regime there are no photons in the ground state, while for the super-radiant sectors there are mainly photons of the modes  $\Omega_{1k}$ , with  $k = 2, \dots, n$  or of the type  $\Omega_{jk}$  with  $j > 1$  and  $k > j$ . These regions constitute the quantum phase diagram, i.e., the division of parameter space into sectors where the properties of the ground state change significantly.

For values of the dipolar intensities satisfying

$$4\mu_{1k}^2 \leq \omega_k \Omega_{1k}, \quad 4\mu_{jk}^2 \leq (\sqrt{\omega_j} + \sqrt{\omega_k})^2 \Omega_{jk}, \quad (19)$$

the minimum energy value is  $E_{min} = 0$ .



**Figure 2.** Energy surface as a function of the dipolar intensities for atoms in the  $\Lambda$ -configuration. The separatrix (white line) and the order of the transitions are displayed. The normal region  $E_N$  and collective regions  $E_{13}$  and  $E_{23}$  corresponding to the modes  $\Omega_{13}$  and  $\Omega_{23}$ , respectively, are shown. We consider  $\Omega_{13} = 1$ ,  $\Omega_{23} = 0.8$ ,  $\omega_2 = 0.2$  and  $\omega_3 = 1$ .

The points of the quantum phase diagram in the collective region are given by the set of equalities  $E_{jk} = E_{j'k'}$  for the modes  $\Omega_{jk} \neq \Omega_{j'k'}$ :

$$4\mu_{jk}^2 - \frac{\Omega_{jk}}{8\mu_{j'k'}^2\Omega_{j'k'}} \left( \frac{\zeta_j + \zeta_k}{2} \pm \sqrt{\zeta_j \zeta_k} \right) = 0, \quad (20)$$

$$\zeta_\ell = (4\mu_{j'k'}^2 - (\omega_{k'} - \omega_{j'})\Omega_{j'k'})^2 + 8\mu_{j'k'}^2\Omega_{j'k'}(\omega_\ell - \omega_{j'}), \quad \ell = j, k. \quad (21)$$

As an example we show in Fig. 2 the energy surface for the  $\Lambda$  configuration of 3-level atoms interacting with two modes of the electromagnetic field together with the normal and superradiant regimes. The quantum phase diagram appears in white lines and the order of the transitions is also indicated.

The ground state in the normal regime has all the atoms in their lowest energy level and there are no photons, i.e.,

$$|\psi_N\rangle = \frac{\mathbf{b}_1^{\dagger N_a}}{\sqrt{N_a!}} |0\rangle_M \otimes |0\rangle_F. \quad (22)$$

The collective region, denoted by  $S_{1k}$ , with  $k = 2, \dots, n$  takes the form

$$|\Psi_{1k}\rangle = \frac{1}{\sqrt{N_a!}} \left( \frac{\mathbf{b}_1^{\dagger} + \varrho_k^c \mathbf{b}_k^{\dagger}}{\sqrt{1 + \varrho_k^{c2}}} \right)^{N_a} |0\rangle_M \otimes e^{-N_a r_{1k}^{c2}/2} e^{\sqrt{N_a} r_{1k}^c \mathbf{a}_{1k}^{\dagger}} |0\rangle_F, \quad (23)$$

From this expression one can say that the states are equivalent to those associated to a 2-level system with energy levels  $(\omega_1, \omega_k)$  interacting with one-mode of electromagnetic radiation,  $\Omega_{1k}$ .

For the monochromatic regions  $S_{jk}$ , with  $j < k = 3, \dots, n$  one obtains a similar result

$$|\Psi_{jk}\rangle = \frac{1}{\sqrt{N_a!}} \left( \frac{\mathbf{b}_j^{\dagger} + \eta_k^c \mathbf{b}_k^{\dagger}}{\sqrt{1 + \eta_k^{c2}}} \right)^{N_a} |0\rangle_M \otimes e^{-N_a r_{jk}^{c2}/2} e^{\sqrt{N_a} r_{jk}^c \mathbf{a}_{jk}^{\dagger}} |0\rangle_F, \quad (24)$$

where  $j > 1$  and the expression

$$r_{jk}^c = \lim_{\varrho_j^c \rightarrow \infty} r_{jk} \Big|_{\varrho_k^c = \eta_k^c \varrho_j^c} = \frac{2\mu_{jk}\eta_k^c}{(1 + \eta_k^{c2})\Omega_{jk}} \quad (25)$$



was used. In these cases the 2-level systems are associated to the energy levels  $(\omega_j, \omega_k)$  and mode  $\Omega_{jk}$ .

Observe that the different expressions for the ground states are separable states, implying that there is no entanglement between the matter and the electromagnetic field parts. It is well known in the literature that this is not true [25], at least for 2-level atoms interacting with one radiation mode. This separability is modified by restoring the symmetry of the Hamiltonian, that is, by taking linear combinations of the previous variational solutions to guarantee that they have good parity properties with respect to the operators  $K_j$  defined previously.

From the expressions for the minimum energy value in the different regions and the Ehrenfest classification one finds: second order quantum phase transitions at the crossings  $E_N \rightleftharpoons E_{1k}$ ; first order transitions from the normal to the monochromatic regions  $E_N \rightleftharpoons E_{jk}$ , and for the transitions  $E_{jk} \rightleftharpoons E_{j'k'}$ . A first order transition indicates that at least one physical quantity (expectation value of the number of photons, atomic populations, etc.) has a discontinuity.

## 5. Conclusions

We have determined the phase diagrams for  $n$ -level systems interacting with  $\ell$  modes of electromagnetic radiation, with and without the RWA approximation. We have found that there are first and second order phase transitions between the normal and collective regions while the collective regime is divided into monochromatic regions and all the transitions are of first order. This has been supported by the quantum exact calculations of the minimum energy, and expectation values of the number of photons for the three level atomic configuration  $\Xi$  interacting with two modes of electromagnetic radiation [9]. We have shown that the Hamiltonian system in the RWA approximations has  $n$  constants of motion  $K_j$  which in the case without the RWA approximation divides the Hilbert space into  $2^{n-1}$  orthogonal subspaces associated to the parity of  $n - 1$  constants of motion. For each one of the monochromatic regions from the variational procedure we have found analytic expressions for the eigenstates with minimum energy, which are separable states (Vide Eqs. (23,24)). For this reason the statistical properties of the variational solutions have Poisson distributions for the photon-modes while binomial distributions for the matter parts. The parity symmetry can be restored and we hope to recover properly the entanglement properties between the matter and the modes of the electromagnetic field. This work is in progress.

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