Hidden Correlations in Indivisible Qudits as a Resource for Quantum Technologies on Examples of Superconducting Circuits

M A Man'ko[1] **and V I Man'ko**[1,2]

[1] Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia
[2] Moscow Institute of Physics and Technology (State University), Dolgoprudnyi, Moscow Region 141700, Russia

E-mail: `mmanko@sci.lebedev.ru`

Abstract. We show that the density-matrix states of noncomposite qudit systems satisfy entropic and information relations like the subadditivity condition, strong subadditivity condition, and Araki–Lieb inequality, which characterize hidden quantum correlations of observables associated with these indivisible systems. We derive these relations employing a specific map of the entropic inequalities known for density matrices of multiqudit systems to the inequalities for density matrices of single-qudit systems. We present the obtained relations in the form of mathematical inequalities for arbitrary Hermitian $N \times N$-matrices. We consider examples of superconducting qubits and qudits. We discuss the hidden correlations in single-qudit states as a new resource for quantum technologies analogous to the known resource in correlations associated with the entanglement in multiqudit systems.

1. Introduction

Quantum correlations associated, for example, with the phenomenon of entanglement [1] which exists in bipartite and multipartite systems, like systems of qubits or qudits, provide a resource for development of quantum technologies [2]. The properties of correlations can be characterized by different kinds of entropic and information relations known for both classical probability distributions and classical observables (see, for example, [3]) and quantum density matrices of composite systems, like bipartite and tripartite systems, where entropic inequalities given as the subadditivity and strong subadditivity conditions provide information on the presence and degree of correlations [4–6].

Among quantum systems studied in connection with possible applications in quantum technologies, the superconducting circuits based on Josephson junctions are considered from both theoretical and experimental points of view [7–21].

Recently, it was pointed out that quantum correlations known for bipartite and multipartite systems also exist in the systems without subsystems (indivisible or noncomposite systems) [22–25]. We called such correlations in noncomposite systems the hidden correlations. For example, analogs of the entropic and information inequalities like the subadditivity condition, the strong subadditivity condition, and the Araki–Lieb inequality [26] expressed in the form of matrix relations for density matrices of single qudit systems exist for indivisible systems as well.

The aim of this paper is to review the approach developed in [23, 24] for obtaining new entropic inequalities for systems without subsystems and extend this approach for obtaining new entropic inequalities not only for the state density matrices but also for Hermitian matrices of observables. The idea of the approach is based on the fact that any Hermitian matrix can be mapped on the nonnegative matrix with unit trace. The corresponding tool was applied in [27] to obtain the relation of entropy S and energy E of qudit systems, $E + S \leq \ln Z(-1)$, where $Z(\beta)$ is the system partition function, with β being the inverse temperature.

The approach formulated provides the possibility to extend all density-matrix inequalities known for multipartite systems to density-matrix inequalities for indivisible systems as well as to all observables. Here, we concentrate on particular examples of superconducting qudits and finite-dimensional systems.

This paper is organized as follows. In section 2, we review the properties of superconducting circuit states modeled by a parametric quantum oscillator with vibrating voltage and current. In section 3, we present a new inequality for the classical single qudit system in the form of subadditivity condition. In section 4, we discuss the approach to study Hermitian matrices using a map of the matrices on nonnegative matrices to derive new inequalities for Hermitian matrices. In section 5, we consider examples of inequalities for classical probabilities and classical observables. In section 6, we obtain new inequalities for quantum observables on the example of artificial atom realized by a superconducting qudit with $j = 3/2$. In section 7, we give the conclusions and prospectives.

2. Superconducting qubits

Superconducting devices based on application of Josephson junctions are discussed [7–10] as possible technical instruments for developing new technologies, for example, quantum computing. The idea of such applications is related to the fact that the Josephson junction realizes a model of the electric circuit with inductance L and capacitance C [11], i.e., the current and voltage in devices with Josephson junctions vibrate as the momentum and position of a mechanical oscillator does (see, for example, [28]). The frequency of vibrations ω in the electric circuit is determined by a factor proportional to $(LC)^{-1/2}$, and for high frequencies and low temperatures, such that $\hbar\omega \geq T$, the oscillator behaves as a quantum oscillator called the superconducting circuit, where the current and voltage satisfy the Heisenberg uncertainty relation [29], the Schrödinger–Robertson uncertainty relation [30, 31], and the purity-dependent uncertainty relation [32].

If the circuit parameters L and C are constant, the stationary states of the quantum oscillator correspond to the energy levels E_n $(n = 0, 1, 2, \ldots)$. If the oscillator excitations are such that only finite number of levels N is involved, the stationary states of the superconducting circuit are identified with the qudit states of $j = (N-1)/2$. For example, if only two oscillator energy levels are involved, the superconducting circuit realizes a superconducting qubit and, in this case, a set of Josephson junctions models a multiqubit system, which can be employed to provide quantum information devices.

To realize the dynamical (nonstationary) Casimir effect, in [11–13] it was suggested to use the Josephson junction with time-dependent parameters [inductance $L(t)$ and capacitance $C(t)$], which is an analog of the oscillator with time-dependent frequency $\omega(t)$. The current and voltage in parametric superconducting circuits are generated due to temporal variations in the Josephson-junction parameters analogously to the photon in squeezed-states generation in resonators with vibrating boundaries due to the dynamical Casimir effect. The photons created due to the dynamical Casimir effect were registered in the devices where the Josephson junctions are employed [33].

The oscillator with time-dependent frequency is described by the Hamiltonian (we use

dimensionless variables $\hbar = m = \omega(0) = 1$)

$$\hat{H} = \frac{\hat{p}^2}{2} + \omega^2(t)\frac{\hat{q}^2}{2} \,. \tag{1}$$

In [34], it was found that the parametric oscillator has linear in the position and momentum integrals of motion called dynamical invariants (see, for example, [32, 35, 36]) of the form

$$\hat{a}(t) = -\frac{i}{\sqrt{2}}\left(\dot{\mathcal{E}}(t)\hat{q} - \mathcal{E}(t)\hat{p}\right), \tag{2}$$

where the complex function $\mathcal{E}(t)$ satisfies the classical equation of motion for a parametric oscillator $\ddot{\mathcal{E}}(t) + \omega^2(t)\mathcal{E}(t) = 0$ under the initial conditions $\mathcal{E}(t) = 1$, $\dot{\mathcal{E}}(0) = i$, and $\hat{a}(0) = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$. The linear invariants $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfy the commutation relation $\left[\hat{a}(t), \hat{a}^\dagger(t)\right] = 1$. For $\omega(t) = 1$, the integral of motion $\hat{a}(t) = e^{it}\hat{a}(0)$.

An analog of the ground state of the stationary oscillator (superconducting circuit), being the squeezed state, has the wave function

$$\psi_0(x, t) = \left(\pi\mathcal{E}^2(t)\right)^{-1/4} \exp\frac{i\dot{\mathcal{E}}(t)x^2}{2\mathcal{E}(t)} \,. \tag{3}$$

Dispersions of the position $\sigma_{xx}(t)$ and momentum $\sigma_{pp}(t)$ (current and voltage in a superconducting circuit) are determined by the function $\mathcal{E}(t)$ as follows:

$$\sigma_{xx}(t) = \frac{|\mathcal{E}(t)|^2}{2}\,, \qquad \sigma_{pp}(t) = \frac{|\dot{\mathcal{E}}(t)|^2}{2} \,. \tag{4}$$

The correlation coefficient $r(t)$ of the current and voltage $r(t) = \frac{\sigma_{xp}(t)}{\sqrt{\sigma_{xx}(t)\sigma_{pp}(t)}}$ is given by the bound in the Schrödinger–Robertson uncertainty relation

$$\sigma_{xx}(t)\sigma_{pp}(t) \geq \left[4\left(1 - r^2(t)\right)\right]^{-1}, \tag{5}$$

which provides the equality

$$|\mathcal{E}(t)\dot{\mathcal{E}}(t)|^2 = \left[4 - r^2(t)\right]^{-1}. \tag{6}$$

The eigenfunctions of time-dependent invariants $\hat{n}(t) = \hat{a}^\dagger(t)\hat{a}(t)$ are analogs of stationary states of superconducting circuits with time-dependent parameters L and C. The eigenvalues of the integral of motion $\hat{n}(t)$ do not depend on time and take the values $n = 0, 1, 2, \ldots, \infty$.

For finite number of excited states $\mid n\rangle$ ($n = 0, 1, 2, \ldots, N$) of a parametric superconducting circuit, the states can be considered as an approximation of qudit states with $j = (N-1)/2$. These states can be used analogously to the states of stationary superconducting circuits in quantum technologies. For example, if $j = 1/2$, a parametric superconducting qubit can be realized.

The symplectic tomogram (probability distribution) [37] of the squeezed vacuum state (3) of the superconducting circuit

$$w(X, t \mid \mu, \nu) = \frac{1}{2\pi|\nu|}\left|\int\psi_0(y, t)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2 \tag{7}$$

reads

$$w(X, t \mid \mu, \nu) = \frac{1}{\sqrt{2\pi\sigma_{xx}(t)}}\exp\left(-\frac{X^2}{2\sigma_{xx}(t)}\right), \tag{8}$$

where the dispersion $\sigma_{xx}(t)$ of the homodyne quadrature X is

$$\sigma_{xx}(t) = \mu^2 \frac{|\mathcal{E}(t)|^2}{2} + \nu^2 \frac{|\dot{\mathcal{E}}(t)|^2}{2} + \frac{1}{2}\mu\nu\left(|\mathcal{E}(t)\dot{\mathcal{E}}(t)|^2 - 1\right), \tag{9}$$

with μ and ν being the real parameters.

The optical tomogram $w(X, t \mid \theta)$ of squeezed vacuum state of the superconducting circuit has the form (8) with the dispersion $\sigma_{xx}(t)$ of the homodyne quadrature X dependent on the local oscillator phase as

$$\sigma_{xx}(t) = \frac{\cos^2\theta|\mathcal{E}(t)|^2}{2} + \frac{\sin^2\theta|\dot{\mathcal{E}}(t)|^2}{2} + \frac{\sin 2\theta}{2}\left(|\mathcal{E}(t)\dot{\mathcal{E}}(t)|^2 - 1\right). \tag{10}$$

The squeezed coherent state of the parametric superconducting circuit $\mid \alpha, t\rangle$ is such that $\hat{a}(t) \mid \alpha, t\rangle = \alpha \mid \alpha, t\rangle$ has the symplectic tomogram in the form of normal distribution with dispersion $\sigma_{xx}(t)$ given by (9) and the mean value of the homodyne quadrature X

$$\langle X(t)\rangle = \mu\frac{\mathcal{E}^\star(t)\alpha + \mathcal{E}(t)\alpha^\star}{\sqrt{2}} + \nu\frac{\dot{\mathcal{E}}^\star(t)\alpha + \dot{\mathcal{E}}(t)\alpha^\star}{\sqrt{2}}. \tag{11}$$

Thus, the optical tomogram of the squeezed coherent state of the superconducting circuit reads

$$w_\alpha(X, t \mid \theta) = \frac{1}{\sqrt{2\pi\sigma_{xx}(t)}}\exp\left(-\frac{X - \bar{X}_\alpha^2(t)}{2\sigma_{xx}(t)}\right), \tag{12}$$

where $\sigma_{xx}(t)$ is given by (10), and the mean value of the homodyne quadrature is

$$\bar{X}_\alpha(t) = \sqrt{2}\,\mathrm{Re}\left[\alpha\left(\mathcal{E}^\star(t)\cos\theta + \dot{\mathcal{E}}^\star(t)\sin\theta\right)\right]. \tag{13}$$

One can check that tomogram (12) satisfies the entropic inequality valid for an arbitrary optical tomogram [38]

$$-\int w(X, t \mid \theta)\ln w(X, t \mid \theta)\,dX - \int w(X, t \mid \theta + \pi/2)\ln w(X, t \mid \theta + \pi/2)\,dX \geq \ln(\pi e). \tag{14}$$

For example, the state $\mid 1, t\rangle = \hat{a}^\dagger(t) \mid 0, t\rangle$ with the wave function

$$\psi_1(x, t) = -\frac{i}{\sqrt{2}}\left(\mathcal{E}^\star(t)\hat{p} - \dot{\mathcal{E}}^\star(t)x\right)\psi_0(x, t), \tag{15}$$

which is the second component in the superconducting qubit state with the first component given by (3), has the tomogram

$$w_1(X, t \mid \theta) = \pi^{-1/2}(2\sigma_{xx}(t))^{-3/2}X^2\exp\left(-\frac{X^2}{2\sigma_{xx}(t)}\right), \tag{16}$$

where the dispersion $\sigma_{xx}(t)$ is given by (10). Tomogram (16) satisfies the entropic inequality (14). Optical tomograms of the superconducting circuit states $\mid n, t\rangle = \frac{\hat{a}^\dagger(t)^n}{\sqrt{n!}}\mid 0, t\rangle$ are the probability distributions

$$w_n(X, t \mid \theta) = w_0(X, t \mid \theta)\frac{1}{2^n n!}H_n^2\left(\frac{X}{\sqrt{2\sigma_{xx}(t)}}\right), \tag{17}$$

where $H_n(y)$ are Hermite polynomials. The dependence of the tomogram on the local oscillator phase θ is given by the dependence (10) of the dispersion $\sigma_{xx}(t)$ on this parameter.

If the states $\mid n, t\rangle$ with $n = 0, 1, 2, \ldots, N = 2j+1$ are excited, the parametric superconducting circuit can be interpreted as an artificial atom with N levels or qudit with $j = (N-1)/2$.

3. Information inequalities for single qudit states

In this section, we discuss the entropic and information relations, such as equalities and inequalities, as well as quantum correlations of qudit observables for single qudits. Qudit can be realized either by a spin-j particle or the N-level atom. These systems can also be considered as artificial atoms realized by superconducting circuits. In this consideration, we follow [24, 39–41] where a map of integers $1, 2, \ldots, N = mn$ onto pairs of integers (jk), $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$ or, in the case of $N = n_1 n_2 n_3$, on triples of integers (jkl), $j = 1, 2, \ldots, n_1$, $k = 1, 2, \ldots, n_2$ and $l = 1, 2, \ldots, n_3$ was used. This map provides a possibility to apply the relations like entropic inequalities (known for bipartite and tripartite quantum systems) to indivisible systems like a superconducting-circuit qudit.

We demonstrate the inequalities for the probability distribution $P_s = (P_1, P_2, \ldots, P_N)$, where the even number $N = n_1 n_2$, $n_1 = 2$, and $n_2 = N/2$. Then we label the integers s, where $s = 1, 2, \ldots, N$, by the pairs of integers (jk), where $j = 1, 2$ and $k = 1, 2, \ldots, N/2$. We obtain the same set of numbers $P_s \equiv P_{s(jk)} \equiv P_{jk}$. The inequality, known as the subadditivity condition for the joint probability distribution P_{jk}, for P_s reads

$$
-\sum_{s=1}^{N} P_s \ln P_s \leq -\left(\sum_{s=1}^{N/2} P_s\right) \ln \left(\sum_{s=1}^{N/2} P_s\right) - \left(\sum_{s=N/2}^{N} P_s\right) \ln \left(\sum_{s=N/2}^{N} P_s\right)
$$
$$
-\sum_{k=1}^{N/2} (P_k + P_{k+N/2}) \ln(P_k + P_{k+N/2}). \tag{18}
$$

4. Hermitian matrix inequalities

Given Hermitian $N \times N$-matrix h, where $N = nm$. Let the eigenvalues of the matrix h have values with a minimal eigenvalue h_0. The Hermitian $N \times N$-matrix $\rho(x)$, where $x > |h_0|$ given in the block form with $n \times n$-blocks

$$
\rho(x) = (Nx + \operatorname{Tr} h)^{-1}(h^{jk} + x 1_n \delta^{jk}), \qquad j, k = 1, 2, \ldots, m, \tag{19}
$$

is the nonnegative matrix with $\operatorname{Tr} \rho(x) = 1$, where 1_n is the identity $n \times n$-matrix. For the density matrix $h = \rho$, the map $\rho \to \rho(x)$ given by (19) reflects the fact that the noise contribution is taken into account.

One can check that two Hermitian matrices, namely, $m \times m$-matrix $\rho(1, x)$ with matrix elements

$$
\rho_{jk}(1, x) = (Nx + \operatorname{Tr} h)^{-1} \left[\operatorname{Tr}(h^{jk}) + nx \delta^{jk} \right] \tag{20}
$$

and $n \times n$-matrix $\rho(2, x)$ of the form

$$
\rho(2, x) = (Nx + \operatorname{Tr} h)^{-1} \left(mx 1_n + \sum_{k=1}^{m} h^{kk} \right) \tag{21}
$$

are nonnegative matrices and $\operatorname{Tr} \rho(1, x) = \operatorname{Tr} \rho(2, x) = 1$. The matrices satisfy the entropic inequality for the mutual information $I(x)$ of the form

$$
I(x) = \operatorname{Tr} \rho(x) \ln \rho(x) - \operatorname{Tr} \rho(1, x) \ln \rho(1, x) - \operatorname{Tr} \rho(2, x) \ln \rho(2, x) \geq 0. \tag{22}
$$

If $h_0 \geq 0$, one can assume $x = 0$. In this case, inequality (22) for an arbitrary Hermitian $N \times N$-matrix h reads

$$
I(x) = \operatorname{Tr} \left\{ h \ln(h / \operatorname{Tr} h) \right\} - \operatorname{Tr} \left\{ \left[\operatorname{Tr} h^{jk} \right] \ln \left[\left(\operatorname{Tr} h^{jk} \right) / \operatorname{Tr} h \right] \right\}
$$
$$
- \operatorname{Tr} \left\{ \left[\sum_{k=1}^{m} h^{kk} \right] \ln \left[\left(\sum_{k=1}^{m} h^{kk} \right) / \operatorname{Tr} h \right] \right\} \geq 0. \tag{23}
$$

Now we present this inequality on an example of the 3×3-matrix

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}.$$

Since $N = 3$, we may assume this matrix as $\tilde{h} = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$. Then taking in the previous formulas $m = n = 2$, we obtain the 4×4-matrix $\rho(x)$ (20) as follows:

$$\rho(x) = (3x + h_{11} + h_{22} + h_{33})^{-1} \begin{pmatrix} h^{11} + x1_2 & h^{12} \\ h^{21} & h^{22} + x\mathcal{P}_1 \end{pmatrix}, \tag{24}$$

where $\mathcal{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and 2×2-blocks h^{jk} $(j, k = 1, 2)$ read

$$h^{11} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \quad h^{12} = \begin{pmatrix} h_{13} & 0 \\ 0 & 0 \end{pmatrix}, \quad h^{21} = \begin{pmatrix} h_{31} & 0 \\ 0 & 0 \end{pmatrix}, \quad h^{22} = \begin{pmatrix} h_{33} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the 2×2-matrix $\rho(1, x)$ has the form

$$\rho(1, x) = (3x + h_{11} + h_{22} + h_{33})^{-1} \begin{pmatrix} h_{11} + h_{22} + 2x & h_{13} \\ h_{31} & h_{33} + x \end{pmatrix}, \tag{25}$$

and the 2×2-matrix $\rho(2, x)$ reads

$$\rho(2, x) = (3x + h_{11} + h_{22} + h_{33})^{-1} \begin{pmatrix} h_{11} + h_{33} + 2x & h_{12} \\ h_{21} & h_{22} + x \end{pmatrix}. \tag{26}$$

Now we are in the position to present the entropic inequality which is satisfied by the matrix elements of the 3×3-matrix h for $x > h_0$; it is

$$I = \mathrm{Tr}\left\{ \begin{pmatrix} h_{11} + x & h_{12} & h_{13} \\ h_{21} & h_{22} + x & h_{23} \\ h_{31} & h_{32} & h_{33} + x \end{pmatrix} \ln\left[\begin{pmatrix} h_{11} + x & h_{12} & h_{13} \\ h_{21} & h_{22} + x & h_{23} \\ h_{31} & h_{32} & h_{33} + x \end{pmatrix} (3x + h_{11} + h_{22} + h_{33})^{-1} \right] \right\}$$
$$- \mathrm{Tr}\left\{ \begin{pmatrix} h_{11} + h_{22} + 2x & h_{13} \\ h_{31} & h_{33} + x \end{pmatrix} \ln\left[\begin{pmatrix} h_{11} + h_{22} + 2x & h_{13} \\ h_{31} & h_{33} + x \end{pmatrix} (3x + h_{11} + h_{22} + h_{33})^{-1} \right] \right\}$$
$$- \mathrm{Tr}\left\{ \begin{pmatrix} h_{11} + h_{33} + 2x & h_{13} \\ h_{21} & h_{22} + x \end{pmatrix} \ln\left[\begin{pmatrix} h_{11} + h_{33} + 2x & h_{13} \\ h_{21} & h_{22} + x \end{pmatrix} (3x + h_{11} + h_{22} + h_{33})^{-1} \right] \right\} \geq 0. \tag{27}$$

If the matrix h is nonnegative Hermitian matrix, one has inequality (27), where $x = 0$; this inequality reads

$$I = \mathrm{Tr}\left\{ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \ln\left[\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} (h_{11} + h_{22} + h_{33})^{-1} \right] \right\}$$
$$- \mathrm{Tr}\left\{ \begin{pmatrix} h_{11} + h_{22} & h_{13} \\ h_{31} & h_{33} \end{pmatrix} \ln\left[\begin{pmatrix} h_{11} + h_{22} & h_{13} \\ h_{31} & h_{33} \end{pmatrix} (h_{11} + h_{22} + h_{33})^{-1} \right] \right\}$$
$$- \mathrm{Tr}\left\{ \begin{pmatrix} h_{11} + h_{33} & h_{13} \\ h_{21} & h_{22} \end{pmatrix} \ln\left[\begin{pmatrix} h_{11} + h_{33} & h_{13} \\ h_{21} & h_{22} \end{pmatrix} (h_{11} + h_{22} + h_{33})^{-1} \right] \right\} \geq 0. \tag{28}$$

If in inequality (28) $\mathrm{Tr}\, h = 1$ and $h \geq 0$, one arrives at the inequality obtained for the qutrit density matrix in [42]. The discussed inequalities describe relations for the density matrix of qutrit corresponding to an artificial three-level atom.

5. Classical observables as "classical states"

Given random variables $j = 1, 2, \ldots, N$ and the probability distribution P_j, which we call "the classical state." One has $\sum_{j=1}^{N} P_j = 1$. Also given a real function f_j of random variables, one can interpret the function f_j as an observable. The means and higher moments are

$$\langle f \rangle = \sum_{j=1}^{N} f_j P_j, \qquad \langle f^k \rangle = \sum_{j=1}^{N} f_j^k P_j. \tag{29}$$

One can construct a new function $F_j = (f_j + x)n$, where $x > |f_0|$. The number f_0 is a minimum value of the observable f_j, and it can be negative. The normalization factor $n = \left[\left(\sum_{j=1}^{N} f_j \right) + Nx \right]^{-1}$ and the function F_j is nonnegative and normalized, $\sum_{j=1}^{N} F_j = 1$. We can consider the function F_j as a probability distribution, which we associate with the observable f_j, and this fact provides the possibility to extend some relations known for the probability distributions and apply these relations to the observables. For example, we can introduce the Shannon entropy [43] for the observable f_j as follows:

$$S_f(x) = - \sum_{j=1}^{N} [(f_j + x)n \ln(f_j + x)n] \geq 0. \tag{30}$$

Also we can associated the Tsallis q-entropy [44] with the observable f_j as

$$T_f(x) = \frac{1}{1-q} \left\{ \sum_{j=1}^{N} [(f_j + x)n]^q - 1 \right\}. \tag{31}$$

Then we can rewrite the known inequality for relative entropy [2] associated with two probability distributions for two observables $f_j^{(1)}$ and $f_j^{(2)}$; it reads

$$\sum_{j=1}^{N} \left[\left(f_j^{(1)} + x_1 \right) n_1 \ln \left(f_j^{(1)} + x_1 \right) n_1 \right] - \left[\left(f_j^{(1)} + x_1 \right) n_1 \ln \left(f_j^{(2)} + x_2 \right) n_2 \right] \geq 0. \tag{32}$$

Here, x_1 and x_2 are moduli of minimum nonpositive values of observables $f_j^{(1)}$ and $f_j^{(2)}$, respectively. In particular, one can write the inequality for an arbitrary observable f_j and the probability distribution P_j

$$\sum_{j=1}^{N} [P_j \ln P_j - P_j \ln[(f_j + x)n]] \geq 0. \tag{33}$$

Analogously, we can write inequalities for observables in view of Tsallis entropy.

The subadditivity condition is valid for an arbitrary observable f_j as well; for a single system with $N = 4$, we provide the condition in an explicit form. Thus, given an observable f_j, i.e., four numbers f_1, f_2, f_3, and f_4. Assume the number f_4 to be negative, i.e., $f_4 = -|f_4|$ and take the variable $x > |f_4|$. Now we introduce the function $F_j(x)$ taking four values

$$F_1(x) = (f_1 + x)n_4, \quad F_2(x) = (f_2 + x)n_4, \quad F_3(x) = (f_1 + x)n_4, \quad F_4(x) = (f_4 + x)n_4, \tag{34}$$

where $n_4 = (f_1 + f_2 + f_3 + f_4 + 4x)^{-1}$. Then we have the nonnegativity condition for relative entropy associated with the probability distribution P_j and observable f_j of the form

$$\begin{aligned} & P_1 \ln \left\{ P_1[(f_1 + x)n_4]^{-1} \right\} + P_2 \ln \left\{ P_2[(f_2 + x)n_4]^{-1} \right\} \\ & + P_3 \ln \left\{ P_3[(f_3 + x)n_4]^{-1} \right\} + P_4 \ln \left\{ P_4[(f_4 + x)n_4]^{-1} \right\} \geq 0. \end{aligned} \tag{35}$$

The other known inequality is the subadditivity condition for entropy associated with the observable f_j and two other entropies, which are analogs of "subsystem" states described by two probability distributions

$$\mathcal{P}_1(x) = (f_1 + f_2 + 2x)n_4, \qquad \mathcal{P}_2(x) = (f_3 + f_4 + 2x)n_4$$

and

$$\Pi_1(x) = (f_1 + f_3 + 2x)n_4, \qquad \Pi_2(x) = (f_2 + f_4 + 2x)n_4.$$

The inequality

$$S_1(x) + S_2(x) \geq S(x), \tag{36}$$

where

$$S_1(x) = -\sum_{k=1}^{2} \mathcal{P}_k(x) \ln \mathcal{P}_k(x), \quad S_2(x) = -\sum_{k=1}^{2} \Pi_k(x) \ln \Pi_k(x), \quad S(x) = -\sum_{k=1}^{4} F_k(x) \ln F_k(x),$$

reads

$$\begin{aligned}
&-(f_1 + f_2 + 2x) \ln[(f_1 + f_2 + 2x)n_4] - (f_3 + f_4 + 2x) \ln[(f_3 + f_4 + 2x)n_4] \\
&-(f_1 + f_3 + 2x) \ln[(f_1 + f_3 + 2x)n_4] - (f_2 + f_4 + 2x) \ln[(f_2 + f_4 + 2x)n_4] \\
\geq \;&-(f_1 + x) \ln[(f_1 + x)n_4] - (f_2 + x) \ln[(f_2 + x)n_4] \\
&-(f_3 + x) \ln[(f_3 + x)n_4] - (f_4 + x) \ln[(f_4 + x)n_4].
\end{aligned} \tag{37}$$

Thus, the values of an arbitrary classical observable f_j satisfy the inequality, which is an analog of the subadditivity condition for entropies of bipartite classical systems. Formally, inequality (36) is valid for arbitrary real numbers f_j for large numbers x.

6. Quantum inequalities

Hidden correlations described by classical probability distributions and characterized by entropic inequalities take place in quantum systems. For example, the four-level artificial atom realized in the superconducting circuit as a qudit with $j = 3/2$ is described by the density 4×4-matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{pmatrix}.$$

One can check that this Hermitian matrix satisfies the entropic subadditivity condition in spite of the fact that the system under consideration is not bipartite; the inequality reads

$$-\operatorname{Tr}(\rho \ln \rho) \leq -\operatorname{Tr}[\rho(1) \ln \rho(1)] - \operatorname{Tr}[\rho(2) \ln \rho(2)], \tag{38}$$

where $\rho(1) = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{34} \\ \rho_{31} + \rho_{41} & \rho_{33} + \rho_{44} \end{pmatrix}$ and $\rho(2) = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{42} & \rho_{22} + \rho_{44} \end{pmatrix}$. An observable for the artificial atom is described by the Hermitian 4×4-matrix

$$f = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{14} & f_{24} & f_{34} & f_{44} \end{pmatrix}.$$

The matrix $f(1) = \begin{pmatrix} f_{11} + f_{22} & f_{13} + f_{24} \\ f_{31} + f_{42} & f_{33} + f_{44} \end{pmatrix}$ is the Hermitian 4×4-matrix. An analog of the relative entropy nonnegativity can be given through matrices $\rho(1)$ and $f(1)$ as the following inequality:

$$
\mathrm{Tr}\left[\begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{34} \\ \rho_{31} + \rho_{41} & \rho_{33} + \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{34} \\ \rho_{31} + \rho_{41} & \rho_{33} + \rho_{44} \end{pmatrix} - \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{34} \\ \rho_{31} + \rho_{41} & \rho_{33} + \rho_{44} \end{pmatrix}\right]
$$
$$
\times \quad \ln \left\{(f_{11} + f_{22} + f_{33} + f_{44} + 2x)^{-1} \begin{pmatrix} f_{11} + f_{22} + x & f_{13} + f_{24} \\ f_{31} + f_{42} & f_{33} + f_{44} + x \end{pmatrix}\right\} \geq 0. \tag{39}
$$

This new inequality is valid for large x for an arbitrary state of the superconducting circuit with excited four energy levels and the observable.

A new inequality can also be written for qudit tomograms and tomographic symbols of observables. For example, for the case of four-level artificial atom realized by the superconducting circuit, the tomogram of observable f reads

$$
w_f(m, \vec{n}) = (ufu^\dagger)_{mm}, \tag{40}
$$

where $m = 1, 2, 3, 4$ and u is the unitary 4×4-matrix depending on the unit vector $\vec{n}$. In this case, one has four real numbers: $w_f(1, \vec{n})$, $w_f(2, \vec{n})$, $w_f(3, \vec{n})$, and $w_f(4, \vec{n})$. Then a new quantum inequality for observable f and the state tomogram with the density matrix ρ can be written in the form of tomographic relative entropy nonnegativity as follows:

$$
\sum_{m=1}^{4} \left\{ w_\rho(m, \vec{n}) \ln w_\rho(m, \vec{n}) - w_\rho(m, \vec{n}) \ln \left([w_f(m, \vec{n}) + x] \left[\sum_{m'=1}^{4} [w_f(m', \vec{n}) + x] \right]^{-1} \right) \right\} \geq 0. \tag{41}
$$

Thus, we obtained some new quantum inequalities corresponding to quantum correlations available in superconducting qudits taking into account the properties of observables.

7. Conclusions

To conclude, we point out the main results of our study.

We presented a review of the approach for extending the entropic inequalities known for multipartite systems, both classical and quantum, for example, the nonnegativity condition of relative entropy to obtain new inequalities for indivisible systems, including new inequalities for physical observables.

We discussed the example of superconducting circuit and entropic inequalities for an artificial atom with four energy levels and its parametric analog.

We derive a new inequality for the tomographic symbol of an observable of the four-level atom related to quantum correlations in the system.

The new inequalities obtained can be checked experimentally following the procedure considered in [16], where testing entropic inequalities in superconducting qudits were discussed.

The new entropic and information inequalities reflect the presence of quantum correlations in systems like superconducting single qudits. These correlations can be used as a resource for quantum technologies analogously to the correlations related to the entanglement in multiqubit systems and studied as such a resource. In the future publication, we extend our study to the other systems.

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