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Fredholm partial integral equations of second type with degenerate kernel

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Abstract. In this paper we study the solvability of the Fredholm partial integral equations of second type with degenerate kernels.

1. Introduction

Various problems of quantum mechanics [1], quantum field theory [2], partial differential equations [3], mathematical physics [4], and a number of other problems are reduced to special cases of the following integral equation

$$\begin{aligned} f(x,y) &= \int_{\Omega_1} k_1(x,s,y)f(s,y)d\mu_1(s) + \int_{\Omega_2} k_2(x,t,y)f(x,t)d\mu_2(t) \\ &\quad + \int_{\Omega_1}\int_{\Omega_2} k(x,s;y,t)f(s,t)d\mu_1(s)d\mu_2(t) + g(x,y). \end{aligned} \tag{1.1}$$

where Ω_1 and Ω_2 are sets with a finite Lebesgue measure in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, respectively and $k_1 : \Omega_1^2 \times \Omega_2 \to \mathbb{C}$, $k_2 : \Omega_1 \times \Omega_2^2 \to \mathbb{C}$, $k : \Omega_1^2 \times \Omega_2^2 \to \mathbb{C}$, $g : \Omega_1 \times \Omega_2 \to \mathbb{C}$ are given measurable functions, $\mu_1(\cdot), \mu_2(\cdot)$ are the Lebesgue measures on $\sigma-$ algebras of subsets Ω_1 and Ω_2, respectively.

The equation (1.1) contains a partial integrals, i.e. integrals in which an unknown function $f(x,y)$ is integrated by parts of variables. Therefore, this kind of equations are called *partial integral equations (PIE).*

Solvability and properties of the solutions of PIE (1.1) depend on the spaces in which it is considered. The solvability of PIE (1.1) on the space of continuous functions were investigated in [5]-[8].

In 1975, Likhtarnikov and Vitova [9] studied spectral properties of partial integral operators. In [9], the following restrictions were imposed: $k_1(x,s) \in L_2(\Omega_1 \times \Omega_1)$, $k_2(y,t) \in L_2(\Omega_2 \times \Omega_2)$ and $T_0 = K = 0$. In [10], spectral properties of partial integral operators (PIO) with positive kernels were studied (under restriction $T_0 = K = 0$)). In [11] spectral properties of PIO with kernels of two variables in $L_p, p \geq 1$ are studied. In [8], [12]-[14], for more general PIO's with continuous kernels or kernels in $C(L_1)$, spectral properties of the PIO and solvability of partial integral equations in the space of $C([a,b] \times [c,d])$ were studied. In [3] some applications of partial integral equations and operators in solving problems of continuous mechanics, elasticity problems and other problems were considered. Until now, a solvability of PIE in the space L_2 is left open. The present paper is devoted to this problem. Namely, we investigate a solvability of problem for PIE with degenerate kernels from L_2.

Let $(\Omega, \varepsilon, \mu)$ be a measurable space with a finite measure, $L_0(\Omega)$ be the $*$- algebra of equivalence classes of all complex measurable functions on Ω and $L_\infty(\Omega) \subset L_0(\Omega)$ be a subalgebra of equivalence classes of all bounded measurable functions on Ω. By $[f]$ we denote an equivalence class containing a function $f \in L_0(\Omega)$.

By $L_\infty[L_2(\Omega_1), \Omega_2]$ we denote the set of all complex measurable functions $f(x, y)$ on $\Omega_1 \times \Omega_2$ satisfying the following condition: the integral

$$\varphi_0(y) = \int_{\Omega_1} |f(x, y)|^2 d\mu_1(x)$$

exists for almost all $y \in \Omega_2$ and $\varphi_0 \in L_\infty(\Omega_2)$.

In the $L_\infty[L_2(\Omega_1), \Omega_2]$ we define $L_\infty(\Omega_2)$ - valued inner product $\langle f, g\rangle$ by

$$\langle f, g\rangle = \langle f, g\rangle(y) = \int_{\Omega_1} f(x, y)\overline{g(x, y)} d\mu_1(x).$$

Let $\varphi_i, \psi_i \in L_\infty \left[L_2(\Omega_1), \Omega_2\right], \ i = \overline{1, m}$ and

$$k_1(x, s, y) = \sum_{i=1}^{m} \varphi_i(x, y)\psi_i(s, y), \quad (x, s, y) \in \Omega_1^2 \times \Omega_2.$$

Then the partial integral operator (PIO) T_1 defined by

$$T_1 f(x, y) = \int_{\Omega_1} k_1(x, s, y) f(s, y) d\mu_1(s)$$

is linear and bounded on $L_2(\Omega_1 \times \Omega_2)$.

In this paper we study the solvability of the partial integral equation

$$f - T_1 f = g, \tag{1.2}$$

in space $L_2(\Omega_1 \times \Omega_2)$, where $g = g(x, y) \in L_2(\Omega_1 \times \Omega_2)$ is a given function.

2. Solvability of nonhomogeneous partial integral equation with degenerate kernel

If $g(x, y) \neq \theta$, then PIE (1.2) is called *nonhomogeneous Fredholm PIE (NPIE)* of second type with degenerate kernel. The homogeneous partial integral equation (HPIE) corresponding the NPIE (1.2) has the following form

$$h - T_1 h = \theta. \tag{2.1}$$

We define measurable functions τ_{ij} on Ω_2 by

$$\tau_{ij}(\omega) = \int_{\Omega_1} \psi_i(s, \omega)\varphi_j(s, \omega) d\mu_1(s), \ \ , \ i, j = 1, ..., m.$$

One can see that $\tau_{ij} \in L_\infty(\Omega_2)$.

Let $\mathbf{e}$ be an identity element of the algebra $L_\infty(\Omega_2)$, i.e. $\mathbf{e}(\omega) = 1$ for almost all $\omega \in \Omega_2$.

Now we define $m \times m$ matrices $\mathbb{T}$ and $\mathbb{I}$, respectively, whose entries are elements of $L_0(\Omega_2)$, as follows

$$\mathbb{T} = \mathbb{T}(\omega) = (\tau_{ij}(\omega))_{i,j=\overline{1,m}}, \quad \mathbb{I} = \mathbb{I}(\omega) = (\delta_{ij}(\omega))_{i,j=\overline{1,n}},$$

where $\delta_{ii}(\omega) = \mathbf{e}(\omega)$ and $\delta_{ij}(\omega) = \theta(\omega)$ at the $i \neq j$.

Let $\mathbb{D}_1(\omega)$ be a function on Ω_2 given by

$$\mathbb{D}_1(\omega) = \det(\mathbb{T}(\omega) - \mathbb{I}(\omega)), \quad \omega \in \Omega_2.$$

One can see that $\mathbb{D}_1(\omega)$ is a measurable function. Moreover, we have $\mathbb{D}_1 = \mathbb{D}_1(\omega) \in L_\infty(\Omega_2)$. The function $\mathbb{D}_1$ is called a *determinant Fredholm of PIE* (1.2).

Let $\phi \in L_0(\Omega)$. We define its support by the equality $s(\phi) = s_\phi = [\chi_{\Omega(\phi \neq 0)}]$.

Theorem 2.1. *If $s(\mathbb{D}_1) = \boldsymbol{e}$ (i.e. $\mathbb{D}_1(\omega) \neq 0$ for almost all $\omega \in \Omega_2$), then the HPIE (2.1) has a trivial solution in $L_2(\Omega_1 \times \Omega_2)$, and the NPIE (1.2) has a unique solution in $L_0[L_2(\Omega_1)]$.*

Proof. Suppose that $h(x, y)$ is a solution HPIE (2.1). Let us denote

$$b_i(\omega) = \int_{\Omega_1} \psi_i(s, \omega) h(s, \omega) d\mu_1(s), \quad i = 1, \ldots, m. \tag{2.2}$$

Obviously, $b_i \in L_2(\Omega_2)$ and

$$h(x, y) = \sum_{i=1}^{m} b_i(y) \varphi_i(x, y). \tag{2.3}$$

From the equality (2.2) and (2.3) we obtain the following system of equations for the unknown functions $b_1(y), \ldots, b_m(y)$:

$$b_i(y) = \sum_{j=1}^{m} \tau_{ij}(y) b_j(y), \quad i = 1, \ldots, m.$$

Consequently,

$$\begin{cases} (\tau_{11}(y) - 1) b_1(y) + \tau_{12}(y) b_2(y) + \cdots + \tau_{1m}(y) b_m(y) = \theta(y), \\ \tau_{21}(y) b_1(y) + (\tau_{22}(y) - 1) b_2(y) + \cdots + \tau_{2m}(y) b_m(y) = \theta(y), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \tau_{m1}(y) b_1(y) + \tau_{m2}(y) b_2(y) + \cdots + (\tau_{mm}(y) - 1) b_m(y) = \theta(y). \end{cases} \tag{2.4}$$

We can rewrite the last system (2.4) in matrix form as follows:

$$(\mathbb{T} - \mathbb{I})\mathbf{b} = \theta, \tag{2.5}$$

where $\mathbf{b} = \mathbf{b}(\omega)$ columns of the matrix which consisting of functions $b_{11} = b_1(y), b_{21} = b_2(y), \ \ldots, \ b_{m1} = b_m(y)$.

Let $s_{\mathbb{D}_1} = \mathbf{e}$. Then $\mathbb{D}_1(\omega) \neq 0$ for almost all $\omega \in \Omega_2$. It follows that, for almost all $y \in \Omega_2$ equations (2.4) has only a trivial solution, i.e. the equation (2.5) has only zero solution: $\mathbf{b} = \theta$.

An arbitrary solution of the NPIE (1.2) has the form

$$f(x, y) = g(x, y) + \sum_{i=1}^{m} b_i(y) \varphi_i(x, y),$$

where $b_i(y) = \int_{\Omega_1} \psi_i(s, y) f(s, y) d\mu_1(s)$.

Let

$$a_i(\omega) = \int_{\Omega_1} \psi_i(s, \omega) g(s, \omega) d\mu_1(s).$$

It is clear that $a_i \in L_2(\Omega_2)$. For unknown functions $b_1(y),...,b_m(y)$ we obtain the system of equations

$$\begin{cases} (1-\tau_{11}(\omega))b_1(\omega) - \tau_{12}(\omega)b_2(\omega) - \cdots - \tau_{1m}(\omega)b_m(\omega) = a_1(\omega), \\ -\tau_{21}(\omega)b_1(\omega) - (1-\tau_{22}(\omega))b_2(\omega) + \cdots + \tau_{2m}(\omega)b_m(\omega) = a_2(\omega), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\tau_{m1}(\omega)b_1(\omega) - \tau_{m2}(\omega)b_2(\omega) - \cdots - (1-\tau_{mm}(\omega))b_m(\omega) = a_m(\omega). \end{cases}$$

Put $\widetilde{\mathbb{D}_1}(\omega) = \det(\mathbb{I}(\omega) - \mathbb{T}(\omega)) = -\mathbb{D}_1(\omega)$.

a) if $s(a_i) = \theta$ for all $i \in \{1,...,m\}$, then $g(x,y) = \theta$ is a solution of the HPIE (1.2) and there is not another solution of the equation (1.2).

b) Assume that, for some $i_0 \in \{1,...,m\}$ one has $s(a_{i_0}) \neq \theta$. We define measurable functions $\Delta_1(\omega)$, $\Delta_2(\omega)$, $...,\Delta_m(\omega)$ on Ω_2 as follows: the elements in $k-$th column of the determinant $\det(\mathbb{I}(\omega) - \mathbb{T}(\omega))$ we replace by the functions $a_1(\omega), a_2(\omega),...,a_m(\omega)$ and the resulting determinant is denoted by $\Delta_k(\omega)$. For example,

$$\Delta_1(\omega) = \begin{vmatrix} a_1(\omega) & -\tau_{12}(\omega) & \dots & -\tau_{1m}(\omega) \\ a_2(\omega) & 1-\tau_{22}(\omega) & \dots & -\tau_{2m}(\omega) \\ \dots & \dots & \dots & \dots \\ a_m(\omega) & -\tau_{m2}(\omega) & \dots & 1-\tau_{mm}(\omega) \end{vmatrix}.$$

It is easy to see that $\Delta_k \in L_2(\Omega_2)$, $k \in \{1,...,m\}$.

Let $\omega \in \Omega_2$ is a fixed element. In $L_2(\Omega_1)$ we consider the Fredholm second type equation

$$\varphi(x) - (K_\omega\varphi)(x) = g(x,\omega), \tag{2.6}$$

where

$$K_\omega\varphi(x) = \int_{\Omega_1} k_1(x,s,\omega)\varphi(s)d\mu_1(s).$$

The equation (2.6) for every $\omega \in \Omega_2$ has a unique solution

$$\varphi(x) = \varphi_\omega(x) = g(x,\omega) - \sum_{i=1}^{m} \frac{\Delta_i(\omega)}{\mathbb{D}_1(\omega)}\varphi_i(x,\omega).$$

Clearly, the function

$$f_0(x,y) = g(x,y) - \sum_{i=1}^{m} \frac{\Delta_i(y)}{\mathbb{D}_1(y)}\varphi_i(x,y) \tag{2.7}$$

belongs to $L_0[L_2(\Omega_1),\Omega_2]$ and

$$f_0 - T_1f_0 = g,$$

i.e. f_0 is a solution of (1.2).

Now we prove the uniqueness of the solution f_0 for the equation (1.2). Let $f_1 \in L_0[L_2(\Omega_2)]$ be a solution of (1.2) and $f_1 \neq f_0$. Then from the equality $f_1 - T_1f_1 = g$ for a.e. $\omega \in \Omega_2$ we get

$$f_1(x,\omega) - (T_1f_1)(x,\omega) = g(x,\omega),$$

which means

$$f_1(x,\omega) - (K_\omega f_1)(x,\omega) = g(x,\omega) \ \text{ a.e. } \ \omega \in \Omega_2.$$

By the uniqueness of the solution of the Fredholm equation (2.6), we obtain

$$f_1(x,\omega) = g(x,\omega) - \sum_{i=1}^{m} \frac{\Delta_i(\omega)}{\mathbb{D}_1(\omega)} \varphi_i(x,\omega)$$

for a.e. $\omega \in \Omega_2$, i.e. $f_1(x,y) = f_0(x,y)$.

□

3. Solvability of homogeneous partial integral equation with degenerate kernel

We define linear operators $A_k f(x,y) = \Delta_{f,k}(y)$, $k = \overline{1,m}$, on the space $L_2(\Omega_1 \times \Omega_2)$, here $\Delta_{f,k}(\omega)$ corresponds to the determinant, which instead of k-th column of the determinant $\det(\mathbb{I}(\omega) - \mathbb{T}(\omega))$ for the following functions:

$$a_{f,1}(\omega) = \int_{\Omega_1} \psi_1(s,\omega) f(s,\omega) d\mu_1(s), \ \ldots, \ a_{f,m}(\omega) = \int_{\Omega_1} \psi_m(s,\omega) f(s,\omega) d\mu_1(s),$$

respectively.

By Theorem 2.1 and equality (2.7), we get the following theorem

Theorem 3.1. *Let $s(\mathbb{D}_1) = \boldsymbol{e}$. If $\mathbb{D}_1^{-1} \in L_\infty(\Omega_2)$, then the operator $I - T_1$ (I is the identity operator) is invertible and*

$$(I - T_1)^{-1} f(x,y) = f(x,y) + \frac{1}{\mathbb{D}_1(y)} (S_1 f)(x,y),$$

where

$$S_1 f(x,y) = \sum_{i=1}^{m} \varphi_i(x,y) A_i f(x,y), \quad f \in L_2(\Omega_1 \times \Omega_2).$$

Remark 3.2. *Note that each A_i is a linear bounded operator on $L_2(\Omega_1 \times \Omega_2)$ and it is PIO as well.*

Theorem 3.3. *Let $s(\mathbb{D}_1) \neq \boldsymbol{e}$. Then HPIE (2.1) has a nontrivial solution in the $L_2(\Omega_1 \times \Omega_2)$ and moveover, any solution $h(x,y)$ of the equation (2.1) has the form*

$$h(x,y) = \sum_{j=1}^{m} (\boldsymbol{e}(y) - s_{\mathbb{D}_1}(y)) b_j(y) \varphi_j(x,y),$$

where $b_k \in L_2(\Omega_2)$ is arbitrary, $k \in \{1, ..., m\}$.

Proof. Let $h \in L_2(\Omega_1 \times \Omega_2)$ be a solution of the equation (2.1). Then the function h has a form

$$h(x,y) = \sum_{i=1}^{m} d_i(y) \varphi_i(x,y),$$

where $d_i \in L_2(\Omega_2)$ is arbitrary. Let $\mathcal{D}_0 = \{\omega \in \Omega_2 : \ \mathbb{D}_1(\omega) = 0\}$ and $\mathcal{D}_1 = \Omega_2 \setminus \Omega_0$. For each $\omega \in \Omega_2$ we consider the homogeneous Fredholm's equation of the second type in $L_2(\Omega_1)$:

$$\varphi(x) - (K_\omega \varphi)(x) = \theta. \tag{3.1}$$

By the Fredholm's theorem for all $\omega \in \mathcal{D}_1$ the equation (3.1) has only a trivial solution $\varphi(x) = \varphi_\omega(x) = \theta$, and for all $\omega \in \mathcal{D}_0$ the equation (3.1) has not trivial solution in $L_2(\Omega_1)$. For $\omega \in \mathcal{D}_0$ the solution of the equation (3.1) has a form

$$\varphi(x) = \varphi_\omega(x) = \sum_{i=1}^{m} d_i \varphi_i(x, y),$$

where $d_i(\omega) \in \mathbb{C}$ is an arbitrary number.

Put

$$h(x, y) = \sum_{i=1}^{m} (\mathbf{e}(y) - \mathbb{D}_1(y)) b_i(y) \varphi_i(x, y). \tag{3.2}$$

It is easy to verify that

$$h(x, y) = \begin{cases} \theta, & \text{if} \quad \omega \in \mathcal{D}_1, \\ \sum\limits_{i=1}^{m} b_i(\omega)\varphi_i(x, \omega), & \text{if} \quad \omega \in \mathcal{D}_0. \end{cases}$$

Hence, $h(x, \omega) \in L_2(\Omega_1)$ is a solution of (3.1). It is easy to see that $h(x, y) \in L_2(\Omega_1 \times \Omega_2)$ and the function $h(x, y)$ (3.2) is a solution of the HPIE (2.1). $\square$

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