Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

An improvement of extremality regions for Gibbs measures of the Potts model on a Cayley tree

Farhod Haydarov[1], **Rustam Khakimov**[2]

[1] National University of Uzbekistan, Tashkent, Uzbekistan
[2] Institute of Mathematics, National University of Uzbekistan, 29, Do'rmon Yo'li str., 100125, Tashkent, Uzbekistan

E-mail: haydarov_imc@mail.ru, rustam-7102@rambler.ru

Abstract. We give a condition of extemelity for translation-invariant Gibbs measures of $q-$state Potts model on a Cayley tree. We'll improve the regions of extremality for some measures considered in [14]. Moreover, some results in [14] are generalized.

1. Introduction

The Potts model is a generalization of the Ising model. In [3], [4] the q-state Potts model on a Cayley tree of order $k \geq 2$ was studied, and it has been known for a long time that at sufficiently low temperatures, there are at least $q+1$ translation-invariant Gibbs measures. This measures can be considered as tree-indexed Markov chains. Such translation-invariant tree-indexed measures are equivalently called translation- invariant splitting Gibbs measures (TISGMs).

In [6] the uniqueness of the translation-invariant Gibbs measure of the antiferromagnetic Potts model with an external field is proved. In [7] the Potts model with a countable number of states and nonzero external field on a Cayley tree was considered. In that paper, it was established that the model has a unique translation-invariant Gibbs measure.

In [13] all TISGMs (tree-indexed Markov chains) for the Potts model are found on the Cayley tree of order $k \geq 2$, and it is shown that at sufficiently low temperatures their number is $2^q - 1$. In the case $k = 2$ the explicit formulae for the critical temperatures and all TISGMs are given. Further, in [14] by means of methods and results of [10], [21], [15] it has been found some regions for the temperature parameter ensuring that a given TISGM is (non-)extreme in the set of all Gibbs measures. In particular, it was shown the existence of a temperature interval for which there are at least $2^{q-1} + q$ extreme TISGMs. In case of the order of the tree is two, it was given an explicit formulae and some numerical values of the critical temperature. Note that other properties of the Potts model on a Cayley tree were studied in [1, 5, 8, 9, 11, 12],[17]-[20].

In this paper, we consider the $q-$state Potts model on the Cayley tree of order two. Some results of [14] will be improved. Moreover, we will extend Theorem 6 of [14].

2. Definitions and known facts

A Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Im^k = (V, L, i)$, where V is the set of vertices $\Im^k$, L

Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

the set of edges and i is the incidence function setting each edge $l \in L$ into the correspondence with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then the vertices x and y are called the *nearest neighbors*, denoted by $l = \langle x, y\rangle$. The distance $d(x, y), x, y \in V$ on a Cayley tree is defined by

$$d(x, y) = \min\ \{d | \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \ \text{ such that } \ \langle x_0, x_1\rangle, \ldots, \langle x_{d-1}, x_d\rangle\}.$$

For a fixed $x^0 \in V$ we set $W_n = \ \{x \in V \ \mid \ d(x, x^0) = n\}$,

$$V_n = \ \{x \in V \ \mid \ d(x, x^0) \leq n\}, \ \ L_n = \ \{l = \langle x, y\rangle \in L \ \mid \ x, y \in V_n\}.$$

Put

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \ \ x \in W_n.$$

Namely, $S(x)$ is the set of *direct successors* of x.

We consider the model in which the spin variables take values in the set $\Phi = \ \{1, 2, \ldots, q\}$, $(q \geq 2)$ and which are located at the tree vertices. For $A \subset V$ a *configuration* σ_A on A is an arbitrary function $\sigma_A : A \to \Phi$. Note that $\Omega_A = \Phi^A$ is the set of all configurations. We denote that $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

A Hamiltonian of the Potts model is defined as

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2.1}$$

where $J \in R$ and δ_{ij} is the Kronecker symbol.

In this paper, we restrict ourselves to the case of ferromagnetic interaction $J > 0$.

Define a finite-dimensional distribution of a probability measure μ in the volume V_n by

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right\}, \tag{2.2}$$

where $\beta = 1/T$, $T > 0$-temperature, Z_n^{-1} is the normalizing factor, $\{h_x = (h_{1,x}, \ldots, h_{q,x}) \in R^q, x \in V\}$ is a collection of vectors, and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma(x)\sigma(y)}$$

is the restriction of Hamiltonian to V_n.

The probability distributions (2.2) are called compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}), \tag{2.3}$$

here $\sigma_{n-1} \vee \omega_n$ is the concatenation of configurations. In this case, by the well-known Kolmogorov's extension theorem, there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$

$$\mu(\{\sigma \in \Omega : \ \sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

This measure μ is called a *splitting Gibbs measure* corresponding to the Hamiltonian (2.1) and vector-valued function $h_x, x \in V$.

The following statement describes conditions on $\tilde{h}_x$ guaranteeing compatibility of $\{\mu_n\}$.

Theorem 2.1. *(see [3], [19, p.106]) The probability distributions μ_n, $n = 1, 2, \ldots$, in (2.2) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \tag{2.4}$$

where $F : h = (h_1, \ldots, h_{q-1}) \in R^{q-1} \to F(h, \theta) = (F_1, \ldots, F_{q-1}) \in R^{q-1}$ is defined by

$$F_i = \ln\left(\frac{(\theta - 1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}}\right),$$

$\theta = \exp(J\beta)$, and $h_x = (h_{1,x}, \ldots, h_{q-1,x})$ with

$$h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}, \quad i = 1, \ldots, q-1. \tag{2.5}$$

From Theorem 2.1 it follows that for any $h = \{h_x, \quad x \in V\}$ satisfying (2.4) there exists a unique SGM μ for the Potts model.

Note that a translation-invariant splitting Gibbs measure (TISGM) corresponds to a solution h_x of (2.4) with $h_x = h = (h_1, \ldots, h_{q-1}) \in R^{q-1}$ for all $x \in V$. Then from equation (2.4) we get $h = kF(h, \theta)$, and denoting $z_i = \exp(h_i), i = 1, \ldots, q-1$, the last equation can be written as follows

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j}\right)^k, \quad i = 1, \ldots, q-1. \tag{2.6}$$

From [13] the following facts are known:

1. By solving (2.6) the full set of TISGMs is described. It is shown that any TISGM of the Potts model corresponds to a solution of the following equation

$$z = f_m(z) \equiv \left(\frac{(\theta + m - 1)z + q - m}{mz + q - m - 1 + \theta}\right)^k, \tag{2.7}$$

for some $m = 1, \ldots, [q/2]$.

2. Let $\theta_m = 1 + 2\sqrt{m(q-m)}$, $m = 1, \ldots, q-1$.. If $\theta < \theta_1$ then there exists a unique TISGM for $k \geq 2$, $J > 0$. Moreover, each θ_m is a critical value for the change of the number of TISGMs.

3. Extremity conditions

Following [14], to check the extremity of the Gibbs measure, we apply arguments of a reconstruction on trees [2], [10], [16].

From [14] it is known that for each fixed m, the equation (2.7) has up to three solutions: $z_0 = 1, z_i = z_i(\theta, q, m), i = 1, 2$, with $z_1 < z_2$ (see [13, Step 1 of the proof of Theorem 1]). Denote by $\mu_i = \mu_i(\theta, m)$ the TISGM of the Potts model which corresponds to the solution z_i.

For $l = (\underbrace{z, z, \ldots, z}_{m}, \underbrace{1, 1, \ldots, 1}_{q-m})$ a TISGM corresponding to a vector $l \in R^q$ is a tree-indexed Markov chain with states $\{1, 2, \ldots, q\}$ and transition probabilities matrix $\mathbb{P} = (P_{ij})$ with

$$P_{ij} = \frac{l_j \exp(J\beta\delta_{ij})}{\sum_{r=1}^{q} l_r \exp(J\beta\delta_{ir})}. \tag{3.1}$$

Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

From (3.1) we get

$$P_{ij} = \begin{cases} \theta z/Z_1, & \text{if } \ i=j,\, i\in\{1,\dots,m\} \\ z/Z_1, & \text{if } \ i\neq j,\, i,j\in\{1,\dots,m\} \\ 1/Z_1, & \text{if } \ i\in\{1,\dots,m\}, j\in\{m+1,\dots,q\} \\ z/Z_2, & \text{if } \ i\in\{m+1,\dots,q\}, j\in\{1,\dots,m\} \\ \theta/Z_2, & \text{if } \ i=j,\, i\in\{m+1,\dots,q\} \\ 1/Z_2, & \text{if } \ i\neq j,\, i,j\in\{m+1,\dots,q\}, \end{cases} \tag{3.2}$$

where

$$Z_1 = (\theta+m-1)z + q - m, \quad Z_2 = mz + \theta + q - m - 1.$$

Let us first give some necessary definitions from [15]. Considering finite complete subtrees $\mathcal{T}$ that are initial points of Cayley tree Γ^k, i.e. share the same root; if $\mathcal{T}$ has depth d (i.e. the vertices of $\mathcal{T}$ are within distance $\leq d$ from the root) then it has $(k^{d+1}-1)/(k-1)$ vertices, and its boundary $\partial\mathcal{T}$ consists the neighbors (in $\Gamma^k \setminus \mathcal{T}$) of its vertices, i.e., $|\partial\mathcal{T}| = k^{d+1}$. We identify subgraphs of $\mathcal{T}$ with their vertex sets and write $E(A)$ for the edges within a subset A and ∂A for the boundary of A, i.e., the neighbors of A in $(\mathcal{T}\cup\partial\mathcal{T})\setminus A$.

For a given subtree $\mathcal{T}$ of Γ^k and a vertex $x\in\mathcal{T}$, we write $\mathcal{T}_x$ for the (maximal) subtree of $\mathcal{T}$ rooted at x. When x is not the root of $\mathcal{T}$, let $\mu^s_{\mathcal{T}_x}$ denote the (finite-volume) Gibbs measure in which the parent of x has its spin fixed to s and the configuration on the bottom boundary of $\mathcal{T}_x$ (i.e., on $\partial\mathcal{T}_x \setminus \{\text{parent of } x\}$) is specified by τ.

For two measures μ_1 and μ_2 on Ω, $\|\mu_1-\mu_2\|_x$ denotes the variation distance between the projections of μ_1 and μ_2 onto the spin at x, i.e.,

$$\|\mu_1-\mu_2\|_x = \frac{1}{2}\sum_{i=1}^{q} |\mu_1(\sigma(x)=i) - \mu_2(\sigma(x)=i)|.$$

Let $\eta^{x,s}$ be the configuration η with the spin at x set to s.

Following [15] we define

$$\kappa \equiv \kappa(\mu) = \sup_{x\in\Gamma^k} \max_{x,s,s'} \|\mu^s_{\mathcal{T}_x} - \mu^{s'}_{\mathcal{T}_x}\|_x;$$

$$\gamma \equiv \gamma(\mu) = \sup_{A\subset\Gamma^k} \max \|\mu^{\eta^{y,s}}_A - \mu^{\eta^{y,s'}}_A\|_x,$$

where the maximum is taken over all boundary conditions η, all sites $y\in\partial A$, all neighbors $x\in A$ of y, and all spins $s,s'\in\{1,\dots,q\}$.

We apply [15, Theorem 9.3], which says that for an arbitrary channel $\mathbb{P} = (P_{ij})_{i,j=1}^q$ on a tree reconstruction of the corresponding tree-indexed Markov chain (splitting Gibbs measure) is impossible if $k\kappa\gamma < 1$.

Note that κ has the particularly simple form (see [15])

$$\kappa = \frac{1}{2}\max_{i,j}\sum_l |P_{il} - P_{jl}| \tag{3.3}$$

and γ is a constant which does not have a clean general formula, but can be estimated in specific models (as Ising, Hard-Core etc.). For example, if $\mathbb{P}$ is the symmetric channel of the Potts model (i.e. corresponding to the solution $z=1$) then $\gamma \leq \frac{\theta-1}{\theta+1}$ [15, Theorem 8.1].

Using (3.3) and (3.2) for $i \neq j$ we get (see [14])

$$\frac{1}{2}\sum_{l=1}^{q}|P_{il}-P_{jl}| = \begin{cases} a, & \text{if } \ i,j=1,\dots,m \\ b, & \text{if } \ i,j=m+1,\dots,q \\ c, & \text{otherwise,} \end{cases}$$

where a and b are defined by

$$a=\frac{(\theta-1)z}{Z_1}, \quad b=\frac{(\theta-1)\sqrt[k]{z}}{Z_1}, \tag{3.4}$$

$$Z_1=(\theta+m-1)z+q-m, \quad Z_2=mz+\theta+q-m-1,$$
$$c=\frac{1}{2Z_1}\Big(z|\theta-\sqrt[k]{z}|+|1-\theta\sqrt[k]{z}|+(z(m-1)+q-m-1)|1-\sqrt[k]{z}|\Big).$$

Clearly,

$$\kappa=\begin{cases} \max\{b,c\}, & \text{if } \ m=1 \\ \max\{a,b,c\} & \text{if } \ m\geq 2. \end{cases} \tag{3.5}$$

We consider the case $z \neq 1$ (where $z=x^2$ and x is a solution to (2.7)) and fix the solution of (2.6), which has the form $(\underbrace{z,z,\dots,z}_{m},1,\dots,1)$ and the corresponding matrix is $\mathbb{P}$.

For $p_1,p_2,u\geq 0$, $p_1+p_2+u\leq 1$, define the following functions

$$K_1(p_1,p_2,u)=\frac{\theta z p_1}{(\theta-1)zp_1+(1-z)u+z}-\frac{zp_1}{(\theta-1)zp_2+(1-z)u+z};$$

$$K_2(p_1,p_2,u)=\frac{\theta z p_1}{(\theta-1)zp_1+(1-z)u+z}-\frac{zp_1}{(\theta-1)p_2+(1-z)u+z};$$

$$K_3(p_1,p_2,u)=\frac{\theta p_1}{(\theta-z)p_1+(1-z)u+z}-\frac{p_1}{(\theta-1)zp_2+(1-z)(u+p_1)+z};$$

$$K_4(p_1,p_2,u)=\frac{\theta p_1}{(\theta-z)p_1+(1-z)u+z}-\frac{p_1}{(\theta-1)p_2+(1-z)(u+p_1)+z}.$$

Proposition 3.1. *[14]*

1) If $z\geq 1$ then

$$\gamma \leq \max_{\substack{p_1,p_2,u\geq 0:\\ p_1+p_2+u\leq 1}}\{K_1(p_1,p_2,u),K_3(p_1,p_2,u)\}\leq\frac{\theta-1}{\theta+1}. \tag{3.6}$$

2) If $z\leq 1$ then

$$\gamma \leq \max_{\substack{p_1,p_2,u\geq 0:\\ p_1+p_2+u\leq 1}}\{K_2(p_1,p_2,u),K_4(p_1,p_2,u)\}\leq\frac{\theta-1}{\theta+1}+1-z. \tag{3.7}$$

Remark 3.2. *The function $K_1(p_1,p_2,u)$ $(K_3(p_1,p_2,u))$ reaches its maximum for $u=0, p_1=p_2=\frac{1}{2}$, i.e. if $K_1\leq f(\theta,z)$ $(K_3\leq f(\theta,z))$, then $f(\theta,z)\geq\frac{\theta-1}{\theta+1}$ for $z\geq 1$.*

The following Proposition improves the part 2) of Proposition 1.

Proposition 3.3. *If $z \leq 1$ then*

$$\gamma \leq \max_{\substack{p_1,p_2,u\geq 0:\\ p_1+p_2+u\leq 1}} \{K_2(p_1,p_2,u), K_4(p_1,p_2,u)\} \leq \frac{\sqrt{\theta(\theta-1)+\theta z}-\sqrt{z}}{\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}}. \tag{3.8}$$

Proof. We shall find maximum values of functions $K_2(p_1,p_2,u)$ and $K_4(p_1,p_2,u)$. We consider

$$K_2(p_1,p_2,u) = \frac{\theta z p_1}{(\theta-1)zp_1+(1-z)u+z} - \frac{zp_1}{(\theta-1)p_2+(1-z)u+z},$$

where $p_1,p_2,u \geq 0,\ p_1+p_2+u \leq 1,\ z>0,\ \theta>1$. Let be $p_1+p_2+u=\alpha\leq 1$. Note that the function $K_2(p_1,p_2,u)$ is an increasing function of p_2. Then $K_2(p_1,p_2,u) \leq K_2(p_1,p_2+1-\alpha,u)$. Hence, it is sufficient to find the maximum of function $K_2(p_1,p_2,u)$ for $\alpha=1$.

$$K_2(p_1,p_2,u) = \frac{\theta p_1}{\theta p_1+z_1u+p_2} - \frac{p_1}{[(\theta-1)z_1+1]p_2+z_1u+p_1},$$

where $z_1=\frac{1}{z},\ 0<z\leq 1$. Let $\frac{u}{p_1}=x,\ \frac{p_2}{p_1}=y$. Then

$$K_2(p_1,p_2,u) = \frac{\theta}{\theta+z_1x+y} - \frac{1}{1+z_1x+[(\theta-1)z_1+1]y}.$$

Denote $L(x,y)=\frac{\theta}{\theta+z_1x+y}-\frac{1}{1+z_1x+[(\theta-1)z_1+1]y}$. By inequalities $(L(x,y))'_x \leq 0$ (resp.$(L(x,y))'_x \geq 0$) we obtain following inequalities

$$\sqrt{\theta}-(1+(\theta+\sqrt{\theta})z_1)y \leq z_1x, \tag{3.9}$$

$$(resp.\sqrt{\theta}-(1+(\theta+\sqrt{\theta})z_1)y \geq z_1x) \tag{3.10}$$

If (x,y) satisfies (3.9) for some θ and z_1, then

$$L(x,y) \leq \frac{\theta}{\theta+y} - \frac{1}{1+(\theta-1)z_1y+y} = f(y) \leq \max L(0,y).$$

If (3.10) holds then from $0\leq y\leq \frac{\sqrt{\theta}}{1+(\theta+\sqrt{\theta})z_1}$ it follows that

$$L(x,y) \leq L\left(\frac{\sqrt{\theta}-(1+(\theta+\sqrt{\theta})z_1)y}{z_1}, y\right) = \frac{\sqrt{\theta}-1}{(\sqrt{\theta}+1)(1-z_1y)}.$$

Denote $g(y)=\frac{\sqrt{\theta}-1}{(\sqrt{\theta}+1)(1-z_1y)}$. It is easy to check that $g'(y)>0$ for $\theta>1$. Hence the function $g(y)$ is an increasing. Since $y=\frac{\sqrt{\theta}}{1+(\theta+\sqrt{\theta})z_1}<\frac{1}{z_1}$ we have

$$L\left(\frac{\sqrt{\theta}-(1+(\theta+\sqrt{\theta})z_1)y}{z_1}, y\right) \leq L\left(0, \frac{\sqrt{\theta}}{1+(\theta+\sqrt{\theta})z_1}\right) \leq \max L(0,y).$$

Consequently, $L(x,y)\leq \max L(0,y)$. We consider

$$L'(0,y) = -\frac{\theta}{(\theta+y)^2} + \frac{(\theta-1)z_1+1}{(1+[(\theta-1)z_1+1]y)^2} = 0 \ \Leftrightarrow$$

$$\frac{\sqrt{\theta}}{\theta+y} = \frac{\sqrt{(\theta-1)z_1+1}}{1+[(\theta-1)z_1+1]y} \;\Rightarrow y = \frac{\sqrt{\theta}}{\sqrt{(\theta-1)z_1+1}}.$$

Hence

$$\max L(0,y) = L\left(0, \frac{\sqrt{\theta}}{\sqrt{(\theta-1)z_1+1}}\right) = \frac{\sqrt{\theta(\theta-1)z_1+\theta}-1}{\sqrt{\theta(\theta-1)z_1+\theta}+1}.$$

By $z_1 = \frac{1}{z}$ we obtain

$$\max K_2(p_1,p_2,u) = \frac{\sqrt{\theta(\theta-1)+\theta z}-\sqrt{z}}{\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}}.$$

Analogously, we get

$$\max K_4(p_1,p_2,u) = \frac{\sqrt{\theta(\theta-1)+\theta z}-\sqrt{z}}{\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}}.$$

□

Remark 3.4. *The function $K_2(p_1,p_2,u)$ ($K_4(p_1,p_2,u)$) reaches its maximum for $p_1+p_2 = 1$, $\frac{p_2}{p_1} = \frac{\sqrt{\theta}z}{z+\theta+\sqrt{\theta}}$, $u=0$ ($p_1+p_2=1$, $\frac{p_2}{p_1} = \sqrt{\frac{\theta}{z(z+\theta-1)}}$, $u=0$), i.e. if $K_2(p_1,p_2,u) \le f(\theta,z)$ ($K_4(p_1,p_2,u) \le f(\theta,z)$), then $f(\theta,z) \ge \frac{\sqrt{\theta(\theta-1)+\theta z}-\sqrt{z}}{\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}}$ for $z<1$.*

Lemma 3.5. *The following inequality holds*

$$\frac{\sqrt{\theta(\theta-1)+\theta z}-\sqrt{z}}{\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}} \le \frac{\theta-1}{\theta+1}+1-z \tag{3.11}$$

for $0<z\le 1$, $\theta>1$.

Proof. Clearly, (3.11) is equivalent to

$$\frac{-2\sqrt{z}}{\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}} \le \frac{\theta-1}{\theta+1}-z.$$

If $z \le \frac{\theta-1}{\theta+1}$, then the inequality is hold. Let $z > \frac{\theta-1}{\theta+1}$. Then

$$2\sqrt{z} \ge \left(z-\frac{\theta-1}{\theta+1}\right)(\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}).$$

From $\sqrt{\theta(\theta-1)+\theta z} < \theta$ we get

$$\left(z-\frac{\theta-1}{\theta+1}\right)(\sqrt{\theta(\theta-1)+\theta z}+\sqrt{z}) \le (\theta+1)z-(\theta-1).$$

We will prove $(\theta+1)z-(\theta-1) \le 2\sqrt{z}$. Indeed

$$\theta z-\theta+z+1 \le 2\sqrt{z} \;\Leftrightarrow (1-\sqrt{z})^2 \le \theta(1-\sqrt{z})(1+\sqrt{z}).$$

As $1-\sqrt{z} \le 1+\sqrt{z} \le \theta(1+\sqrt{z})$ for $0<z\le 1$ and $\theta>1$, the last inequality is hold. The equality in (3.11) is only right for $z=1$. □

Theorem 3.6. *[14] If $k=2$, $m=1$ then* *(a) There exists $\theta^{**} > \theta_c = q+1$ such that the measure $\mu_1(\theta,1)$ is extreme for any $\theta \in [1+2\sqrt{q-1}, \theta^{**})$, $q \geq 2$.*

(b) The measure $\mu_2(\theta,1)$ is extreme for any $\theta \geq 1+2\sqrt{q-1}$, $q \geq 2$.

From [14] it is known that if $\theta > \theta_c = q+1$ then $z_1 < 1$.

Theorem 3.7. *If $m=1, k=2$ then there exist $\tilde{\theta} > \theta_c = q+1$ such that the measure $\mu_1(\theta,1)$ is extreme for any $\theta \in (1+2\sqrt{q-1},\ \tilde{\theta})$, $q \geq 2$.*

Proof. Let $z_1 < 1$. In this case independently on $q \geq 2$ we get $c = \frac{\theta\sqrt{z_1}-1}{\theta z_1+q-1}$. Using $z_1 \leq 1$ (i.e. $\theta \in [\theta_c, +\infty)$) we get $b \geq c$. Consequently, $\kappa = b$. Hence by Proposition 3.3 we should check

$$2\gamma\kappa \leq 2b\left(\frac{\sqrt{\theta(\theta-1)+\theta z_1} - \sqrt{z_1}}{\sqrt{\theta(\theta-1)+\theta z_1} + \sqrt{z_1}}\right) < 1, \tag{3.12}$$

here $b = \frac{(\theta-1)\sqrt{z_1}}{\theta z_1+q-1}$ for $m=1$ and z_1 is a solution to $z_1 - (\theta-1)\sqrt{z_1} + q - 1 = 0$. Denote

$$f(\theta,q) = \frac{2(\theta-1)\sqrt{z_1}}{\theta z_1+q-1} \cdot \frac{\sqrt{\theta(\theta-1)+\theta z_1} - \sqrt{z_1}}{\sqrt{\theta(\theta-1)+\theta z_1} + \sqrt{z_1}} - 1.$$

It is easy to check

$$f(\theta_c,q) = \frac{\theta-1}{\theta+1} - 1 < 0.$$

Since

$$\sqrt{z_1} = \frac{2(q-1)}{\theta - 1 + \sqrt{(\theta-1)^2 - 4(q-1)}},$$

we get

$$\lim_{\theta\to+\infty} f(\theta,q) > 0.$$

Hence there exist $\theta' \in (\theta_c, +\infty)$ such that $f(\theta',q) = 0$, $\tilde{\theta} = \min\{\theta' : \ f(\theta',q) = 0\}$. Then $f(\theta,q) < 0$ for any $\theta \in [\theta_c, \tilde{\theta})$. Consequently the measure $\mu_1(\theta,1)$ is extreme for any $\theta \in [\theta_c, \tilde{\theta})$.

For $q=3$ ($m=1$) a numerical analysis shows that $\theta^* - \tilde{\theta} = 6.243 - 5.079 = 1.164$ (see Fig.1), where $\theta^* = 1 + (\sqrt{2}+1)q - 2m$ is the critical value of θ above of which the measure $\mu_1(\theta,1)$ is not extreme (see [14]). $\square$

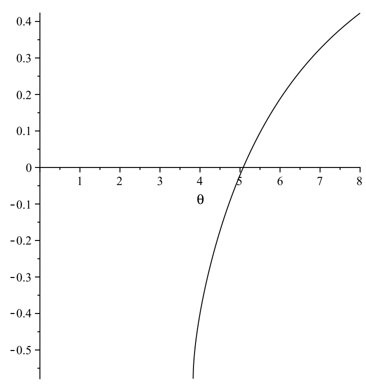


Fig. 1. The graph of the function $f(\theta,q)$ for $q=3$. The solution of $f(\tilde{\theta},3)=0$ is $\tilde{\theta} \approx 5.079$.

Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

Remark 3.8. *1. In [14] for $q = 3$, $z_1 < 1$ the measure $\mu_1(\theta, 1)$ was extreme for $\theta \in (4, \theta^{**})$ and it was non-extreme for $\theta \in (\theta^*, 2 + 3\sqrt{2})$. The difference is $\theta^* - \theta^{**} = 2.0149$. In our case this difference is equal to $\theta^* - \tilde{\theta} = 1.1636$, i.e. it was reduced the interval where it is not known the extremality of the measure $\mu_1(\theta, 1)$.*

Theorem 3.9. *[14]*

(i) If $m = 2$ then for each $q = 4, 5, 6, 7, 8$ there exists $\breve{\theta} > q + 1$ such that the measure $\mu_1(\theta, 2)$ is extreme for any $\theta \in [\theta_2, \breve{\theta})$. Moreover, if $q = 9, 10, 11, 12, 13$ then there exists $\acute{\theta} = \acute{\theta}(q)$ such that $\theta_2 < \acute{\theta} < q + 1$ and $\mu_1(\theta, 2)$ is extreme for $\theta \in [\acute{\theta}, \breve{\theta})$.

(ii) If $m = 2$ then for each $q = 4, 5, 6, 7, 8$ there exists $\grave{\theta} = \grave{\theta}(q)$ such that $\theta_2 < \grave{\theta} \leq q + 1$ and $\mu_2(\theta, 2)$ is extreme for $\theta \in [\theta_2, \grave{\theta})$ (see Fig.7).

(iii) If $q < \frac{m+1}{2m}\left[3m + 1 + \sqrt{m^2 + 6m + 1}\right]$, $m \geq 2$ then the measure $\mu_1(\theta_m, m) = \mu_2(\theta_m, m)$ is extreme.

The following theorem is the generalization of Theorem 3.9

Theorem 3.10. *Let $k = 2$, $m \geq 2$. Then*

1. If $2m \leq q < \frac{m+1}{2m}[3m + 1 + \sqrt{m^2 + 6m + 1}]$, then there exists $\ddot{\theta} > q + 1$ such that the measure $\mu_1(\theta, m)$ is extreme for any $\theta \in [\theta_m, \ddot{\theta})$;

2. If $q > \frac{m+1}{2m}[3m + 1 + \sqrt{m^2 + 6m + 1}]$, then there exists $\overline{\theta} \in (\theta_m, \theta_c)$ such that the measure $\mu_1(\theta, m)$ is extreme for any $\theta \in (\overline{\theta}, \ddot{\theta})$;

3. If $2m \leq q < m + \frac{1}{4m}(m + 1 + \sqrt{m^2 + 2m + 7})^2 = \zeta(m)$, then there exists $\overline{\overline{\theta}} \in (\theta_m, +\infty)$ such that the measure $\mu_2(\theta, m)$ is extreme for any $\theta \in [\theta_m, \overline{\overline{\theta}})$.

Proof. 1. Denote $\frac{m+1}{2m}[3m+1+\sqrt{m^2 + 6m + 1}] = \alpha(m)$. We'll check extremality for the measure $\mu_1(\theta, m)$ with $z_1 \geq 1$. In this case $\theta \in [\theta_m, \theta_c]$. Then it is easy to check that $\kappa = a$. Consequently

$$2\kappa\gamma \leq 2a\frac{\theta - 1}{\theta + 1} < 1.$$

Since

$$\sqrt{z_1} = \frac{2(q - m)}{\theta - 1 + \sqrt{(\theta - 1)^2 - 4m(q - m)}},$$

we get

$$\theta^2 - (2m + 4)\theta - 2m + 3 - (\theta - 3)\sqrt{(\theta - 1)^2 - 4m(q - m)} < 0.$$

Consequently

$$u(\theta) = \theta^3 - (q + 3)\theta^2 + (8q - 10m - 1)\theta - (9q - 8m - 12) > 0. \tag{3.13}$$

We note that $u(\theta_m) > 0$ for $q < \alpha(m)$. Moreover the function $u(\theta)$ increases in $[\theta_m^{(1)}, +\infty]$,

$$\theta_m^{(1)} = \frac{q + 3 + \sqrt{(q + 3)^2 - 3(8q - 10m - 1)}}{3}.$$

From $\left[\frac{q}{2}\right] \geq m$ it follows that $\theta_m^{(1)} \leq \theta_m$. Indeed,

$$\frac{q + 3 + \sqrt{(q + 3)^2 - 3(8q - 10m - 1)}}{3} \leq 1 + 2\sqrt{m(q - m)},$$

which is equivalent to

$$\xi(q) = 2q\sqrt{m(q - m)} + 5m + 2 - 3q - 6m(q - m) \leq 0. \tag{3.14}$$

Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

We compute the derivative

$$\xi'(q) = 2\sqrt{m(q-m)} + \frac{mq}{\sqrt{m(q-m)}} - 3 - 6m.$$

For $m \le \left[\frac{q}{2}\right]$ and $q < \frac{m+1}{2m}[3m+1+\sqrt{m^2+6m+1}]$ the function $\xi(q)$ is increasing. By the last inequality we get

$$q < \frac{2(m+1)^2}{m} \le 2(m+3).$$

Hence $\xi(q) < \xi(2(m+3))$. For $m \ge 2$ we can see easily $\xi(2(m+3)) < 0$ and $\theta_m^{(1)} \le \theta_m$. Thus the function $u(\theta)$ increases on the segment $[\theta_m, \theta_c]$, i.e. $u(\theta) > 0$ for any $\theta \in [\theta_m, \theta_c]$. Consequently the measure $\mu_1(\theta, m)$ is extreme for $z_1 > 1$ and $\theta \in [\theta_m, \theta_c]$.

Case: $z_1 < 1$. In this case we have, independently on the values of q and m,

$$c = \frac{1}{Z_1}\left(\theta\sqrt{z_1} - 1\right).$$

It is easy to see that $b \ge c$ for $z_1 < 1$, i.e. for $\theta > \theta_c$. Consequently, $\kappa = b$ and we should check

$$2\kappa\gamma \le \frac{2(\theta-1)\sqrt{z_1}}{\theta z_1 + q - 1} \cdot \frac{\sqrt{\theta(\theta-1)+\theta z_1} - \sqrt{z_1}}{\sqrt{\theta(\theta-1)+\theta z_1} + \sqrt{z_1}} < 1.$$

Denote

$$g(\theta, q, m) = \frac{2(\theta-1)\sqrt{z_1}}{\theta z_1 + q - 1} \cdot \frac{\sqrt{\theta(\theta-1)+\theta z_1} - \sqrt{z_1}}{\sqrt{\theta(\theta-1)+\theta z_1} + \sqrt{z_1}} - 1.$$

From

$$\sqrt{z_1} = \frac{2(q-m)}{\theta - 1 + \sqrt{(\theta-1)^2 - 4m(q-m)}},$$

we have

$$g(\theta_c, q, m) = \frac{\theta-1}{\theta+1} - 1 < 0, \quad \lim_{\theta\to+\infty} g(\theta, q, m) > 0.$$

Hence there exists $\theta'' \in (\theta_c, +\infty)$ such that $g(\theta'', q, m) = 0$, $\ddot{\theta} = \min\{\theta'' : g(\theta'', q, m) = 0\}$. Then $g(\theta, q, m) < 0$ for any $\theta \in [\theta_c, \ddot{\theta})$, i.e. the measure $\mu_1(\theta, m)$ is extreme for this condition.

Note that for $q = 6$, $m = 2$ we have $\theta^* = 3(1+2\sqrt{2}) \approx 11.485$ and a numerical analysis shows that $\ddot{\theta} \approx 8.22$ (see Fig.2). So $\theta^* - \ddot{\theta} \approx 3.265$.

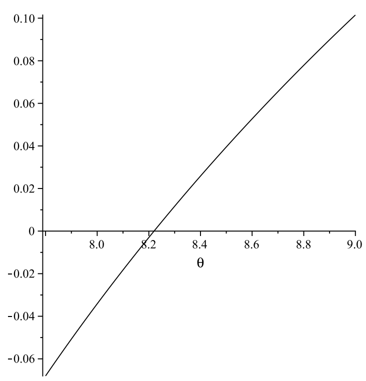


Fig. 2. The graph of the functions $g(\theta, q, m)$ for $q = 6, m = 2$. $\theta^* - \ddot{\theta} \approx 3.265$.

Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

2. The case $z_1 \geq 1$. If $q > \frac{m+1}{2m}[3m+1+\sqrt{m^2+6m+1}]]$, then $u(\theta_m) < 0$. We note that $u(\theta_c) > 0$. Moreover the function $u(\theta)$ increases for $[\theta_m, \theta_c]$. Hence there exists a unique $\overline{\theta} \in (\theta_m, \theta_c]$ such that $u(\overline{\theta}) = 0$. From the condition of extremality (3.13) we get the measure $\mu_1(\theta, m)$ is extreme for $(\overline{\theta}, \theta_c)$.

The case $z_1 < 1$ is similar to the proof of part 1 of the theorem. That's why the measure $\mu_1(\theta, m)$ is extreme for $(\overline{\theta}, \ddot{\theta})$.

3. We have $z_2 \geq 1$ for $\theta \geq \theta_m$. We check the condition of extremality of the measure $\mu_2(\theta, m)$

$$2\kappa\gamma \leq 2c\frac{\theta-1}{\theta+1} < 1,$$

which is equivalent to

$$v(\theta) = \theta^2 - (2m+4)\theta - 2m + 3 + (\theta-3)\sqrt{(\theta-1)^2 - 4m(q-m)} < 0. \tag{3.15}$$

The inequality (3.15) has a solution if the inequality

$$\theta^2 - (2m+4)\theta - (2m-3) < 0,$$

has a solution. The solution of the last inequality has following form

$$\theta < m + 2 + \sqrt{m^2+2m+7}.$$

Since $\theta \geq \theta_m = 1 + 2\sqrt{m(q-m)}$ we obtain

$$1 + 2\sqrt{m(q-m)} < m + 2 + \sqrt{m^2+2m+7}.$$

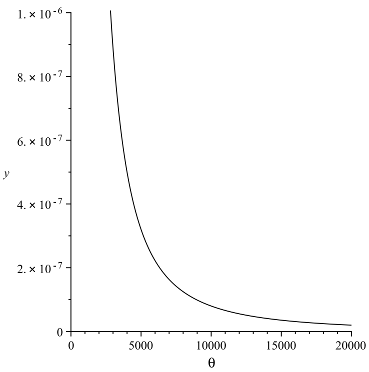
Hence

$$2m \leq q < m + \frac{1}{4m}(m+1+\sqrt{m^2+2m+7})^2.$$

For this condition it is easy to check that $\lim_{\theta\to+\infty} v(\theta) = +\infty$ for this condition. Consequently there exist $\overline{\overline{\theta}} \in (\theta_m, +\infty)$ such that $v(\overline{\overline{\theta}}) = 0$. So the measure $\mu_2(\theta, m)$ is extreme for $\theta \in (\theta_m, \overline{\overline{\theta}})$. □

Remark 3.11. *1. If $q > m + \frac{1}{4m}(m+1+\sqrt{m^2+2m+7})^2$, then the condition of extremality (3.15) does not satisfy for any θ, i.e. in this case it is not known the extremality of the measure $\mu_2(\theta, m)$.*

2. Using a graphs we can see that derivatives $f'(\theta, q) > 0$, $g'(\theta, q, m) > 0$ for finite q and m (see Fig 3.), i.e. functions $f(\theta, q)$ and $g(\theta, q, m)$ are increasing (by analytical method the proof of this statement is very difficult). So values θ', θ'' are uniquely determined by $\tilde{\theta} = \theta'$, $\ddot{\theta} = \theta''$.



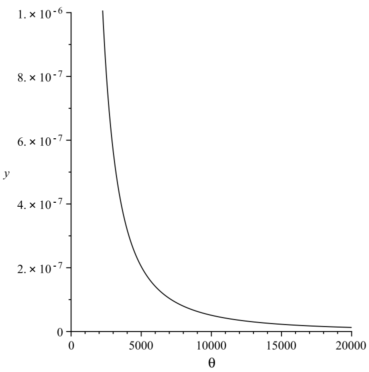


Fig. 3. The graph of the functions $f'(\theta, q)$ for $q = 5, m = 1$ (on the left). The graph of the functions $g'(\theta, q, m) > 0$ for $q = 8, m = 3$ (on the right).

Journal of Physics: Conference Series **697** (2016) 012019 doi:10.1088/1742-6596/697/1/012019

References

[1] Brightwell G R and Winkler P 1999 *J. Combin. Theory Ser.B* **77**(2) 221–262
[2] Formentin M and Külske C 2009 *Electron. Commun. Probab.* 14 587–596
[3] Ganikhodjaev N N 1990 *Theor. Math. Phys.* **85**(2) 1125–1134
[4] Ganikhodjaev N N 1992 *Dokl. AN Uzbekistan* No. 6-7 4–7
[5] Ganikhodjaev N N, Mukhamedov F M and Mendes J F F 2006 *Jour. Stat. Mech.* P08012
[6] Ganikhodzhaev N N and Rozikov U A 1997 *Theor. Math. Phys.* **111**(1) 480–486
[7] Ganikhodzhaev N N and Rozikov U A 2006 *Lett. Math. Phys.* **75**: 2 99–109
[8] Häggström O 1996 *Probab. Theory and Relat. Fields* **104** 231–253
[9] Häggström O and Külske C 2004 *Markov Proc. Rel. Fields* **10**(3) 477–506
[10] Kesten H and Stigum B P 1966 *Ann. Math. Statist.* **37** 1463–1481
[11] Khakimov R M 2014 *Journal of Siberian Federal University. Mathematics and Phisics* **7**(3) 297–304
[12] Khakimov R M 2014 *Theor. and Math. Phys.* **179**(1) 405–415
[13] Külske C, Rozikov U A and Khakimov R M *Jour. Stat. Phys.* **156**(1) 189–200
[14] Külske C and Rozikov U A 2014 arXiv:1403.5775v1
[15] Martinelli F, Sinclair A and Weitz D 2007 *Random Structures and Algoritms* **31** 134–172
[16] Mossel E 2004 In book: *Graphs, morphisms and statistical physics* **63** (New-York: AMS) 155–170
[17] Mukhamedov F M and Rozikov U A 2001 *Theor. Math. Phys.* **126** 169–174.
[18] Mukhamedov F M and Rozikov U A 2000 *Theor. Math. Phys.* **124** 1202–1209.
[19] Rozikov U A 2013 *Gibbs measures on Cayley trees* (Singapore: World Scientific)
[20] Rozikov U A and Khakimov R M 2013 *Theor. Math. Phys.* **175**(2) 699–709
[21] Sly A 2011 *Ann. Probab.* **39** 1365–1406