Quadratic Stochastic Operators with Countable State Space

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Abstract. In this paper, we provide the classes of Poisson and Geometric quadratic stochastic operators with countable state space, study the dynamics of these operators and discuss their application to economics.

1. Introduction

Let $(X, \mathcal{F})$ be a measurable space, $S(X, \mathcal{F})$ be the set of all probability measures on $(X, \mathcal{F})$, and $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ be a family of functions on $X \times X \times \mathcal{F}$ that satisfy the following conditions:

(i) $P(x, y, \cdot) \in S(X, \mathcal{F})$, for any fixed $x, y \in X$;

(ii) $P(x, y, A)$ regarded as a function of two variables x and y with fixed $A \in \mathcal{F}$ is measurable function on $(X \times X, \mathcal{F} \otimes \mathcal{F})$;

(iii) $P(x, y, A) = P(y, x, A)$ for any $x, y \in X, A \in \mathcal{F}$.

We consider a nonlinear transformation (quadratic stochastic operator) $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$ defined by

$$(V\lambda)(A) = \int_X \int_X P(x, y, A)d\lambda(x)d\lambda(x), \tag{1}$$

where $\lambda \in S(X, \mathcal{F})$ is an arbitrary initial probability measure and $A \in \mathcal{F}$ is an arbitrary measurable set.

Note that the third condition $P(x, y, A) = P(y, x, A)$ is not overloaded, since otherwise one can determine a new function

$$Q(x, y, A) = \frac{P(x, y, A) + P(y, x, A)}{2}$$

with preserving quadratic stochastic operator (qso) V, i.e.

$$(V\lambda)(A) = \int_X \int_X P(x, y, A)d\lambda(x)d\lambda(x) = \int_X \int_X Q(x, y, A)d\lambda(x)d\lambda(x).$$

Definition 1.1 *An element $\mu \in S(X, \mathcal{F})$ is called a fixed point of a qso V if $V\mu = \mu$.*

Let $\{V^n\lambda : n = 0,1,2,\cdots\}$ be the trajectory of the initial point $\lambda \in S(X,\mathcal{F})$, where $V^{n+1}\lambda = V(V^n\lambda)$ for all $n = 0,1,2,\cdots$.
In measure theory, there are various notions of the convergence of measures: weak convergence, strong convergence, total variation convergence. Below we consider strong convergence.

Definition 1.2 *For $(X,\mathcal{F})$ a measurable space, a sequence $\{\mu_n\}$ is said to converge strongly to a limit μ if*

$$\lim_{n\to\infty}\mu_n(A) = \mu(A)$$

for every set $A \in \mathcal{F}$.

Definition 1.3 *A qso V is called regular if for any initial point $\mu \in S(X,\mathcal{F})$ the strong limit*

$$\lim_{n\to\infty} V^n(\mu)$$

exists.

If a state space $X = \{1,2,\cdots,m\}$ is finite and the corresponding σ-algebra is the power set $\mathcal{P}(X)$, i.e. the set of all subsets of X, then the set of all probability measures on $(X,\mathcal{P}(X))$ coincides with

$$S^{m-1} = \{\mathbf{x} = (x_1,x_2,\cdots,x_m) \in R^m : x_i \geq 0 \text{ for any } i, \text{ and } \sum_{i=1}^m x_i = 1\} \tag{2}$$

and corresponding qso V has the following form

$$(V\mathbf{x})_k = \sum_{i,j=1}^m P_{ij,k}x_ix_j, \tag{3}$$

where $P_{ij,k} \equiv P(i,j,\{k\})$.
Such operators can be reinterpreted in terms of evolutionary operator of free population, evolutionary games and a gene conversion and in those forms it has a fair history. Note that the theory of qso on finite state space is well developed. The detailed exposure of the theory of quadratic stochastic operators is presented in [1]-[2], [4]-[5], [11-30]. In [6]-[10], [20] it was studied qso defined on countable and continual state spaces.

In statistical mechanics the ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time averages may be replaced by space averages. For nonlinear dynamical systems Ulam [29] suggested as analogue of measure-theoretic ergodicity, following ergodic hypothesis:

Definition 1.4 *A nonlinear operator V defined on $S(X,\mathcal{F})$ is called ergodic, if the limit*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}V^k\lambda$$

exists for any $\lambda \in S(X,\mathcal{F})$.

On the basis of numerical calculations for quadratic stochastic operators defined on $S(X, \mathcal{F})$ with finite X, Ulam conjectured [28] that the ergodic theorem holds for any such qso V. In 1977 Zakharevich [29] proved that this conjecture is false in general. He considered following operator on S^2

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2, \\ x_2' &= x_2^2 + 2x_2x_3 \\ x_3' &= x_3^2 + 2x_1x_3 \end{aligned}$$

and proved that it is non-ergodic transformation. Later in [13] it was established sufficient condition to be non-ergodic transformation for qso defined on S^2 and in [5] described non-ergodic qso defined on S^3 and S^4. In next sections we will show that Ulam's conjecture is true for some classes of qso defined on infinite state space.

2. Countable State Space

In this section we consider non-linear transformations defined on countable state space and investigate their trajectory behavior. Let $X = \{0, 1, \cdots\}$ be a countable sample space and corresponding σ-algebra $\mathcal{F}$ be the power set $\mathcal{P}(X)$. Remind that to define a probability measure μ on countable sample space X it is enough to define the measure $\mu(\{k\})$ of each singleton $\{k\}, k = 0, 1, \cdots$. Below we will write $\mu(k)$ instead of $\mu(\{k\})$. For countable state space X a qso (1) has the following form:

$$V\mu(k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) \tag{4}$$

where $\mu \in S(X, \mathcal{F}), P_{ij,k} \equiv P(j, i, k)$, and $k \in X$.

2.1. Poisson QSO

Remind that a Poisson distribution P_λ with a positive real parameter λ is defined on X by the equation

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in X$$

Definition 2.1 *A quadratic stochastic operator V (4) is called a* Poisson qso *if for any $i, j \in X$, the probability measure $P(i, j, \cdot)$ is the Poisson distribution $P_{\lambda(i,j)}$ with positive real parameter $\lambda(i, j)$.*
We select the following class of a Poisson qso.

Definition 2.2 *A qso V is called m-Poisson, where m is a positive integer, if $\{\lambda(i, j) : i, j \in X\} = \{\lambda_0, \lambda_1, \cdots, \lambda_{m-1}\}$.*

One can define m-Poisson qso as follows. Assume that for any $i, j \in X$, with $i + j \equiv s(mod\ m)$ we have $\lambda(i, j) = \lambda_s$, and

$$P_{ij,k} = e^{-\lambda_s}\frac{\lambda_s^k}{k!} \tag{5}$$

for any $k \in X$ with $k \equiv s(mod\ m)$, where $s = 0, 1, \cdots, m-1$. Then corresponding qso V is a m-Poisson qso.
For any initial measure $\mu \in S(X, \mathcal{F})$ let

$$A_s(\mu) = \sum_{n=0}^{\infty} \mu(mn + s),$$

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where $s = 0, 1, \cdots, m-1$. It is easy to compute $A_s(P_\lambda)$ for the Poisson distribution P_λ. Let

$$B_q(P_\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{nm+q}}{(nm+q)!}$$

with $q = 0, 1, \cdots, m-1$, where $\sum_{q=0}^{m-1} B_q(P_\lambda) = e^\lambda$ and $A_s(P_\lambda) = e^{-\lambda} B_s(P_\lambda)$. Let us consider the equation $\nu^m = 1$ with roots $\nu_q = \cos\frac{2\pi q}{m} + i \sin\frac{2\pi q}{m}$, where $q = 0, 1, \cdots, m-1$. Then

$$e^{\nu_q \lambda} = e^{\cos\frac{2\pi q}{m}} \left(\cos\left(\lambda \sin\frac{2\pi q}{m}\right) + i \sin\left(\lambda \sin\frac{2\pi q}{m}\right) \right)$$

and

$$e^{\nu_q \lambda} = \sum_{s=0}^{m-1} \nu_q^s B_s(P_\lambda),$$

where $q = 0, 1, \cdots, m-1$.
For any initial measure $\mu \in S(X, \mathcal{F})$ we have

$$\begin{aligned} V\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) \\ &= \sum_{s=0}^{m-1} e^{-\lambda_s} \frac{\lambda_s^k}{k!} \left[\sum_{p+q \equiv s (mod\ m)}^{m-1} A_p(\mu) A_q(\mu) \right] \end{aligned}$$

and

$$\begin{aligned} V^2\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k} V\mu(i) V\mu(j) \\ &= \sum_{s=0}^{m-1} e^{-\lambda_s} \frac{\lambda_s^k}{k!} \left[\sum_{p+q \equiv s (mod\ m)}^{m-1} A_p(V\mu) A_q(V\mu) \right] \end{aligned}$$

By simple calculations we have

$$A_t(V\mu) = \sum_{s=0}^{m-1} A_t(P_{\lambda_s}) \left[\sum_{p+q \equiv s (mod\ m)}^{m-1} A_p(\mu) A_q(\mu) \right] \tag{6}$$

Thus by induction for the sequence $V^n(\mu)$ we produce the following recurrent equation

$$V^{n+1}\mu(k) = \sum_{s=0}^{m-1} e^{-\lambda_s} \frac{\lambda_s^k}{k!} \left[\sum_{p+q \equiv s (mod\ m)}^{m-1} A_p(V^n\mu) A_q(V^n\mu) \right] \tag{7}$$

where $n = 0, 1, \cdots,$ and for parameters $A_t(V^n\mu), t = 0, 1, \cdots, m-1$ we have the following recurrent equations

$$A_t(V^{n+1}\mu) = \sum_{s=0}^{m-1} A_t(P_{\lambda_s}) \left[\sum_{p+q \equiv s (mod\ m)}^{m-1} A_p(V^n\mu) A_q(V^n\mu) \right] \tag{8}$$

Let us put $x_p = A_p(V^n\mu)$, where $p = 0, 1, \cdots, m-1$. It is evident that $(x_0, x_1, \cdots, x_{m-1}) \in S^{m-1}$, such that the equations (8) one can rewrite as

$$x'_t = \sum_{s=0}^{m-1} A_t(P_{\lambda_s}) \left[\sum_{p+q\equiv s(mod\ m)}^{m-1} x_p x_q \right], \tag{9}$$

where $t = 0, 1, \cdots, m-1$. Then these system of equations (9) is a qso on finite dimensional simplex S^{m-1} which is denoted by W_1. It is evident that if W_1 is a regular transformation, then from (7) it follows that the limit $\lim_{n\to\infty} V^{n+1}\mu(k)$ exists for any $k \in X$. Thus the problem of investigating limit behaviour of the trajectory qso (4) is reduced to similar problem for qso defined on finite dimensional simplex. In next subsection we consider Geometric qso.

2.2. Geometric qso

A Geometric distribution G_α with a real parameter α, $0 < \alpha < 1$, is defined on countable set X by the equation

$$G_\alpha(k) = \alpha(1-\alpha)^k, \quad k \in X.$$

Definition 2.3 *A quadratic stochastic operator V (4) is called a* Geometric qso *if for any $i, j \in X$, the probability measure $P(i, j, \cdot)$ is the Geometric distribution with parameter $\alpha(i, j)$. We select the following class of a Geometric qso.*

Definition 2.4 *A qso V is called m-Geometric, where m is a positive integer, if $\{\alpha(i, j) : i, j \in X\} = \{\alpha_0, \alpha_1, \cdots, \alpha_{m-1}\}$.*

One can define m-Geometric qso as follows. Assume that for any $i, j \in X$, with $i+j \equiv s(mod\, m)$ we have $\alpha(i, j) = \alpha_s$, and

$$P_{ij,k} = \alpha_s(1-\alpha_s)^k$$

for any $k \in X$, where $s = 0, 1, \cdots, m-1$. Then the corresponding qso V is a m-Geometric one. As above one can produce the following recurrent equations

$$V^{n+1}\mu(k) = \sum_{s=0}^{m-1} \alpha_s(1-\alpha_s)^k \left[\sum_{p+q\equiv s(mod\ m)}^{m-1} A_p(V^n\mu)A_q(V^n\mu) \right] \tag{10}$$

where $n = 0, 1, \cdots,$ and for parameters $A_t(V^n\mu), t = 0, 1, \cdots, m-1$ we have the following recurrent equations

$$A_t(V^{n+1}\mu) = \sum_{s=0}^{m-1} A_t(G_{\alpha_s}) \left[\sum_{p+q\equiv s(mod\ m)}^{m-1} A_p(V^n\mu)A_q(V^n\mu) \right], \tag{11}$$

where

$$A_t(G_{\alpha_s}) = \alpha_s \frac{(1-\alpha_s)^t}{1-(1-\alpha_s)^m}.$$

Now we assume $x_p = A_p(V^n\mu)$, where $p = 0, 1, \cdots, m-1$. Then $(x_0, x_1, \cdots, x_{m-1}) \in S^{m-1}$, and the equations (8) can be rewritten as

$$x'_t = \sum_{s=0}^{m-1} A_t(G_{\alpha_s}) \left[\sum_{p+q\equiv s(mod\ m)}^{m-1} x_p x_q \right], \tag{12}$$

where $t = 0, 1, \cdots, m-1$. These system of equations (11) is a qso on finite dimensional simplex S^{m-1} which is denoted by W_2. It is evident that if W_2 is regular, then from (10) one finds that the limit $\lim_{n\to\infty} V^{n+1}\mu(k)$ exists for any $k \in X$.

In next Section 3 we show the regularity of W_1 and W_2.

3. Regularity of quadratic stochastic operators W_1 and W_2

First we recall a construction of qso [30]. Let G be a finite commutative group and $S(G)$ be a set of all probability measures on G. It is evident, that if $|G| = m$, then $S(G)$ coincides with S^{m-1}.

Let further $H \subset G$ be a subgroup of G and $\{g + H : \ g \in G\}$ be the cosets of H in G. Assume $\mu \in S(G)$ is a fixed positive measure, that is $\mu(g) > 0$ for any $g \in G$. Then we define the coefficients $p_{fg,h}$, where $f, g, h \in G$ in the following way:

$$p_{fg,h} = \begin{cases} \frac{\mu(h)}{\mu(f+g+H)} & \text{if } h \in f + g + H; \\ 0 & \text{otherwise,} \end{cases}$$

where $p_{fg,h} = p_{gf,h}, \ \ p_{fg,h} > 0$ for any f, g, h and $\sum_{h\in G} p_{fg,h} = 1$. It is evident that if $H = \{e\}$, where e is the neutral element of group G, then

$$p_{fg,h} = \begin{cases} 1 & \text{if } h = f + g; \\ 0 & \text{otherwise} \end{cases}$$

and corresponding qso V is defined as

$$(Vx)_h = \sum_{f,g\in G, f+g=h} x_f x_g. \tag{13}$$

In [31] the authors studied qso (13) and proved the following statement.

Theorem 3.1 *Almost all orbits of the qso V tend to the center of the simplex.*

Below we apply this construction to investigate the operators W_1 and W_2, respectively.

Let $G = Z_m$ and Q be a qso defined on $S(Z_m)$ by the trivial subgroup. Namely, the qso Q is defined as follows:

$$(Qx)_k = \sum_{i,j\in Z_m: i+j\equiv k (mod m)} x_i x_j, \tag{14}$$

It is evident that any heredity coefficient $p_{ij,k}$ of the operator Q is equal to either 0 or 1.

Let $\Pi = ||p_{ij}||_{i,j=0}^{m-1}$ be a right stochastic matrix with each row summing to 1, and $W = \Pi Q$ be a composition of the operators Q and Π, i.e.

$$(Wx)_k = \sum_{i=0}^{m-1} p_{ki}(Qx)_i, \tag{15}$$

where Qx and Wx are column vectors. One can see that W is also a qso such that, for any fixed k its heredity coefficient satisfies $P_{ij,k} \in \{p_{k0}, p_{k1}, \cdots, p_{k,m-1}\}$. Note that the quadratic stochastic operators (9) and (12) have the form (15) with right stochastic matrix $\Pi = ||p_{ij}||_{i,j=0}^{m-1}$, where $p_{ki} = A_k(G_{\lambda_i})$ and $p_{ki} = A_k(G_{\alpha_i})$, respectively, for all $i, j = 0, 1, \cdots, m-1$.

In [18] and [19] the authors proved that if all heredity coefficients of qso are positive, and $\frac{p_{i_1k,j}}{p_{i_1k,j}} \leq \mu < 3$, then the qso has a single fixed point, and all trajectories converge to this fixed point.

It is evident that for any Geometric distribution G_α we have $A_0(G_\alpha) > A_1(G_\alpha) > A_2(G_\alpha) > \cdots > A_{m-1}(G_\alpha)$ and $\frac{A_0(G_\alpha)}{A_{m-1}(G_\alpha)} < 3$ if $\alpha < 1 - \frac{1}{\sqrt[m]{3}}$. Thus we have the following statement.

Theorem 3.2 *Let V be an $m-$Geometric qso with parameters $\alpha_0, \alpha_1, \cdots, \alpha_{m-1}$. If $\alpha_0, \alpha_1, \cdots, \alpha_{m-1} < 1 - \frac{1}{\sqrt[m]{3}}$, then V is a regular qso.*

As shown above for any Poisson distribution P_λ one can compute explicitly $A_0(P_\lambda), A_1(P_\lambda), \cdots, A_{m-1}(P_\lambda)$ for any positive integer m. Then one can show that there exists λ^* such that $\frac{A_i(P_\lambda)}{A_j(P_\lambda)} < 3$ for any $\lambda > \lambda^*$. For example, if $m = 2$ we have $A_0(P_\lambda) = \frac{1+e^{-2\lambda}}{2}$, $A_1(P_\lambda) = \frac{1-e^{-2\lambda}}{2}$ and $\frac{A_0(P_\lambda)}{A_1(P_\lambda)} < 3$ if $\lambda > \frac{\ln 2}{2}$, i.e. $\lambda^* = \frac{\ln 2}{2}$. Hence, we have the following statement.

Theorem 3.3 *Let V be an $m-$Poisson qso with parameters $\lambda_0, \lambda_1, \cdots, \lambda_{m-1}$. If $\lambda_0, \lambda_1, \cdots, \lambda_{m-1} > \lambda^*$, then V is a regular qso.*

Remark 3.4 *In [19] Lyubich conjectured that any qso V with positive heredity coefficients has a unique or at least finitely many fixed points. By numerical analysis for small m ([6]-[9]) we have checked uniqueness of fixed point for the considered qso. Therefore, for any m, one can expect that $m-$Geometric and $m-$Geometric qso's have unique fixed points.*

4. Applications

In this section we consider an application to economics of qso with infinite state space, using a model introduced by Föllmer [3]. He considered a countably set A of economic agents (as in physical examples the number of agents considered is often quite large, and therefore, it was considered an infinite case to approximate such situations), each being in a state s specified by his preferences and resources. He then allowed an interaction between different agents in the following manner. First, the environment e of the economic agent a is a configuration on $A - a$ which specifies the states of the other agents. The collection of local (microeconomic) characteristics of the form $r_a(s|e)$ can then be given as the conditional probability that a is in state s given the environment e. Then any probability measure μ which possesses the local characteristics is given by r called a *global phase* of the economy.

A price is then defined as a vector $p = (p_1, \cdots, p_k)$. Then, based on some maximization scheme using the agent's preferences, a well-defined *individual excess demand* $\zeta(\omega(a), p)$ is determined. The individual's excess demand is to be thought of as the difference between his demand and what he already has. The price p is said to stabilize the global phase μ of the economy E if

$$\lim \frac{1}{|A_n|} \sum_{a\in A_n} \zeta(\omega(a), p) = 0, \tag{16}$$

whenever A_n is an increasing sequence of finite subsets of A which exhausts A. Equation (16) is interpreted as having the per capita excess demand going to 0. Föllmer [3] showed that for the Ising model, if there are two pure phases then there is no price which makes (16) true. In [4] the authors proposed a construction of quadratic stochastic operator V_μ using Gibbs state μ of some lattice model. Applying this construction for Föllmer model one can define qso on infinite state space, namely countable infinite set A of economic agents and investigate ergodicity of this qso. The question then consideredas follows: is there any correlation between ergodicity of the qso and stabilization the price (16)?

Theorem 4.1 *If for the Ising model there are two pure phases μ^* and μ^{**} then corresponding qso V_{μ^*} and $V_{\mu^{**}}$ are non ergodic transformation.*

The Sketch of the Proof Let $\{\mu_n\}$ be a sequence of conditional Gibbs measures that converges to pure phase μ. Using the construction suggested in [4] one can define the sequence of qso $\{V_{\mu_n}\}$ defined on finite state space and show that for $T < T_c$ (see [4]) qso V_{μ_n} is non ergodic transformation. By the standard calculus arguments, one can prove that there exists limit of the sequence qso $\{V_{\mu_n}\}$ and limit transformation is non ergodic qso. Thus if for the Ising model occurs the phase transition, then there is no price which makes (16) true and respectively the corresponding qso is non ergodic transformation.

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5. References

[1] Akin E and Losert V 1984 *J. Math. Biology* **20** 231
[2] Bernstein S 1924 *Uchn. Zapiski. NI Kaf. Ukr. Otd. Mat.* **1** 83 (Russian)
[3] Föllmer H 1974 *Jour. Math. Economics***1** 51
[4] Ganikhodjaev N N and Rozikov U A 2006 *Regular Chaotic Dyn.* **11** 467
[5] Ganikhodjaev N, Ganikhodjaev R and Jamilov U 2015 *Ergodic Th. Dyn. Sys.* **35** 1443
[6] Ganikhodjaev N and Hamzah N Z A 2014 *The Scientific World Jour.* **2014** ID 832861
[7] Ganikhodjaev N and Hamzah N Z A *Third International Conf. Math. Appl. Engineering (ICMAE14)* (to appear)
[8] Ganikhodjaev N and Hamzah N Z A 2015 *AIP Conf. Proc.* **1643** 706
[9] Ganikhodjaev N and Hamzah N Z A *Simposium Kebangsaan Sains Matematik ke-22" (SKSM22)* (to appear)
[10] Ganikhodjaev N and Hamzah N Z A 2015 *AIP Conf. Proc.* **1660** 050025
[11] Ganikhodjaev N, Saburov M and Jamilov U 2013 *App Math Info Sci* **7** 1721
[12] Ganikhodjaev N, Saburov M and Nawi A M 2014 *The Scientific World Journal* **2014** 1
[13] Ganikhodjaev N and Zanin D 2004 *Russian Math. Surveys* **59** 571
[14] Ganikhodzhaev R 1989 *Dokl. Akad. Nauk Uz* **1** 3
[15] Ganikhodzhaev R 1993 *Acad. Sci. Sb.Math.* **76** 489
[16] Ganikhodzhaev R 1994 *Math. Notes* **56** 1125
[17] Ganikhodzhaev R, Mukhamedov F and Rozikov U 2011 *Infinite Dim. Anal. Quantum Probab. Related Topics* **14** 279
[18] Kesten H 1972 *Adv. App. Prob.* **2** 1
[19] Lyubich Yu 1992 *Mathematical structures in population genetics* (Berlin: Springer)
[20] Mukhamedov F, Akin H and Temir S. 2005 *J. Math. Anal. Appl.* **310** 533
[21] Mukhamedov F and Ganikhodjaev N 2015 *Lect. Notes Math.* **2133** (Berlin: Springer)
[22] Mukhamedov F and Saburov M 2014 *Bull. Malays. Math. Sci. Soc* **37** 59
[23] Nagylaki T 1983 *Proc. Natl. Acad. Sci. USA* **80** 5941
[24] Nagylaki T 1983 *Proc. Natl. Acad. Sci. USA* **80** 6278
[25] Saburov M 2007 *Dokl. Acad. N. Rep. Uz.* **6** 8
[26] Saburov M 2012 *Proc. Inter. Conf. Appl. Sci. & Tech.* 54
[27] Saburov M 2012 *Dokl. Acad. Nauk Rep. Uzb.* **3** 9
[28] Saburov M 2013 *World Applied Science Journal* **21** 94
[29] Ulam S 1960 *A collection of mathematical problems* (New-York: Interscience)
[30] Zakharevich M 1978 *Russian Math. Surv.* **33** 265
[31] Ganikhodjaev N, Wahiddin M R B and Zanin D 2007 arXiv:0708.0697