Journal of Physics: Conference Series **697** (2016) 012014 doi:10.1088/1742-6596/697/1/012014

Dynamics of Double Stochastic Operators

Mansoor Saburov

Department of Computational & Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, P.O. Box, 25200, Kuantan, Pahang, MALAYSIA

E-mail: msaburov@iium.edu.my, msaburov@gmail.com

Abstract. A double stochastic operator is a generalization of a double stochastic matrix. In this paper, we study the dynamics of double stochastic operators. We give a criterion for a regularity of a double stochastic operator in terms of absences of its periodic points. We provide some examples to insure that, in general, a trajectory of a double stochastic operator may converge to any interior point of the simplex.

1. Introduction

A notion of a double stochastic operator (in short DSO) was firstly introduced in the paper [7] as a generalization of a double stochastic matrix in a class of nonlinear operators. For reasons of self-exposition, it is convenient to provide some necessary notations and notions in the theory of majorization (for a detail, see [1, 2, 13]).

Let $\|\mathbf{x}\|_1 = \sum\limits_{k=1}^{m} |x_k|$ be a norm of a vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$. We say that $\mathbf{x} \geq 0$ (resp. $\mathbf{x} > 0$) if $x_k \geq 0$ (resp. $x_k > 0$) for all $k = \overline{1,m}$. Let $S^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 = 1, \ \mathbf{x} \geq 0\}$ be the $(m-1)-$dimensional standard simplex. An element of the simplex S^{m-1} is called *a stochastic vector*. Recall that a square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is called *stochastic* if every row is a stochastic vector. A square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is said to be *double stochastic* if every row and column are stochastic vectors. For a given vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$, let $x_{[1]} \geq \cdots \geq x_{[m]}$ denote the components of $\mathbf{x}$ in a non-increasing order and $\mathbf{x}_{\downarrow} = (x_{[1]}, \cdots, x_{[m]})$. We set $\mathbb{R}^m_{\downarrow} = \{x \in \mathbb{R}^m : x_1 \geq \cdots \geq x_m\}$.

We say that $\mathbf{x}$ is majorized by $\mathbf{y}$ written $\mathbf{x} \prec \mathbf{y}$ if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad \forall\ k = \overline{1, m-1,} \qquad \sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]}.$$

The following results are the classical results in the theory of majorization [13].

Theorem 1.1. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ be vectors. The following statements are equivalent:*

(i) The vector $\mathbf{x}$ is majorized by $\mathbf{y}$, i.e., $\mathbf{x} \prec \mathbf{y}$;

(ii) One has $D\mathbf{y} = \mathbf{x}$ for some double stochastic matrix D;

(iii) The vector $\mathbf{x}$ is in the convex hull of the $m!$ permutations of $\mathbf{y}$.

(iv) One has $\sum \phi(x_i) \leq \sum \phi(y_i)$ for all convex continuous functions ϕ;

We shall interchangeably use three equivalent statements $(i) - (iii)$ throughout this paper. A set of all $m!$ permutations of x and its convex hull are denoted, respectively, by $Per_{m!}(\mathbf{x})$ and $Conv(Per_{m!}(\mathbf{x}))$. Due to Theorem 1.1, the conditions (i) and (ii) are equivalent. Therefore, we can give an equivalent definition of a double stochastic matrix which is convenient in nonlinear settings: a matrix D is said to be *double stochastic* if one has that $D\mathbf{x} \prec \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^m$.

Any mapping $V : S^{m-1} \to S^{m-1}$ is called *a stochastic operator.*

Definition 1.2 ([7])**.** *A stochastic operator $V : S^{m-1} \to S^{m-1}$ is said to be double stochastic if one has that $V(\mathbf{x}) \prec \mathbf{x}$ for any $\mathbf{x} \in S^{m-1}$.*

Proposition 1.3. *Let $V : S^{m-1} \to S^{m-1}$ be a stochastic operator. Then the following statements are equivalent:*

(i) V is a double stochastic operator;
(ii) One has that $V(Conv(Per_{m!}(\mathbf{x}))) \subset Conv(Per_{m!}(\mathbf{x}))$ for any $\mathbf{x} \in S^{m-1}$.

Proof. It follows from Theorem 1.1 that one has that $\mathbf{x} \prec \mathbf{y}$ if and only if $Conv(Per_{m!}(\mathbf{x})) \subset Conv(Per_{m!}(\mathbf{y}))$. Consequently, one has that $V(\mathbf{x}) \prec \mathbf{x}$ for any $\mathbf{x} \in S^{m-1}$ if and only if $V(Conv(Per_{m!}(\mathbf{x}))) \subset Conv(Per_{m!}(\mathbf{x}))$ for any $\mathbf{x} \in S^{m-1}$. □

Sometimes, the following geometric definition of a DSO is very useful in a practice.

Definition 1.4. *A stochastic operator $V : S^{m-1} \to S^{m-1}$ is said to be double stochastic if one has that $V(Conv(Per_{m!}(\mathbf{x}))) \subset Conv(Per_{m!}(\mathbf{x}))$ for any $\mathbf{x} \in S^{m-1}$.*

Throughout this paper, we shall consider a continuous DSO without mentioning "continuity".

A cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is called *stochastic* if $\sum\limits_{k=1}^m p_{ijk} = 1,\ p_{ijk} \geq 0,\ \forall i,j,k = \overline{1,m}.$ Every cubic stochastic matrix is associated with a quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ as follows

$$(V(\mathbf{x}))_k = \sum_{i,j=1}^m x_i x_j p_{ijk}, \quad \forall\ k = \overline{1,m}. \tag{1.1}$$

A Birkhoff theorem states that a set of extreme points of a set of double stochastic matrices coincides with a set of all permutations matrices. One of the main purposes of studying DSO was to solve Birkhoff's problem in a class of quadratic double stochastic operators. However, Birkhoff's problem remains open in the class of quadratic double stochastic operators [9, 10].

By being the simplest nonlinear mapping, a quadratic stochastic operator has an incredible application in population genetics[3, 5, 6, 11, 12], control systems [23, 24]. In population genetics, the quadratic stochastic operator describes a distribution of the next generation of the system if the current distribution is given [12, 26]. In this sense, the quadratic stochastic operator is a primary source for investigations of evolution of population genetics. The detailed exposure of the theory of quadratic stochastic operators is presented in [8, 10],[14]-[22].

Let $\mathbf{Fix}(V) = \{\mathbf{x} \in S^{m-1} : V(\mathbf{x}) = \mathbf{x}\}$ be a fixed point set. Due to Brouwer's theorem, $\mathbf{Fix}(V) \neq \emptyset$. In this paper, we study the dynamics of double stochastic operators. We give a criterion for a regularity of a double stochastic operator in terms of absences of its periodic points. This answers Problem 2.35 mentioned in the survey paper [8]. We provide some examples to insure that, in general, a trajectory of a double stochastic operator may converge to any interior point of the simplex.

2. Examples for Double Stochastic Operators

Let S^1 be 1D simplex, $e_1 = (1,0)$ and $e_2 = (0,1)$ be vertexes of the simplex S^1. Let $[a,b] = \{(1-\lambda)a + \lambda b\}_{0\le\lambda\le 1}$ be a closed line segment connecting points $a, b \in S^1$. For any $a = (a_1; a_2) \in S^1$ we set $a' = (a_2; a_1) \in S^1$. Let us fix a point $a_\circ = (a_1^\circ; a_2^\circ) \in S^1$ where $a_1^\circ \le a_2^\circ$. Throughout this paper, for any mapping V, we denote by $V^{\circ n} = \underbrace{V \circ \cdots \circ V}_{n}$.

Example 2.1. *We define an operator $V_{S^1} : S^1 \to S^1$ as follows:*

$$V_{S^1}(x) = \begin{cases} \dfrac{x + a_\circ}{2} & \text{if} \quad x \in [e_2, a_\circ] \\ x & \text{if} \quad x \in [a_\circ, a'_\circ] \\ \dfrac{x + a'_\circ}{2} & \text{if} \quad x \in [a'_\circ, e_1] \end{cases}.$$

It is clear that for any $x \in S^1$ one has that $V_{S^1}(x) \in [x, x']$. This means that V_{S^1} is a continuous DSO and $Fix(V_{S^1}) = [a_\circ, a'_\circ]$. Moreover, we get that

$$V_{S^1}^{\circ n}(x) = \begin{cases} \dfrac{x - a_\circ}{2^n} + a_\circ & \text{if} \quad x \in [e_2, a_\circ] \\ x & \text{if} \quad x \in [a_\circ, a'_\circ] \\ \dfrac{x - a'_\circ}{2^n} + a'_\circ & \text{if} \quad x \in [a'_\circ, e_1] \end{cases}.$$

Therefore, $V_{S^1}^{\circ n}(x) \to a_\circ$ if $x \in [e_2, a_\circ]$ and $V_{S^1}^{\circ n}(x) \to a'_\circ$ if $x \in [a'_\circ, e_1]$.

This example shows that the trajectory of DSO may converge to any interior point of S^1.

Example 2.2. *We define an operator $W_{S^1} : S^1 \to S^1$ as follows:*

$$W_{S^1}(x) = \begin{cases} \dfrac{x' + a'_\circ}{2} & \text{if} \quad x \in [e_2, a_\circ] \\ x' & \text{if} \quad x \in [a_\circ, a'_\circ] \\ \dfrac{x' + a_\circ}{2} & \text{if} \quad x \in [a'_\circ, e_1] \end{cases}.$$

It is clear that for any $x \in S^1$ one has that $W_{S^1}(x) \in [x, x']$. This means that W_{S^1} is a continuous DSO. The set of periodic-2 points of W_{S^1} is $[a_\circ, a'_\circ]$. Moreover, we have that

$$W_{S^1}^{\circ(2n+1)}(x) = \begin{cases} \dfrac{x' - a'_\circ}{2^{2n+1}} + a'_\circ & \text{if} \quad x \in [e_2, a_\circ] \\ x' & \text{if} \quad x \in [a_\circ, a'_\circ] \\ \dfrac{x' - a_\circ}{2^{2n+1}} + a_\circ & \text{if} \quad x \in [a'_\circ, e_1] \end{cases},$$

$$W_{S^1}^{\circ(2n)}(x) = \begin{cases} \dfrac{x - a_\circ}{2^{2n}} + a_\circ & \text{if} \quad x \in [e_2, a_\circ] \\ x & \text{if} \quad x \in [a_\circ, a'_\circ] \\ \dfrac{x - a'_\circ}{2^{2n}} + a'_\circ & \text{if} \quad x \in [a'_\circ, e_1] \end{cases}.$$

This yields that the trajectory of W_{S^1} converges to periodic-2 points, i.e., $\omega(x^{(0)}) = \{a_\circ, a'_\circ\}$ whenever $x^{(0)} \in [e_2, a_\circ] \cup [a'_\circ, e_1]$.

We see that a trajectory of a DSO might converge to its periodic points.

Now, we shall provide an example for a DSO in which the center of the simplex S^1 is its isolated fixed point. To do so, we are going to study the dynamics of the following function $f_{a_\circ} : [a_1^\circ, a_2^\circ] \to [a_1^\circ, a_2^\circ]$

$$f_{a_\circ}(x) = (x - a_1^\circ)\left(x - \frac{1}{2}\right)(x - a_2^\circ) + x, \tag{2.1}$$

where $a_\circ = (a_1^\circ; a_2^\circ) \in S^1$ and $a_1^\circ \leq a_2^\circ$.

Proposition 2.3. *Let $f_{a_\circ} : [a_1^\circ, a_2^\circ] \to [a_1^\circ, a_2^\circ]$ be a function given by (2.1). Then the following statements hold true:*

(i) The function $f_{a_\circ}$ is increasing;

(ii) One has that $Fix(f_{a_\circ}) = \left\{a_1^\circ, \frac{1}{2}, a_2^\circ\right\}$;

(iii) If $x \in \left[a_1^\circ, \frac{1}{2}\right]$ then $f_{a_\circ}(x) \geq x$ and if $x \in \left[\frac{1}{2}, a_2^\circ\right]$ then $f_{a_\circ}(x) \leq x$;

(iv) One has that $\omega_{f_{a_\circ}}(x_\circ) = \left\{\frac{1}{2}\right\}$ for any $x_\circ \in (a_1^\circ, a_2^\circ)$.

The proof of the proposition is straightforward.

Example 2.4. *We define an operator $Z_{S^1} : S^1 \to S^1$ as follows:*

$$Z_{S^1}(x) = \begin{cases} \dfrac{x + a_\circ}{2} & \textit{if} \quad x \in [e_2, a_\circ] \\ F_{a_\circ}(x) & \textit{if} \quad x \in [a_\circ, a_\circ'] \\ \dfrac{x + a_\circ'}{2} & \textit{if} \quad x \in [a_\circ', e_1] \end{cases},$$

where $F_{a_\circ}(x) = (f_{a_\circ}(x_1), 1 - f_{a_\circ}(x_1))$, a function $f_{a_\circ}$ is defined by (2.1), and $x = (x_1, x_2) \in [a_\circ, a_\circ'] \subset S^1$. This operator is continuous. We want to show that it is a DSO. Thanks to Proposition 2.3 (iii), one has that $F_{a_\circ}(x) \in [x, x']$ for any $x \in [a_\circ, a_\circ']$. This yields that $Z_{S^1}(x) \in [x, x']$ for any $x \in S^1$. Due to Definition 1.4, the operator Z_{S^1} is a DSO. Moreover, we have that $Fix\,(Z_{S^1}) = \{a_\circ, a_\circ', c\}$, where $c = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the center of the simplex S^1. One can easily see that

$$Z_{S^1}^{\circ n}(x) = \begin{cases} \dfrac{x - a_\circ}{2^n} + a_\circ & \textit{if} \quad x \in [e_2, a_\circ] \\ F_{a_\circ}^{\circ n}(x) & \textit{if} \quad x \in [a_\circ, a_\circ'] \\ \dfrac{x - a_\circ'}{2^n} + a_\circ' & \textit{if} \quad x \in [a_\circ', e_1] \end{cases},$$

where $F_{a_\circ}^{\circ n}(x) = \left(f_{a_\circ}^{\circ n}(x_1), 1 - f_{a_\circ}^{\circ n}(x_1)\right)$. Consequently, we have that

$$\omega\left(x^{(0)}\right) = \begin{cases} \{a_\circ\} & \textit{if} \quad x^{(0)} \in [e_2, a_\circ] \\ \{c\} & \textit{if} \quad x^{(0)} \in (a_\circ, a_\circ') \\ \{a_\circ'\} & \textit{if} \quad x^{(0)} \in [a_\circ', e_1] \end{cases}.$$

In all examples, operators were defined on S^1. However, we can get similar pictures in the higher dimensional simplex.

Let S^{m-1} be an $(m-1)-$dimensional simplex, $e_1, \cdots, e_m$ be vertexes of the simplex S^{m-1}, and $c = \left(\frac{1}{m}, \cdots, \frac{1}{m}\right)$ be a center of the simplex S^{m-1}, where $e_i = \left(0, \cdots, 0, \underbrace{1}_{i}, 0, \cdots, 0\right)$.

Similarly, we define $[a,b]$ as a closed line segment connecting points $a,b\in S^{m-1}$, i.e., $[a,b]=\{(1-\lambda)a+\lambda b\}_{0\le\lambda\le1}$.

Let us fix an interior point $a_\circ=(a_1^\circ,\cdots,a_m^\circ)\in intS^{m-1}$. Let $Per_{m!}(a_\circ)$ be a set of all permutation of $a_\circ$ and $Conv(Per_{m!}(a_\circ))$ be a convex hull of $Per_{m!}(a_\circ)$. Let $\partial(Conv(Per_{m!}(a_\circ)))$ be a boundary of the set $Conv(Per_{m!}(a_\circ))$. One can easily check that for any point $x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))$ the set $[x,c]\cap\partial(Conv(Per_{m!}(a_\circ)))$ is a singleton. We denote it by x^*, i.e., $x^*=[x,c]\cap\partial(Conv(Per_{m!}(a_\circ)))$.

Example 2.5. *We define an operator $V_{S^{m-1}}:S^{m-1}\to S^{m-1}$ as follows:*

$$V_{S^{m-1}}(x)=\begin{cases}\dfrac{x+x^*}{2} & if \quad x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))\\ x & if \quad x\in Conv(Per_{m!}(a_\circ))\end{cases}.$$

This operator is continuous. Let us show that $V_{S^{m-1}}$ is a DSO. Since x, x^*, and $\frac{x+x^*}{2}$ belong to $Conv(Per_{m!}(x))$, one has that $V_{S^{m-1}}(Conv(Per_{m!}(x)))\subset Conv(Per_{m!}(x))$ for any $x\in S^{m-1}$. Due to Definition 1.4, the operator $V_{S^{m-1}}$ is a DSO. Moreover, we have that $Fix(V_{S^{m-1}})=Conv(Per_{m!}(a_\circ))$.

One can easily check that for any $x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))$ one has that $V_{S^{m-1}}([x,c])\subset[x,c]$ and $(V_{S^{m-1}}(x))^*=x^*$, where c is the center of the simplex. Therefore, we get that

$$V^{\circ n}_{S^{m-1}}(x)=\begin{cases}\dfrac{x-x^*}{2^n}+x^* & if \quad x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))\\ x & if \quad x\in Conv(Per_{m!}(a_\circ))\end{cases}.$$

Consequently, $V^{\circ n}_{S^{m-1}}(x)\to x^*$ for any $x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))$.

This examples shows that any interior point of the simplex S^{m-1} might be a limiting point of a trajectory of some DSO defined on S^{m-1}.

Let us give an example for a DSO defined on the simplex S^{m-1} in which its omega limiting set is not a singleton.

Example 2.6. *We define an operator $W_{S^{m-1}}:S^{m-1}\to S^{m-1}$ as $W_{S^{m-1}}(x)=V_{S^{m-1}}(P_\circ x)$ for any $x\in S^{m-1}$, where $P_\circ$ is some permutation matrix and $V_{S^{m-1}}$ is a DSO defined in Example 2.5.*

One can easily see that $P_\circ\circ V_{S^{m-1}}=V_{S^{m-1}}\circ P_\circ$. In fact, we have for any $x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))$ that

$$P_\circ(V_{S^{m-1}}(x))=P_\circ\left(\frac{x+x^*}{2}\right)=\frac{P_\circ x+P_\circ x^*}{2}=\frac{P_\circ x+(P_\circ x)^*}{2}=V_{S^{m-1}}(P_\circ(x)).$$

Therefore, we get that $W^{\circ n}_{S^{m-1}}(x)=P_\circ^{\circ n}(V^{\circ n}_{S^{m-1}}(x))$ for any $x\in S^{m-1}$.

It follows from Example 2.5 that

$$Fix(V_{S^{m-1}})=Conv(Per_{m!}(a_\circ)),\quad \omega_{V_{S^{m-1}}}(x)=\{x^*\}$$

for any $x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))$. Consequently, we have that

$$Per_k(W_{S^{m-1}})=Conv(Per_{m!}(a_\circ)),\quad \omega_{W_{S^{m-1}}}(x)=\{P_\circ^{\circ i}x^*\}_{i=1}^k$$

for any $x\in S^{m-1}\setminus Conv(Per_{m!}(a_\circ))$, where $P_\circ^{\circ k}=\mathbb{I}$ and $\mathbb{I}$ is an identity matrix.

Example 2.7. *The following quadratic operator $V : S^2 \to S^2$ was studied in the paper [25]*

$$V : \begin{cases} (V(x))_1 = x_1x_2 + x_1x_3 + x_3^2 \\ (V(x))_2 = x_2^2 + x_1x_2 + x_2x_3 \\ (V(x))_3 = x_3x_2 + x_1^2 + x_1x_3. \end{cases}$$

Let $I(x) = (x_1, x_2, x_3)$ $\Pi(x) = (x_3, x_2, x_1)$ be the identity and permutation operators. It is clear that, one has for any $x \in S^2$ that

$$V(x) = x_2 I(x) + (x_1 + x_3)\Pi(x). \tag{2.2}$$

This means that $V(x) \in Conv(Per_{3!}(x))$ for any $x \in S^2$, i.e., V is a double stochastic operator. Due to (2.2), if $x_1 = x_3$ then $I(x) = \Pi(x)$ and $V(x) = x_2 I(x) + (x_1 + x_3)I(x) = I(x) = x$, if $x_2 = 0$ then $V(x) = (x_1 + x_3)\Pi(x) = \Pi(x)$. Therefore, $\Gamma_{13} = \{x \in S^2 : x_1 = x_3\}$ is a fixed point set and $\Gamma_2 = \{x \in S^2 : x_2 = 0\}$ is a periodic-2 point set.

Theorem 4 of the paper [25] says that if $x^{(0)} \in S^2 \setminus \Gamma_2$ then the trajectory $\{V^{\circ n}(x^{(0)})\}_{n\in\mathbb{N}}$ converges to the center $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of the simplex S^2. However, this is obviously wrong because of $\left(x^{(n)}\right)_2 = \left(V^{\circ n}(x^{(0)})\right)_2 = x_2^{(0)}$ for any $n \in \mathbb{N}$.

Since $x_2^{(n)} = x_2^{(0)}$ and $\left|x_1^{(n+1)} - x_3^{(n+1)}\right| = \left|2x_2^{(0)} - 1\right| \left|x_1^{(n)} - x_3^{(n)}\right|$ for any $n \in \mathbb{N}$, we have that

$$\omega_V(x^{(0)}) = \left\{ \left(\frac{1 - x_2^{(0)}}{2}, x_2^{(0)}, \frac{1 - x_2^{(0)}}{2} \right) \right\}$$

for any $x^{(0)} \in S^2 \setminus \Gamma_2$.

These examples show that, in general, a trajectory of DSO may converge to any interior point of the simplex. In the next section, we provide a criterion for a regularity of a double stochastic operator in terms of absences of its periodic points. This answers Problem 2.35 mentioned in the survey paper [8]. Some concrete examples to Problem 2.35 are also given in the papers [23, 24].

3. Regularity of Double Stochastic Operators

Let (X, d) be a compact convex manifold and $V : X \to X$ be a continuous operator. Let $\{x^{(n)}\}_{n=0}^{\infty}$, where $x^{(n)} = V(x^{(n-1)}) = V^{\circ n}(x^{(0)})$, $x^{(0)} \in X$, be a trajectory of V starting from an initial point $x^{(0)}$. An operator V is called *regular* if its trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ converges for all $x^{(0)} \in X$ (a limiting point might be depended on an initial point). It is clear that if V is regular then V does not have any order periodic points except fixed points. One of the fascinating results in 1D dynamical system is that *if $X = [a, b]$ then V is regular if and only if it does not have any order periodic points except fixed points* (see [4]). It is natural to seek an analogy of this incredible result in the higher dimensional case. However, in general, this result does not hold true in the higher dimensional case. In the paper [20], it was shown that, in the class of Volterra-QSO, absences of periodic points do not imply a regularity of Volterra-QSO. It turns out that, in the class of DSO, the regularity of DSO can be described in terms of absences of periodic points of DSO.

Theorem 3.1. *Let $V : S^{m-1} \to S^{m-1}$ be a DSO. Then V is regular if and only if it does not have any order periodic points except fixed points.*

Proof. Let $V : S^{m-1} \to S^{m-1}$ be a DSO. It is clear that if V is regular then it does not have any order periodic points except fixed points. Let us prove the *"if"* part of the theorem.

The "if" part. Suppose that V does not have any order periodic points except fixed points. Let $\{x^{(n)}\}_{n=0}^{\infty}$ be a trajectory of V starting from any initial point $x^{(0)} \in S^{m-1}$. Since $V : S^{m-1} \to S^{m-1}$ is a DSO, we have that

$$\sum_{i=1}^{k} x_{[i]}^{(n+1)} \leq \sum_{i=1}^{k} x_{[i]}^{(n)}, \quad \forall\ k = \overline{1, m-1}, \tag{3.1}$$

$$\sum_{i=1}^{m} x_{[i]}^{(n+1)} = \sum_{i=1}^{m} x_{[i]}^{(n)} = 1. \tag{3.2}$$

It follows from (3.1) that $\left\{\sum\limits_{i=1}^{k} x_{[i]}^{(n)}\right\}_{n=0}^{\infty}$ is a bounded decreasing sequence (therefore it is convergent) for every $k = \overline{1, m-1}$.

Consequently, it follows from $x_{[k]}^{(n)} = \sum\limits_{i=1}^{k} x_{[i]}^{(n)} - \sum\limits_{i=1}^{k-1} x_{[i]}^{(n)}$ and (3.2) that a sequence $\{x_{[k]}^{(n)}\}_{n=0}^{\infty}$ is convergent for any $k = \overline{1, m}$. This means that the sequence $x_{\downarrow}^{(n)} = \left(x_{[1]}^{(n)}, x_{[2]}^{(n)}, \cdots, x_{[m]}^{(n)}\right)$ converges to some point $x^* = (x_1^*, \cdots, x_m^*)$ whenever $n \to \infty$.

Let $\omega\left(x^{(0)}\right)$ be an omega limiting set of the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ and $x^{\circ} \in \omega\left(x^{(0)}\right)$ be any point. Then there is a sub-sequence $\{x^{(n_k)}\}_{k=0}^{\infty}$ of the sequence $\{x^{(n)}\}_{n=0}^{\infty}$ such that $x^{(n_k)} \to x^{\circ}$ whenever $k \to \infty$. Since a down arrow mapping $\downarrow : \mathbb{R}^m \to \mathbb{R}^m_{\downarrow}$, where $\downarrow(x) = x_{\downarrow}$, is continuous, we get that $x_{\downarrow}^{(n_k)} \to x_{\downarrow}^{\circ}$ whenever $k \to \infty$. On the other hand, we know that $x_{\downarrow}^{(n)} \to x^*$ whenever $n \to \infty$. Therefore, we have that $x_{\downarrow}^{\circ} = x^*$. This means that there is a permutation matrix P such that $x^{\circ} = Px^*$. As a conclusion, we may say that any omega limiting point is some permutation of x^*. It means that $\omega\left(x^{(0)}\right)$ is a finite set, i.e., $\omega\left(x^{(0)}\right) = \{P_i x^*\}_{i=1}^{r}$ for some permutation matrices P_i, where $i = \overline{1, r}$.

Since the simplex S^{m-1} is compact and V is continuous, we obtain that $V\left(\omega\left(x^{(0)}\right)\right) = \omega\left(x^{(0)}\right)$ and every omega limiting point is a periodic point of (minimum) period $|\omega\left(x^{(0)}\right)|$. We know that V does not have any order periodic points except fixed points, therefore, we get that $|\omega\left(x^{(0)}\right)| = 1$ for any $x^{(0)} \in S^{m-1}$. This means that V is regular. □

If a DSO is regular then a limiting point of its trajectory is a fixed point. The structure of the fixed point set can be arbitrary. The fixed point set can be finite as well as infinite (see examples section). It is clear that the center $c = (\frac{1}{m}, \cdots, \frac{1}{m})$ of the simplex S^{m-1} is always the fixed point of any DSO. In fact, one has that $c \prec V(c) \prec c$. This means that $V(c) = Pc$ for some permutation matrix P. On the other hand, $Pc = c$ for any permutation matrix P. Therefore, $V(c) = c$. Suppose that the center c of the simplex is the isolated fixed point, i.e., there is a neighborhood U_c of the point c such that U_c does not contain any periodic and fixed points of DSO except c. We then have the following result.

Proposition 3.2. *Let $V : S^{m-1} \to S^{m-1}$ be a DSO. If the center of the simplex is the isolated fixed point then it is locally attracting.*

Proof. Let $V : S^{m-1} \to S^{m-1}$ be a DSO and the center $c = (\frac{1}{m}, \cdots, \frac{1}{m})$ of the simplex be its isolated fixed point. This means that, without loss of any generality, there is a convex symmetric neighborhood U_c of c (a set is symmetric if it is invariant under any permutation matrix) such that U_c does not contain any periodic and fixed points of DSO except c. Due to Definition 1.4, we have that $V(U_c) \subset U_c$. Moreover, since the set U_c does not contain any periodic points of DSO, due to Theorem 3.1, V is regular in U_c. We know that the only fixed point in the set U_c is the center of the simplex, therefore, any trajectory of V starting from any point in U_c converges to the center of the simplex. It means that the center of the simplex is locally attracting. □

Therefore, the dynamics of DSO are diverse. It is worth pointing out that a convergence criterion and several properties of Schur decreasing sequences were studied in [10] while a trajectory of DSO is but an example for Schur decreasing sequences.

Acknowledgments
This work was partially supported by the MOHE grant FRGS14-141-0382. The Author is grateful to the Junior Associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

References

[1] Ando T 1989 *Linear Algebra and Its Application* **118** 163–248
[2] Ando T 1994 *Linear Algebra and its Application* **199** 17–67
[3] Bernstein S 1942 *Annals of Mathematical Statistics* **13** 53–61
[4] Coppel W A 1955 *Proc. Cambridge Philos. Soc.* **51** 41–43
[5] Ganikhodjaev N, Saburov M and Jamilov U 2013 *App. Math. Info. Sci.* **7**(5) 1721–1729
[6] Ganikhodjaev N, Saburov M and Nawi A M 2014 *The Scientific World Journal* **2014** 1–11
[7] Ganikhodzhaev R 1992 *Russ. Math. Surveys* **48** 244–246
[8] Ganikhodzhaev R, Mukhamedov F and Rozikov U 2011 *Inf. Dim. Anal. Quan. Prob. Rel. Top.* **14**(2) 279–335
[9] Ganikhodzhaev R, Mukhamedov F and Saburov M 2012 *Linear Algebra and its Application* **436** 1344–1366
[10] Ganikhodjaev R, Saburov M and Saburov Kh 2013 *AIP Conference Proceedings* **1557** 108–111
[11] Kesten H 1970 *Adv. App. Prob.* **2** 1–82
[12] Lyubich Yu I 1992 *Mathematical structures in population genetics* (Berlin:Springer–Verlag)
[13] Marshall A, Olkin I and Arnold B 2011 *Inequalities: Theory of majorization and its applications* (Springer)
[14] Mukhamedov F and Ganikhodjaev N 2015 *Lect. Notes Math.* **2133** (Berlin: Springer)
[15] Mukhamedov F, Akin H and Temir S 2005 *J. Math. Anal. Appl.* **310** 533–556
[16] Mukhamedov F and Embong A F 2015 *J. Inequal. Appl.* **2015** 226
[17] Mukhamedov F and Saburov M 2010 *App. Math. Info. Sci.* **4** 47–62
[18] Mukhamedov F and Saburov M 2014 *Bull. Malay. Math. Sci. Soc* **37** 59–64
[19] Mukhamedov F, Saburov M and Qaralleh I 2013 *Abst. Appl. Anal.* **2013** 942038
[20] Saburov M 2013 *World Applied Science Journal* **21** 94–97
[21] Saburov M 2015 *Math Notes* **97**(5-6) 759–763
[22] Saburov M 2015 *Ann. Funct. Anal.* **6**(4) 247–254
[23] Saburov M, Saburov Kh 2014 *ScienceAsia* **40** (4) 306–312
[24] Saburov M and Saburov Kh 2014 *Inter. Jour.Cont. Auto. Sys.* 12(6) 1276–1282
[25] Shahidi F, Ganikhodzhaev R and Abdulghafor R 2013 *Middle-East Jour.f Sci. Res.* **13** 59–63
[26] Ulam S 1960 *A collection of mathematical problems* (New-York, London)