

# On Quadratic Stochastic Operators Having Three Fixed Points

Mansoor Saburov<sup>1</sup>, Nur Atikah Yusof<sup>2</sup>

<sup>1,2</sup>Department of Computational & Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, P.O. Box, 25200, Kuantan, Pahang, MALAYSIA

E-mail: msaburov@iiu.edu.my, msaburov@gmail.com, y.nuratikah@gmail.com

**Abstract.** We knew that a trajectory of a linear stochastic operator associated with a positive square stochastic matrix starting from any initial point from the simplex converges to a unique fixed point. However, in general, the similar result for a quadratic stochastic operator associated with a positive cubic stochastic matrix does not hold true. In this paper, we provide an example for the quadratic stochastic operator with positive coefficients in which its trajectory may converge to different fixed points depending on initial points.

## 1. Introduction

Let  $\|\mathbf{x}\|_1 = \sum_{k=1}^m |x_k|$  be a norm of a vector  $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ . We say that  $\mathbf{x} \geq 0$  (resp.  $\mathbf{x} > 0$ ) if  $x_k \geq 0$  (resp.  $x_k > 0$ ) for all  $k = \overline{1, m}$ . Let  $\mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 = 1, \mathbf{x} \geq 0\}$  be the  $(m-1)$ -dimensional standard simplex. An element of the simplex  $\mathbb{S}^{m-1}$  is called a *stochastic vector*. Recall that a square matrix  $\mathbb{P} = (p_{ij})_{i,j=1}^m$  is called *stochastic* if every row is a stochastic vector. A square stochastic matrix  $\mathbb{P} = (p_{ij})_{i,j=1}^m$  is called *positive* if  $p_{ij} > 0, \forall i, j = \overline{1, m}$ . The classical Perron-Frobenius theorem states that any positive square stochastic matrix has a unique fixed point in the simplex. However, in general, the similar result for higher dimensional hyper-matrices (tensors) does not hold true. In this paper, we discuss this problem for cubic stochastic matrices.

A cubic matrix  $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$  is called *stochastic* if  $\sum_{k=1}^m p_{ijk} = 1, p_{ijk} \geq 0, \forall i, j, k = \overline{1, m}$ .

Every cubic stochastic matrix is associated with a quadratic stochastic operator  $\mathcal{Q} : \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$  as follows

$$(\mathcal{Q}(\mathbf{x}))_k = \sum_{i,j=1}^m x_i x_j p_{ijk}, \quad \forall k = \overline{1, m}. \quad (1.1)$$

By being the simplest nonlinear mapping, a quadratic stochastic operator has an incredible application in population genetics [1, 2, 3, 6, 8], control systems [18, 19]. In population genetics, the quadratic stochastic operator describes a distribution of the next generation of the system if the current distribution is given [8, 21]. In this sense, the quadratic stochastic operator is a primary source for investigations of evolution of population genetics. The detailed exposure of the theory of quadratic stochastic operators is presented in [4, 5],[10]-[17].



A cubic stochastic matrix  $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$  is said to be *positive* (written  $\mathcal{P} > 0$ ) if  $p_{ijk} > 0$ ,  $\forall i, j, k = \overline{1, m}$ . A quadratic stochastic operator associated with a positive cubic stochastic matrix is called *positive*. Let  $\mathbf{Fix}(\mathcal{Q}) = \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathcal{Q}(\mathbf{x}) = \mathbf{x}\}$  be a fixed point set. Due to Brouwer's theorem,  $\mathbf{Fix}(\mathcal{Q}) \neq \emptyset$ . Meanwhile, a fixed point of the quadratic operator is an equilibrium for the system. In contrast to the linear case, the fixed point set of the quadratic operator is sophisticated. In general, if  $\mathcal{P} > 0$  then it is not necessary to be true that  $|\mathbf{Fix}(\mathcal{Q})| = 1$ . The first attempt to give an example for such kind of quadratic operators was done by A. A. Krapivin [7] and Yu. I. Lyubich [8]. However, it turns out that their examples are wrong. In fact, we shall show that Krapivin's example as well as Lyubich's example has a unique fixed point in the simplex. At the same time, we shall also provide an example for the quadratic operator associated with the positive cubic stochastic matrix which has three fixed points in the simplex.

## 2. Krapivin's Example

A. A. Krapivin has considered the following quadratic operator  $V_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ ,  $V_\varepsilon(\mathbf{x}) = \mathbf{x}' = (x'_1, x'_2, x'_3)$  in his paper [7]

$$V_\varepsilon : \begin{cases} x'_1 = (1 - 4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1 x_2 + (1 + 4\varepsilon)x_1 x_3 + 8\varepsilon x_2 x_3 \\ x'_2 = 2\varepsilon x_1^2 + (1 - 3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2} + 2\varepsilon)x_1 x_2 + 2\varepsilon x_1 x_3 + (1 + 8\varepsilon)x_2 x_3 \\ x'_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1 - 11\varepsilon)x_3^2 + (\frac{3}{2} - 6\varepsilon)x_1 x_2 + (1 - 6\varepsilon)x_1 x_3 + (1 - 16\varepsilon)x_2 x_3 \end{cases}$$

A.A. Krapivin claimed [7] that the quadratic operator  $V_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  has two fixed points on the line segment  $L = \left\{ \left( \frac{t}{2}, \frac{1-t}{2}, \frac{1}{2} \right) \right\}_{0 \leq t \leq 1}$  where  $0 < \varepsilon < \frac{1}{100}$ . However, this claim is wrong.

**Proposition 2.1.** *The quadratic operator  $V_\varepsilon$  does not have any fixed point on  $L$ .*

*Proof.* We search for a fixed point  $\mathbf{x}_0 = (\frac{t}{2}, \frac{1-t}{2}, \frac{1}{2})$  on the line segment  $L$ . We should have that  $V_\varepsilon(\mathbf{x}_0) = \mathbf{x}_0$ . After some algebraic calculations, we obtain the following system of equations

$$\begin{cases} (1 - 6\varepsilon)t^2 - (1 + 4\varepsilon)t + 20\varepsilon = 0 \\ (1 - 6\varepsilon)t^2 - (1 - 4\varepsilon)t + 12\varepsilon = 0 \\ (1 - 6\varepsilon)t^2 - (1 + \frac{4}{3}\varepsilon)t + \frac{52}{3}\varepsilon = 0 \end{cases} \quad (2.1)$$

If we take a difference of the first two equations of the system (2.1) we then get that  $8\varepsilon t = 8\varepsilon$  or  $t = 1$ . However, if we substitute  $t = 1$  into the third equation in the system (2.1) we then have that  $10\varepsilon = 0$  which contradicts to the condition  $0 < \varepsilon < \frac{1}{100}$ . Therefore, the system (2.1) does not have any solution. This completes the proof.  $\square$

Now, we are aiming to prove that  $|\mathbf{Fix}(V_\varepsilon)| = 1$ .

It is clear that  $V_\varepsilon(\mathbb{S}^2) \subset \text{int}\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{S}^2 : x_1 x_2 x_3 > 0\}$ . Hence,  $\mathbf{Fix}(V_\varepsilon) \subset \text{int}\mathbb{S}^2$ .

In order to find all fixed points, we have to solve the system of equations

$$\begin{cases} x_1 = (1 - 4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1 x_2 + (1 + 4\varepsilon)x_1 x_3 + 8\varepsilon x_2 x_3 \\ x_2 = 2\varepsilon x_1^2 + (1 - 3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2} + 2\varepsilon)x_1 x_2 + 2\varepsilon x_1 x_3 + (1 + 8\varepsilon)x_2 x_3 \\ x_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1 - 11\varepsilon)x_3^2 + (\frac{3}{2} - 6\varepsilon)x_1 x_2 + (1 - 6\varepsilon)x_1 x_3 + (1 - 16\varepsilon)x_2 x_3 \end{cases}$$

**Proposition 2.2.** *One has that  $\xi_1 \neq \eta_1$ ,  $\xi_2 \neq \eta_2$ ,  $\xi_3 \neq \eta_3$  for  $\xi, \eta \in \mathbf{Fix}(V_\varepsilon)$  and  $\xi \neq \eta$ .*

*Proof.* Let  $\xi, \eta$  be two distinct solutions of the system given above (if any). Since  $x_3 = 1 - x_1 - x_2$ , we can rewrite the system in terms of  $x_1$  and  $x_2$  as follows

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 - 10\varepsilon x_2^2 - \left(\frac{1}{2} + 6\varepsilon\right)x_1x_2 + 6\varepsilon x_2 + \varepsilon = 0 \end{cases} \quad (2.2)$$

**Case  $\xi_1 \neq \eta_1$**  Our aim is to show that  $\xi_1 \neq \eta_1$ . We suppose the contrary that is  $\xi_1 = \eta_1$ . Since  $\xi \neq \eta$ , we must have that  $\xi_2 \neq \eta_2$ . This means that for  $x_1 = \xi_1 = \eta_1$ , the following two quadratic equations (with respect to  $x_2$ )

$$\begin{aligned} 4\varepsilon x_2^2 + ((12\varepsilon - 1)\xi_1 - 12\varepsilon)x_2 + (2\xi_1^2 - 16\xi_1 + 10)\varepsilon &= 0, \\ -10\varepsilon x_2^2 + \left[-\left(\frac{1}{2} + 6\varepsilon\right)\xi_1 + 6\varepsilon\right]x_2 + (\xi_1^2 + 1)\varepsilon &= 0 \end{aligned}$$

must have two distinct common roots  $\xi_2$  and  $\eta_2$ . Consequently, we should have that

$$\frac{4\varepsilon}{-10\varepsilon} = \frac{(12\varepsilon - 1)\xi_1 - 12\varepsilon}{-\left(\frac{1}{2} + 6\varepsilon\right)\xi_1 + 6\varepsilon} = \frac{2\xi_1^2 - 16\xi_1 + 10}{\xi_1^2 + 1}.$$

It follows from the first equality that  $\xi_1 = \frac{8\varepsilon}{8\varepsilon - 1}$ . Since  $0 < \varepsilon < \frac{1}{100}$ , we get that  $\xi_1 < 0$  which is a contradiction.

**Case  $\xi_2 \neq \eta_2$**  Our aim is to show that  $\xi_2 \neq \eta_2$ . We again suppose the contrary that is  $\xi_2 = \eta_2$ . Since  $\xi \neq \eta$ , we must have that  $\xi_1 \neq \eta_1$ . It follows from (2.2) that for  $x_2 = \xi_2 = \eta_2$ , the following two quadratic equations (with respect to  $x_1$ )

$$\begin{aligned} 2\varepsilon x_1^2 + ((12\varepsilon - 1)\xi_2 - 16\varepsilon)x_1 + (4\xi_2^2 - 12\xi_2 + 10)\varepsilon &= 0, \\ \varepsilon x_1^2 - \left(\frac{1}{2} + 6\varepsilon\right)\xi_2 x_1 + (-10\xi_2^2 + 6\xi_2 + 1)\varepsilon &= 0 \end{aligned}$$

must have two distinct common roots  $\xi_1$  and  $\eta_1$ . Consequently, we should have that

$$\frac{2\varepsilon}{\varepsilon} = \frac{(12\varepsilon - 1)\xi_2 - 16\varepsilon}{-\left(\frac{1}{2} + 6\varepsilon\right)\xi_2} = \frac{4\xi_2^2 - 12\xi_2 + 10}{-10\xi_2^2 + 6\xi_2 + 1}.$$

It follows from the first equality that  $\xi_2 = \frac{2}{3}$ . By substituting  $\xi_2 = \frac{2}{3}$  into the last fraction, we get that  $\frac{4\xi_2^2 - 12\xi_2 + 10}{-10\xi_2^2 + 6\xi_2 + 1} = \frac{34}{5} \neq 2 = \frac{2\varepsilon}{\varepsilon}$ . This is again contradiction.

**Case  $\xi_3 \neq \eta_3$**  Our aim is to show that  $\xi_3 \neq \eta_3$ . We again suppose the contrary that is  $\xi_3 = \eta_3$ . Since  $\xi \neq \eta$ , we must have that  $\xi_1 \neq \eta_1$ . In this case, we can rewrite the system of equations in terms of  $x_1$  and  $x_3$  as follows

$$\begin{cases} (1 - 6\varepsilon)x_1^2 + 4\varepsilon x_3^2 + (1 - 4\varepsilon)x_1x_3 - x_1 + 4\varepsilon x_3 + 2\varepsilon = 0 \\ (9\varepsilon - \frac{3}{2})x_1^2 + 6\varepsilon x_3^2 + (18\varepsilon - \frac{3}{2})x_1x_3 + (\frac{3}{2} - 8\varepsilon)x_1 - 18\varepsilon x_3 + \varepsilon = 0 \end{cases}.$$

This yields that for  $x_3 = \xi_3 = \eta_3$ , the following two quadratic equations (with respect to  $x_1$ )

$$\begin{aligned} (1 - 6\varepsilon)x_1^2 + ((1 - 4\varepsilon)\xi_3 - 1)x_1 + (4\xi_3^2 + 4\xi_3 + 2)\varepsilon &= 0, \\ (9\varepsilon - \frac{3}{2})x_1^2 + [(18\varepsilon - \frac{3}{2})\xi_3 + (\frac{3}{2} - 8\varepsilon)]x_1 + (6\xi_3^2 - 18\xi_3 + 1)\varepsilon &= 0 \end{aligned}$$

must have two distinct common roots  $\xi_1$  and  $\eta_1$ . Consequently, we should have that

$$\frac{1-6\varepsilon}{9\varepsilon-\frac{3}{2}} = \frac{(1-4\varepsilon)\xi_3-1}{(18\varepsilon-\frac{3}{2})\xi_3+\frac{3}{2}-8\varepsilon} = \frac{4\xi_3^2+4\xi_3+2}{6\xi_3^2-18\xi_3+1}.$$

It follows from the first equality that  $\xi_3 = \frac{2}{3}$ . By substituting  $\xi_3 = \frac{2}{3}$  into the last fraction, we get that  $\frac{4\xi_3^2+4\xi_3+2}{6\xi_3^2-18\xi_3+1} = -\frac{58}{75} \neq -\frac{2}{3} = \frac{1-6\varepsilon}{9\varepsilon-\frac{3}{2}}$ . This is again contradiction.

This completes the proof.  $\square$

**Theorem 2.3.** Let  $V_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  be the quadratic operator given above. Then for sufficiently small  $\varepsilon$ , one has that  $\mathbf{Fix}(V_\varepsilon) = \left\{ \left( \frac{3a_0^2-3a_0+1}{2-3a_0}, a_0, \frac{1-2a_0}{2-3a_0} \right) \right\}$  where  $a_0$  is the unique positive root in  $(0, \frac{1}{2})$  of the following quartic equation

$$(9-54\varepsilon)a^4 + (132\varepsilon-15)a^3 + (9-68\varepsilon)a^2 - (12\varepsilon+2)a + 10\varepsilon = 0.$$

*Proof.* Since  $x_3 = 1 - x_1 - x_2$ , it is enough to find all solutions  $(x_1, x_2)$  of the following system of equations

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon-1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 - 10\varepsilon x_2^2 - (\frac{1}{2}+6\varepsilon)x_1x_2 + 6\varepsilon x_2 + \varepsilon = 0 \end{cases} \quad (2.3)$$

which satisfy the conditions  $0 < x_1, x_2 < 1$  and  $0 < x_1 + x_2 < 1$ .

Let  $x_1 = x$  be a variable and  $x_2 = a$  be a parameter. Then the system (2.3) takes the following form

$$x^2 + \frac{(12\varepsilon-1)a-16\varepsilon}{2\varepsilon}x + (2a^2-6a+5) = 0, \quad (2.4)$$

$$x^2 + \frac{a(-1-12\varepsilon)}{2\varepsilon}x + (-10a^2+6a+1) = 0. \quad (2.5)$$

Due to Proposition 2.2, these two quadratic equations cannot have two common roots. Hence, the system (2.3) has a solution  $(x_1, x_2)$  with  $0 < x_1, x_2 < 1$ ,  $0 < x_1 + x_2 < 1$  if and only if two quadratic equations (2.4) and (2.5) must have a unique common root in  $(0, 1)$  for  $a \in (0, 1)$ . We know (see [20]) that two quadratic equations (2.4) and (2.5) have a unique common root if and only if their resultant is equal to zero, i.e.,

$$(9-54\varepsilon)a^4 + (132\varepsilon-15)a^3 + (9-68\varepsilon)a^2 + (-12\varepsilon-2)a + 10\varepsilon = 0. \quad (2.6)$$

In this case,  $x = \frac{3a^2-3a+1}{2-3a}$  is the unique common root of two quadratic equations (2.4) and (2.5). In order to have conditions  $0 < x_1, x_2 < 1$ ,  $0 < x_1 + x_2 < 1$ , we have to solve the following system of inequalities

$$\begin{cases} 0 < \frac{3a^2-3a+1}{2-3a} < 1 \\ 0 < a < 1 \\ 0 < a + \frac{3a^2-3a+1}{2-3a} < 1. \end{cases} \quad (2.7)$$

The solution of the system (2.7) is  $a \in (0, \frac{1}{2})$ . Therefore, the total number of solutions  $(x_1, x_2)$ ,  $0 < x_1, x_2 < 1$ ,  $0 < x_1 + x_2 < 1$  of the system (2.3) is the same as the total number of roots of the quartic equation (2.6) in the interval  $(0, \frac{1}{2})$ . Moreover, there is one-to-one correspondence

between a root  $a_0 \in (0, \frac{1}{2})$  of the quartic equation (2.6) and a fixed point  $(\frac{3a_0^2-3a_0+1}{2-3a_0}, a_0, \frac{1-2a_0}{2-3a_0})$  of the quadratic operator  $V_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ .

Now, we want to show that the quartic equation (2.6) has a unique root in the interval  $(0, \frac{1}{2})$  for sufficiently small  $\varepsilon$ . To do so, we have to apply the Sturm theorem for the quartic equation (2.6) in  $(0, \frac{1}{2})$  (see [20]).

Let  $p(a) = (9 - 54\varepsilon)a^4 + (132\varepsilon - 15)a^3 + (9 - 68\varepsilon)a^2 + (-12\varepsilon - 2)a + 10\varepsilon$  be a quartic polynomial. Let  $\{p_0(a), p_1(a), p_2(a), p_3(a), p_4(a)\}$  be a Sturm sequence of the quartic polynomial  $p(a)$ . Let  $\sigma(\xi)$  be the number of sign changes (ignoring zero terms) in the sequence  $p_0(\xi), p_1(\xi), p_2(\xi), p_3(\xi), p_4(\xi)$ . Then due to the Sturm theorem, the number of roots of the quartic polynomial  $p(x)$  in the interval  $(0, \frac{1}{2})$  is equal to  $\sigma(0) - \sigma(\frac{1}{2})$ . Simple calculations show that for sufficiently small  $\varepsilon$ , one has that

$$\begin{aligned} p_0(0) &= 10\varepsilon > 0, & p_0(\frac{1}{2}) &= -\frac{1}{16} + \frac{1}{8}\varepsilon < 0, \\ p_1(0) &= -(12\varepsilon + 2) < 0, & p_1(\frac{1}{2}) &= \frac{1}{4} - 8\varepsilon > 0, \\ p_2(0) &\simeq -\frac{5}{24(6\varepsilon - 1)} > 0, & p_2(\frac{1}{2}) &\simeq -\frac{13}{6\varepsilon - 1} > 0, \\ p_3(0) &\simeq \frac{9}{(3 - 344\varepsilon)^2} > 0, & p_3(\frac{1}{2}) &\simeq \frac{9}{(3 - 344\varepsilon)^2} > 0, \\ p_4(0) &\simeq \frac{3}{6\varepsilon - 1} < 0, & p_4(\frac{1}{2}) &\simeq \frac{3}{6\varepsilon - 1} < 0. \end{aligned}$$

Therefore, we get that  $\sigma(0) - \sigma(\frac{1}{2}) = 3 - 2 = 1$ . Consequently, this means that the quartic equation (2.6) has a unique root  $a_0$  in the interval  $(0, \frac{1}{2})$ , or equivalently, the quadratic operator  $V_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  has a unique fixed point  $(\frac{3a_0^2-3a_0+1}{2-3a_0}, a_0, \frac{1-2a_0}{2-3a_0})$  in the simplex  $\mathbb{S}^2$ . This completes the proof.  $\square$

### 3. Lyubich's Example

Yu. I. Lyubich has considered (see [8], page 296) the following quadratic operator  $W_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ ,  $W_\varepsilon(\mathbf{x}) = \mathbf{x}' = (x'_1, x'_2, x'_3)$

$$W_\varepsilon : \begin{cases} x'_1 = (1 - 4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1 x_2 + (1 + 4\varepsilon)x_1 x_3 + 8\varepsilon x_2 x_3 \\ x'_2 = 2\varepsilon x_1^2 + (1 - 3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2} + 2\varepsilon)x_1 x_2 + 2\varepsilon x_1 x_3 + (1 - 12\varepsilon)x_2 x_3 \\ x'_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1 - 11\varepsilon)x_3^2 + (\frac{3}{2} - 6\varepsilon)x_1 x_2 + (1 - 6\varepsilon)x_1 x_3 + (1 + 4\varepsilon)x_2 x_3 \end{cases}$$

where  $0 < \varepsilon < \frac{1}{12}$ . In commentaries and references section, Yu.I. Lyubich wrote that the quadratic operator  $W_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  was constructed by A.A. Krapivin in [7]. However, Krapivin's example  $V_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  considered in the previous section is slightly different from the quadratic operator  $W_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  given in Lyubich's book [8]. Yu. I. Lyubich claimed that if  $0 < \varepsilon < \frac{9-5\sqrt{2}}{124}$  then the quadratic operator  $W_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  has three fixed points in the simplex  $\mathbb{S}^2$ . However, this is wrong. Namely, the quadratic operator  $W_\varepsilon$  has a unique fixed point for any  $0 < \varepsilon < \frac{1}{12}$ . For the sake of argument, we shall present its proof by repeating the same method used in Krapivin's example.

It is clear that  $W_\varepsilon(\mathbb{S}^2) \subset \text{int}\mathbb{S}^2$ . Hence,  $\mathbf{Fix}(W_\varepsilon) \subset \text{int}\mathbb{S}^2$ .

In order to find all fixed points, we have to solve the system of equations

$$\begin{cases} x_1 = (1 - 4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1 x_2 + (1 + 4\varepsilon)x_1 x_3 + 8\varepsilon x_2 x_3 \\ x_2 = 2\varepsilon x_1^2 + (1 - 3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2} + 2\varepsilon)x_1 x_2 + 2\varepsilon x_1 x_3 + (1 - 12\varepsilon)x_2 x_3 \\ x_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1 - 11\varepsilon)x_3^2 + (\frac{3}{2} - 6\varepsilon)x_1 x_2 + (1 - 6\varepsilon)x_1 x_3 + (1 + 4\varepsilon)x_2 x_3 \end{cases}$$

**Proposition 3.1.** *One has that  $\xi_1 \neq \eta_1, \xi_2 \neq \eta_2, \xi_3 \neq \eta_3$  for  $\xi, \eta \in \mathbf{Fix}(W_\varepsilon)$  and  $\xi \neq \eta$ .*

*Proof.* Let  $\xi, \eta$  be two distinct solutions of the system given above (if any). Since  $x_3 = 1 - x_1 - x_2$ , we can rewrite the system of equations in terms of  $x_1$  and  $x_2$  as

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 + 10\varepsilon x_2^2 + (14\varepsilon - \frac{1}{2})x_1x_2 - 14\varepsilon x_2 + \varepsilon = 0 \end{cases} \quad (3.1)$$

**Case  $\xi_1 \neq \eta_1$**  Our aim is to show that  $\xi_1 \neq \eta_1$ . We suppose the contrary that is  $\xi_1 = \eta_1$ . Since  $\xi \neq \eta$ , we must have that  $\xi_2 \neq \eta_2$ . This means that for  $x_1 = \xi_1 = \eta_1$ , the following two quadratic equations (with respect to  $x_2$ )

$$\begin{aligned} 4\varepsilon x_2^2 + ((12\varepsilon - 1)\xi_1 - 12\varepsilon)x_2 + (2\xi_1^2 - 16\xi_1 + 10)\varepsilon &= 0, \\ 10\varepsilon x_2^2 + ((14\varepsilon - \frac{1}{2})\xi_1 - 14\varepsilon)x_2 + (\xi_1^2 + 1)\varepsilon &= 0 \end{aligned}$$

must have two distinct common roots  $\xi_2$  and  $\eta_2$ . Consequently, we should have that

$$\frac{4\varepsilon}{10\varepsilon} = \frac{(12\varepsilon - 1)\xi_1 - 12\varepsilon}{(14\varepsilon - \frac{1}{2})\xi_1 - 14\varepsilon} = \frac{2\xi_1^2 - 16\xi_1 + 10}{\xi_1^2 + 1}.$$

It follows from the first equality that  $\xi_1 = \frac{8\varepsilon}{8\varepsilon - 1}$ . Since  $0 < \varepsilon < \frac{1}{12}$ , we get that  $\xi_1 < 0$  which is a contradiction.

**Case  $\xi_2 \neq \eta_2$**  Our aim is to show that  $\xi_2 \neq \eta_2$ . We again suppose the contrary that is  $\xi_2 = \eta_2$ . Since  $\xi \neq \eta$ , we must have that  $\xi_1 \neq \eta_1$ . It follows from (3.1) that for  $x_2 = \xi_2 = \eta_2$ , the following two quadratic equations (with respect to  $x_1$ )

$$\begin{aligned} 2\varepsilon x_1^2 + ((12\varepsilon - 1)\xi_2 - 16\varepsilon)x_1 + (4\xi_2^2 - 12\xi_2 + 10)\varepsilon &= 0, \\ \varepsilon x_1^2 + (14\varepsilon - \frac{1}{2})\xi_2 x_1 + (10\xi_2^2 - 14\xi_2 + 1)\varepsilon &= 0 \end{aligned}$$

must have two distinct common roots  $\xi_1$  and  $\eta_1$ . Consequently, we should have that

$$\frac{2\varepsilon}{\varepsilon} = \frac{(12\varepsilon - 1)\xi_2 - 16\varepsilon}{(14\varepsilon - \frac{1}{2})\xi_2} = \frac{4\xi_2^2 - 12\xi_2 + 10}{10\xi_2^2 - 14\xi_2 + 1}.$$

It follows from the first equality that  $\xi_2 = -1$  which is a contradiction.

**Case  $\xi_3 \neq \eta_3$**  Our aim is to show that  $\xi_3 \neq \eta_3$ . We again suppose the contrary that is  $\xi_3 = \eta_3$ . Since  $\xi \neq \eta$ , we must have that  $\xi_1 \neq \eta_1$ . In this case, we can rewrite the system of equations in terms of  $x_1$  and  $x_3$  as follows

$$\begin{cases} (1 - 6\varepsilon)x_1^2 + 4\varepsilon x_3^2 + (1 - 4\varepsilon)x_1x_3 - x_1 + 4\varepsilon x_3 + 2\varepsilon = 0 \\ (9\varepsilon - \frac{3}{2})x_1^2 - 14\varepsilon x_3^2 - (\frac{3}{2} + 2\varepsilon)x_1x_3 + (\frac{3}{2} - 8\varepsilon)x_1 + 2\varepsilon x_3 + \varepsilon = 0 \end{cases}$$

This yields that for  $x_3 = \xi_3 = \eta_3$ , the following two quadratic equations (with respect to  $x_1$ )

$$\begin{aligned} (1 - 6\varepsilon)x_1^2 + ((1 - 4\varepsilon)\xi_3 - 1)x_1 + (4\xi_3^2 + 4\xi_3 + 2)\varepsilon &= 0, \\ (9\varepsilon - \frac{3}{2})x_1^2 + [-(\frac{3}{2} + 2\varepsilon)\xi_3 + (\frac{3}{2} - 8\varepsilon)]x_1 + [-14\xi_3^2 + 2\xi_3 + 1]\varepsilon &= 0 \end{aligned}$$

must have two distinct common roots  $\xi_1$  and  $\eta_1$ . Consequently, we should have that

$$\frac{1 - 6\varepsilon}{9\varepsilon - \frac{3}{2}} = \frac{(1 - 4\varepsilon)\xi_3 - 1}{-(\frac{3}{2} + 2\varepsilon)\xi_3 + \frac{3}{2} - 8\varepsilon} = \frac{4\xi_3^2 + 4\xi_3 + 2}{-14\xi_3^2 + 2\xi_3 + 1}.$$

It follows from the first equality that  $\xi_3 = -1$  which is a contradiction.

This completes the proof  $\square$

**Theorem 3.2.** Let  $W_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  be the quadratic operator given above. Then for any  $0 < \varepsilon < \frac{1}{12}$ , one has that  $\mathbf{Fix}(W_\varepsilon) = \left\{ \left( \frac{1+2a_0(1-a_0)}{2(1+a_0)}, a_0, \frac{1-2a_0}{2(1+a_0)} \right) \right\}$  where  $a_0$  is the unique positive root in  $(0, \frac{1}{2})$  of the following quartic equation

$$(2 - 12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon = 0.$$

*Proof.* Since  $x_3 = 1 - x_1 - x_2$ , it is enough to find all solutions  $(x_1, x_2)$  of the following system of equations

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 + 10\varepsilon x_2^2 + (14\varepsilon - \frac{1}{2})x_1x_2 - 14\varepsilon x_2 + \varepsilon = 0 \end{cases} \quad (3.2)$$

which satisfy the conditions  $0 < x_1, x_2 < 1$  and  $0 < x_1 + x_2 < 1$ .

Let  $x_1 = x$  be a variable and  $x_2 = a$  be a parameter. Then the system (3.2) takes the following form

$$x^2 + \frac{(12\varepsilon - 1)a - 16\varepsilon}{2\varepsilon}x + (2a^2 - 6a + 5) = 0, \quad (3.3)$$

$$x^2 + \frac{a(14\varepsilon - \frac{1}{2})}{\varepsilon}x + (10a^2 - 14a + 1) = 0. \quad (3.4)$$

Due to Proposition 3.1, these two quadratic equations cannot have two common roots. Hence, the system (3.2) has a solution  $(x_1, x_2)$  with  $0 < x_1, x_2 < 1$ ,  $0 < x_1 + x_2 < 1$  if and only if two quadratic equations (3.3) and (3.4) must have a unique common root in  $(0, 1)$  for  $a \in (0, 1)$ . We know (see [20]) that two quadratic equations (3.3) and (3.4) have a unique common root if and only if their resultant is equal to zero, i.e.,

$$(2 - 12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon = 0. \quad (3.5)$$

In this case,  $x = \frac{1+2a(1-a)}{2(1+a)}$  is the unique common root of two quadratic equations (3.3) and (3.4). In order to have conditions  $0 < x_1, x_2 < 1$ ,  $0 < x_1 + x_2 < 1$ , we have to solve the following system of inequalities

$$\begin{cases} 0 < \frac{1+2a(1-a)}{2(1+a)} < 1 \\ 0 < a < 1 \\ 0 < a + \frac{1+2a(1-a)}{2(1+a)} < 1. \end{cases} \quad (3.6)$$

The solution of the system (3.6) is  $a \in (0, \frac{1}{2})$ . Therefore, the total number of solutions  $(x_1, x_2)$ ,  $0 < x_1, x_2 < 1$ ,  $0 < x_1 + x_2 < 1$  of the system (3.2) is the same as the total number of roots of the quartic equation (3.5) in the interval  $(0, \frac{1}{2})$ . Moreover, there is one-to-one correspondence between a root  $a_0 \in (0, \frac{1}{2})$  of the quartic equation (3.5) and a fixed point  $\left( \frac{1+2a_0(1-a_0)}{2(1+a_0)}, a_0, \frac{1-2a_0}{2(1+a_0)} \right)$  of the quadratic operator  $W_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ .

We are aiming to study the number of positive roots of the quartic equation (3.5) in the interval  $(0, \frac{1}{2})$ . Let  $f(a) = (2 - 12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon$ . Since  $0 < \varepsilon < \frac{1}{12}$ , it is easy to check that

$$f(0) = 5\varepsilon > 0, \quad f\left(\frac{1}{2}\right) = \frac{18\varepsilon - 9}{8} < 0, \quad f(2) = 18 - 27\varepsilon > 0.$$

This means that the quartic equation (3.5) has at least two positive roots. On the other hand, due to Descartes's theorem, the number of positive roots cannot be more than the number of sign changes between consecutive nonzero coefficients  $2 - 12\varepsilon$ ,  $16\varepsilon$ ,  $16\varepsilon - 3$ ,  $-(16\varepsilon + 1)$ ,  $5\varepsilon$  of the quartic equation (3.5) which is two.

Therefore, the quartic equation (3.5) has exactly two positive roots in which one of them belongs to  $(0, \frac{1}{2})$  and another one belongs to  $(\frac{1}{2}, 2)$ . Hence, for any  $0 < \varepsilon < \frac{1}{12}$ , there exists a unique positive root  $a_0$  of the quartic equation (3.5) in the interval  $(0, \frac{1}{2})$ . Consequently, for any  $0 < \varepsilon < \frac{1}{12}$ , the quadratic operator  $W_\varepsilon$  has a unique fixed point  $\left(\frac{1+2a_0(1-a_0)}{2(1+a_0)}, a_0, \frac{1-2a_0}{2(1+a_0)}\right)$ . This completes the proof  $\square$

#### 4. Positive Quadratic Stochastic Operator Having Three Fixed Points

In this section, we provide an example for a quadratic operator with positive coefficients having three fixed points in the simplex  $\mathbb{S}^2$ .

Let  $\mathbf{A}(0.1, 0.2, 0.7)$ ,  $\mathbf{B}(0.4, 0.3, 0.3)$  and  $\mathbf{C}(0.59, 0.31, 0.1)$  be points in the simplex. We define a positive quadratic operator  $\mathcal{Q}_0 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ ,  $\mathcal{Q}_0(\mathbf{x}) = \mathbf{x}' = (x'_1, x'_2, x'_3)$  as follows

$$\mathcal{Q}_0 : \begin{cases} x'_1 = \frac{232873}{319300}x_1^2 + \frac{4717}{10300}x_2^2 + \frac{207}{63860}x_3^2 + \frac{7}{5}x_1x_2 + \frac{3}{5}x_1x_3 + \frac{1}{50}x_2x_3 \\ x'_2 = \frac{27}{100}x_1^2 + \frac{1}{2}x_2^2 + \frac{3}{20}x_3^2 + \frac{470171}{814300}x_1x_2 + \frac{378421}{407150}x_1x_3 + \frac{158157}{814300}x_2x_3 \\ x'_3 = \frac{54}{79825}x_1^2 + \frac{433}{10300}x_2^2 + \frac{27037}{31930}x_3^2 + \frac{18409}{814300}x_1x_2 + \frac{191589}{407150}x_1x_3 + \frac{1454157}{814300}x_2x_3 \end{cases}.$$

The straightforward calculation shows that  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are fixed points of the quadratic operator  $\mathcal{Q}_0 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ .

We can define another positive quadratic operator  $\mathcal{Q}_1 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ ,  $\mathcal{Q}_1(\mathbf{x}) = \mathbf{x}' = (x'_1, x'_2, x'_3)$  as follows

$$\mathcal{Q}_1 : \begin{cases} x'_1 = \frac{17322871}{22351000}x_1^2 + \frac{990257}{2163000}x_2^2 + \frac{1559}{13410600}x_3^2 + \frac{13}{10}x_1x_2 + \frac{16}{25}x_1x_3 + \frac{11}{500}x_2x_3 \\ x'_2 = \frac{224}{1000}x_1^2 + \frac{488}{1000}x_2^2 + \frac{125}{1000}x_3^2 + \frac{703327}{1017875}x_1x_2 + \frac{19461451}{24429000}x_1x_3 + \frac{8271787}{24429000}x_2x_3 \\ x'_3 = \frac{4301}{4470200}x_1^2 + \frac{117199}{2163000}x_2^2 + \frac{2933179}{3352650}x_3^2 + \frac{18371}{2035750}x_1x_2 + \frac{13761989}{24429000}x_1x_3 + \frac{1601951}{977160}x_2x_3 \end{cases}$$

The straightforward calculation shows that  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are also fixed points of the quadratic operator  $\mathcal{Q}_1 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ .

In addition, if  $\mathcal{Q} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  is a positive quadratic stochastic operator on 2D simplex then  $|\mathbf{Fix}(\mathcal{Q})| = 1$  or 3 (see [8]). Therefore, the quadratic operators  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$  have exactly three fixed points;  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{C}$ .

Now, we can define a family of positive quadratic operators  $\mathcal{Q}_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  as  $\mathcal{Q}_\varepsilon(\mathbf{x}) = (1 - \varepsilon)\mathcal{Q}_0(\mathbf{x}) + \varepsilon\mathcal{Q}_1(\mathbf{x})$  for any  $\mathbf{x} \in \mathbb{S}^2$  and  $0 \leq \varepsilon \leq 1$ . It is clear that  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are also fixed points of the family of positive quadratic operators  $\mathcal{Q}_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ .

In the paper [9], it was conjectured that *if the set of stationary vectors of a second-order Markov chain contains  $k$ -interior points of the  $(k - 1)$ -dimensional face of the simplex then every vector in the  $(k - 1)$ -dimensional face is a stationary vector.*

However, this conjecture is wrong. The family of quadratic stochastic operators  $\mathcal{Q}_\varepsilon : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  defined above are counterexamples to this conjecture.

#### Acknowledgments

This work was partially supported by the MOHE grant FRGS14-141-0382. The first author (M.S.) is grateful to the Junior Associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

## References

- [1] Bernstein S 1942 *Annals Math. Statist.* **13** 53–61
- [2] Ganikhodjaev N, Saburov M and Jamilov U 2013 *App. Math. Info. Sci.* **7**(5) 1721–1729
- [3] Ganikhodjaev N, Saburov M and Nawi A.M. 2014 *The Scientific World Jour.* **2014** 1–11
- [4] Ganikhodzhaev R, Mukhamedov F and Rozikov U 2011 *Inf. Dim. Anal. Quan. Prob. Rel. Top.* **14**(2) 279–335
- [5] Ganikhodjaev R, Saburov M and Saburov M 2013 *AIP Conf. Proc.* **1557** 108–111
- [6] Kesten H 1970 *Adv. App. Prob.* **2** 1–82
- [7] Krapivin A A 1975 *Theor. Funct Funct Anal Appl* **25** 62–67
- [8] Lyubich Yu I 1992 *Mathematical structures in population genetics* (Berlin:Springer–Verlag)
- [9] Li C-K and Zhang S 2015 *Lin. Alg. Appl.* **473** 114–125
- [10] Mukhamedov F and Ganikhodjaev N 2015 *Lect. Notes Math.* **2133** (Berlin: Springer)
- [11] Mukhamedov F, Akin H and Temir S 2005 *J. Math. Anal. Appl.* **310** 533–556
- [12] Mukhamedov F and Saburov M 2010 *App. Math. Info. Sci.* **4** 47–62
- [13] Mukhamedov F and Saburov M 2014 *Bull. Malay. Math. Sci. Soc* **37** 59–64
- [14] Mukhamedov F, Saburov M and Qaralleh I 2013 *Abst. Appl. Anal.* **2013** 942038
- [15] Saburov M 2013 *World Applied Science Jour.* **21** 94–97
- [16] Saburov M 2015 *Math Notes* **97**(5-6) 759–763
- [17] Saburov M 2015 *Ann. Funct. Anal.* **6**(4) 247–254
- [18] Saburov M and Saburov Kh. 2014 *ScienceAsia* **40**(4) 306–312
- [19] Saburov M and Saburov Kh. 2014 *Inter Journal of Cont Auto Systems* **12**(6) 1276–1282
- [20] Sturmfels B 2002 *Solving systems of polynomial equations* (Texas)
- [21] Ulam S 1960 *A collection of mathematical problems* (New-York, London)