$b-$ Bistochastic Quadratic Stochastic Operators and Their Properties

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Abstract. In the present paper, we consider a class of quadratic stochastic operators (q.s.o.) called b-bistochastic q.s.o. We study several descriptive properties of b- bistochastic q.s.o. It turns out that, upper triangular stochastic matrix defines a linear b-stochastic operator. This allowed us to find some sufficient conditions on cubic stochastic matrix to be a $b-$bistochastic q.s.o.

1. Introduction

The history of quadratic stochastic operators (q.s.o.) can be traced back to Bernstein's work [1] where such kind of operators appeared from the problems of population genetics (see also [7]). Such kind of operators describe time evolution of variety species in biology are represented by so-called Lotka-Volterra(LV) systems [20]. Nowadays, scientists are interested in these operators, since they have a lot of applications especially in modelings in many different fields such as biology [5, 15], physics [17, 19], economics and mathematics [7, 15, 19].

Let us recall how q.s.o. appears in biology. The time evolution of species in biology can be comprehended by the following situation. Let $I = \{1, 2, \dots, n\}$ be the n type of species (or traits) in a population and we denote $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$ to be the probability distribution of the species in an early state of that population. By $P_{ij,k}$ we denote the probability of an individual in the i^{th} species and j^{th} species to cross-fertilize and produce an individual from k^{th} species (trait). Given $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$, we can find the probability distribution of the first generation, $x^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})$ by using a total probability, i.e.

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k \in \{1, \dots, n\}.$$

This relation defines an operator which is denoted by V and it is called *quadratic stochastic operator (q.s.o.)*. In other words, each q.s.o. describes the sequence of generations in terms of probabilities distribution, if the values of $P_{ij,k}$ and the distribution of the current generation are given. The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators. Presently, there is only a small number of studies on dynamical phenomena on higher dimensional systems, even though they are very important (see for example, [6, 11, 12, 13, 18]). In case of q.s.o., the difficulty of the problem depends on the given cubic matrix $(P_{ijk})_{i,j,k=1}^{m}$. In [3, 10], it has given along self-contained exposition of the recent achievements and open problems in the theory of the q.s.o.

In [16] a new majorization was introduced, and it opened a path for the study to generalize the theory of majorization by Hardy, Littlewood and Polya [4]. The new majorization has an advantage as compared to the classical one, since it can be defined as a partial order on sequences. While the classical one is not an antisymmetric order (because any sequence is majorized by any2 of its permutations), it is only defined as a preorder on sequence [16]. Most of the works in the mentioned paper were devoted to the investigation of majorized linear operators (see [4, 16]. Therefore, it is natural to study nonlinear majorized operators.

In what follows, to differentiate between the terms majorization and classical majorization that was popularized by Hardy et al.[4], we call majorization as $b-$order (which is denoted as $\leq^b$) while classical majorization as majorization (which is denoted as $\prec$) only. In [2] it was introduced and studied q.s.o. with a property $V(\mathbf{x}) \prec \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$. Such an operator is called *bistochastic*. In [9], it was proposed to a definition of bistochastic q.s.o. in terms of b- order. Namely, a q.s.o. is called $b-$*bistochastic* if $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x}$ taken from the $n-1$-dimensional simplex. In this paper we continue our previous investigations on b-bistochastic operators. We study several descriptive properties of b- bistochastic q.s.o. It turns out that, upper triangular stochastic matrix defines a linear b-stochastic operator. This allowed us to find some sufficient conditions on cubic stochastic matrix to be a $b-$bistochastic q.s.o.

2. Preliminaries

In this section we recall necessary definitions and facts about b-bistochastic operators.

Throughout this paper we consider the simplex

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \quad |x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1 \right\}. \tag{2.1}$$

For each $k \in \{1, \dots, n-1\}$ we define functional $\mathcal{U}_k : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{U}_k(x_1, \dots, x_n) = \sum_{i=1}^{k} x_i. \tag{2.2}$$

Let $\mathbf{x}, \mathbf{y} \in S^{n-1}$. We say that $\mathbf{x}$ is *b-ordered* or *b-majorized* by $\mathbf{y}$ ($\mathbf{x} \leq^b \mathbf{y}$) if and only if $\mathcal{U}_k(\mathbf{x}) \leq \mathcal{U}_k(\mathbf{y})$, for all $k \in \{1, \dots, n-1\}$.

The introduced relation is partial order i.e. it satisfies the following conditions:

(i) For any $\mathbf{x} \in S^{n-1}$ one has $\mathbf{x} \leq^b \mathbf{x}$,

(ii) If $\mathbf{x} \leq^b \mathbf{y}$ and $\mathbf{y} \leq^b \mathbf{x}$ then $\mathbf{x} = \mathbf{y}$,

(iii) If $\mathbf{x} \leq^b \mathbf{y}$, and $\mathbf{y} \leq^b \mathbf{z}$ then $\mathbf{x} \leq^b \mathbf{z}$.

Using the defined order, one can define the classical majorization [8]. First, recall that for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in S^{n-1}$, by $\mathbf{x}_{[\downarrow]} = (x_{[1]}, x_{[2]}, \dots, x_{[n]})$ one denotes the *rearrangement* of $\mathbf{x}$ in non-increasing order, i.e. $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. Take $\mathbf{x}, \mathbf{y} \in S^{n-1}$, then it is said that an element $\mathbf{x}$ *is majorized by* $\mathbf{y}$ and denoted $\mathbf{x} \prec \mathbf{y}$ if $\mathbf{x}_{[\downarrow]} \leq^b \mathbf{y}_{[\downarrow]}$. We refer the reader to [8] for more information regarding to this topic.

Recall that any operator V with $V(S^{n-1}) \subset S^{n-1}$ is called *stochastic*.

Definition 2.1. *A stochastic operator V is called $b-$bistochastic if one has $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$.*

Note that, the simplest nonlinear operators are quadratic ones. Therefore, we restrict ourselves to such kind of operators. Namely, a stochastic operator $V : S^{n-1} \to S^{n-1}$ is called *quadratic stochastic operator (q.s.o.)* if V has the following form:

$$V(\mathbf{x})_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \quad k = 1, 2, \ldots, n, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n) \in S^{n-1}, \tag{2.3}$$

where $\{P_{ij,k}\}$ are the heredity coefficients with the following properties:

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{n} P_{ij,k} = 1, \quad i, j, k = 1, 2, \ldots, n \ . \tag{2.4}$$

Remark 2.2. *If a q.s.o. V satisfies $V(\mathbf{x}) \prec \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$, then it is called bistochastic [2]. In our definition, we are taking $b-$order instead of the majorization. Note that if one takes absolute continuity instead of the $b-$order, then $b-$bistochastic operator reduces to Volterra q.s.o. [14].*

Let V be a q.s.o., then one can define an associated matrix $\mathbb{T}_n(\mathbf{x}) = [T_{ik}(\mathbf{x})]_{i,k=1}^{n}$, $\mathbf{x} \in S^{n-1}$ by

$$T_{ik}(\mathbf{x}) = \sum_{j=1}^{n} P_{ij,k} x_j, \ \mathbf{x} = (x_i) \in S^{n-1}, \tag{2.5}$$

where $\{P_{ij,k}\}$ are the heredity coefficients. One can see that $\mathbb{T}_n(\mathbf{x})$ is a stochastic matrix. Moreover, one has $V(\mathbf{x}) = \mathbf{x}\mathbb{T}_n(\mathbf{x})$, and $\mathbb{T}_n(\mathbf{x}) = \sum\limits_{l=1}^{n} x_l \mathbb{T}_n(\mathbf{e}_l)$. Hence, each q.s.o. V can be uniquely defined by stochastic matrices, i.e.

$$V = \{\mathbb{T}_n(\mathbf{e}_1), \mathbb{T}_n(\mathbf{e}_2), \ldots, \mathbb{T}_n(\mathbf{e}_n)\} . \tag{2.6}$$

3. Description of b-bistochastic q.s.o.

In this section, we are going to provide some general properties of $b-$bistochastic q.s.o. In [9], we have proved the following fact.

Theorem 3.1. *[9] Let V be a $b-$bistochastic q.s.o. defined on S^{n-1}, then the following statements hold:*

(i) $\sum\limits_{m=1}^{k} \sum\limits_{i,j=1}^{n} P_{ij,m} \leq kn, \ k \in \{1, \ldots, n\}$

(ii) $P_{ij,k} = 0$ *for all* $i, j \in \{k+1, \ldots, n\}$ *where* $k \in \{1, \ldots, n-1\}$

(iii) $P_{nn,n} = 1$

(iv) for every $\mathbf{x} \in S^{n-1}$ *one has*

$$V(\mathbf{x})_k = \sum_{l=1}^{k} P_{ll,k} x_l^2 + 2 \sum_{l=1}^{k} \sum_{j=l+1}^{n} P_{lj,k} x_l x_j \quad \textit{where} \quad k = \overline{1, n-1}$$
$$V(\mathbf{x})_n = x_n^2 + \sum_{l=1}^{n-1} P_{ll,n} x_l^2 + 2 \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} P_{lj,n} x_l x_j.$$

(v) $P_{lj,l} \leq \dfrac{1}{2}$ *for all* $j \geq l+1, \ l \in \{1, \ldots, n-1\}$.

This properties are only necessary conditions for a $b-$bistochastic q.s.o.. Indeed, consider the following example.

Example 3.2. *Let $V : S^3 \to S^3$ be a q.s.o given by the following heredity coefficients*

$$\left\{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & P_{4,1,3} & 1-P_{4,1,3} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & P_{3,2,3} & 1/2-P_{3,2,3} \\ 0 & 1/2 & P_{4,2,3} & 1/2-P_{4,2,3} \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/2 & P_{3,2,3} & 1/2-P_{3,2,3} \\ 0 & 0 & P_{3,3,3} & 1-P_{3,3,3} \\ 0 & 0 & P_{4,3,3} & 1-P_{4,3,3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & P_{4,1,3} & 1-P_{4,1,3} \\ 0 & 1/2 & P_{4,2,3} & 1/2-P_{4,2,3} \\ 0 & 0 & P_{4,3,3} & 1-P_{4,3,3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \right\}$$

where $2P_{4,1,3} + 2P_{3,2,3} + 2P_{4,2,3} + P_{3,3,3} + 2P_{4,3,3} \le 4$. Note that $P_{ij,k} = P_{ji,k}$. One can see that such an operator is not a $b-$bistochastic q.s.o., since for $\mathbf{x} = (0.1, 0.1, 0.8, 0)$ we have

$$\begin{aligned} V(\mathbf{x})_1 + V(\mathbf{x})_2 &= P_{1,1,1}{x_1}^2 + 2\,P_{1,2,1}x_1x_2 + 2\,P_{1,3,1}x_1x_3 + 2\,P_{1,4,1}x_1\,(1-x_1-x_2-x_3) \\ &\quad + P_{1,1,2}{x_1}^2 + 2\,P_{1,2,2}x_1x_2 + 2\,P_{1,3,2}x_1x_3 + 2\,P_{1,4,2}x_1\,(1-x_1-x_2-x_3) \\ &\quad + P_{2,2,2}{x_2}^2 + 2\,P_{2,3,2}x_2x_3 + 2\,P_{2,4,2}x_2\,(1-x_1-x_2-x_3) \\ &= 0 + 0.28 \ge 0.1 + 0.1. \end{aligned}$$

Moreover, under the condition $2P_{4,1,3} + 2P_{3,2,3} + 2P_{4,2,3} + P_{3,3,3} + 2P_{4,3,3} \le 4$, the property (i) in Theorem 3.1 holds. The other properties also satisfy accordingly due to the construction of the q.s.o.

In what follow, we want to recall a fascinating result on $b-$bistochastic linear operators. Let T be a linear stochastic operator $T : S^{n-1} \to S^{n-1}$ such that

$$T(\mathbf{x})_k = \sum_{i=1}^{n} t_{ik}x_i \quad \text{where} \quad t_{ik} \ge 0, \ \sum_{k=1}^{n} t_{ik} = 1, \quad \mathbf{x} = (x_1, \ldots, x_n) \in S^{n-1}. \tag{3.1}$$

In [16], it was showed the simplified form of linear $b-$bistochastic operators. Namely,

Theorem 3.3. *[16] Let T be a linear stochastic operator defined on S^{n-1}. Then T is a $b-$bistochastic if and only if $\mathbb{T}$ is an upper triangular stochastic matrix, i.e.*

$$\mathbb{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \ldots & t_{1n} \\ 0 & t_{22} & t_{23} & \ldots & t_{2n} \\ 0 & 0 & t_{33} & \ldots & t_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

From this theorem, it is interesting to know the following question: if we take $\{\mathbb{T}_n(\mathbf{e}_k)\}$ (in the representation (2.6)) all the stochastic matrices to be upper triangular, then would V be a b-stochastic q.s.o.?

For each $j \in \{1, \ldots, n\}$ let $\mathbb{T}_n(\mathbf{e}_j)$ be a stochastic matrix given be

$$\mathbb{T}_n(\mathbf{e}_j) = \left(a_{ik}^{(j)}\right)_{i,k=1}^{n}; \quad j = 1, 2, \ldots, n. \tag{3.2}$$

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We define a quadratic operator $V : S^{n-1} \to S^{n-1}$ by

$$V(\mathbf{x})_k = \sum_{i,j}^{n} a_{ik}^{(j)} x_i x_j; \quad k = \overline{1,n}, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n) \in S^{n-1}. \tag{3.3}$$

Let us assume that, $\mathbb{T}_n(\mathbf{e}_j)$ is taking upper triangular stochastic form i.e.

$$\mathbb{T}_n(\mathbf{e}_j) = \begin{bmatrix} a_{11}^{(j)} & a_{12}^{(j)} & a_{13}^{(j)} & \ldots & a_{1n}^{(j)} \\ 0 & a_{22}^{(j)} & a_{23}^{(j)} & \ldots & a_{2n}^{(j)} \\ 0 & 0 & a_{33}^{(j)} & \ldots & a_{3n}^{(j)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \tag{3.4}$$

for every $j = 1, 2, \ldots, n$.

Theorem 3.4. *Let $\{\mathbb{T}_n(\mathbf{e}_j)\}$, $j = 1, 2, \ldots, n$ be a collection of upper triangular stochastic matrices and V be the associated q.s.o. defined by* (3.3)*. Then V is a b-bistochastic operator.*

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Using the fact

$$V(\mathbf{x})_k = \sum_{i,j=1}^{n} a_{ik}^{(j)} x_i x_j = \sum_{i=1}^{k} \sum_{j=1}^{n} a_{ik}^{(j)} x_i x_j$$

one gets

$$\begin{aligned}
\sum_{l=1}^{k} V(\mathbf{x})_l &= \sum_{j=1}^{n} a_{11}^{(j)} x_1 x_j + \sum_{i=1}^{2} \sum_{j=1}^{n} a_{i2}^{(j)} x_i x_j + \cdots + \sum_{i=1}^{k} \sum_{j=1}^{n} a_{ik}^{(j)} x_i x_j \\
&= x_1 \sum_{j=1}^{n} a_{11}^{(j)} x_j + x_1 \sum_{j=1}^{n} a_{12}^{(j)} x_j + x_2 \sum_{j=1}^{n} a_{22}^{(j)} x_j + \cdots + \\
&\quad x_1 \sum_{j=1}^{n} a_{1k}^{(j)} x_j + x_2 \sum_{j=1}^{n} a_{2k}^{(j)} x_j + \cdots + x_k \sum_{j=1}^{n} a_{kk}^{(j)} x_j \\
&= x_1 \left(\sum_{l=1}^{k} \sum_{j=1}^{n} a_{1l}^{(j)} x_j \right) + x_2 \left(\sum_{l=2}^{k} \sum_{j=1}^{n} a_{2l}^{(j)} x_j \right) + \cdots + x_k \sum_{j=1}^{n} a_{kk}^{(j)} x_j \\
&= \sum_{i=1}^{k} x_i \left(\sum_{j=1}^{n} \sum_{l=i}^{k} a_{il}^{(j)} x_j \right).
\end{aligned}$$

We know that $\sum_{l=i}^{k} a_{il}^{(j)} \leq 1$ for each i, hence

$$\sum_{l=1}^{k} V(x)_l \leq \sum_{i=1}^{k} x_i \left(\sum_{j=1}^{n} x_j \right) = \sum_{i=1}^{k} x_i$$

This completes the prove. $\square$

Remark 3.5. *It is worth to note, in general the necessary and sufficient conditions of $b-$bistochastic quadratic operators may not need to be upper triangular stochastic matrices. Let us choose the following quadratic operators (indeed it is a q.s.o.):*

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}. \tag{3.5}$$

Let $\mathbf{x} = (x_1, x_2, x_3) \in S^2$, then one has $V(\mathbf{x})_1 = x_1x_2$ and $V(\mathbf{x})_2 = 0$. In addition, the statement $x_2 \leq 1$ and $x_1x_2 \leq x_1$ is always true for any $\mathbf{x} = (x_1, x_2, x_3) \in S^2$. Therefore, one finds that

$$V(\mathbf{x})_1 = x_1x_2 \leq x_1, \quad V(\mathbf{x})_1 + V(\mathbf{x})_2 = x_1x_2 \leq x_1 + x_2.$$

which shows q.s.o. given by (3.5) *is a $b-$bistochastic q.s.o..*

Furthermore, due to Theorem 3.1, we introduce a stochastic cubic matrix such that

$$\left\{ \begin{bmatrix} a^{(1)}_{11} & a^{(1)}_{12} & a^{(1)}_{13} & \dots & a^{(1)}_{1n} \\ a^{(1)}_{21} & a^{(1)}_{22} & a^{(1)}_{23} & \dots & a^{(1)}_{2n} \\ a^{(1)}_{31} & a^{(1)}_{32} & a^{(1)}_{33} & \dots & a^{(1)}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{(1)}_{n1} & a^{(1)}_{n2} & a^{(1)}_{n3} & \dots & a^{(1)}_{nn} \end{bmatrix}, \begin{bmatrix} a^{(2)}_{11} & a^{(2)}_{12} & a^{(2)}_{13} & \dots & a^{(2)}_{1n} \\ 0 & a^{(2)}_{22} & a^{(2)}_{23} & \dots & a^{(2)}_{2n} \\ 0 & a^{(2)}_{32} & a^{(2)}_{33} & \dots & a^{(2)}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a^{(2)}_{n2} & a^{(2)}_{n3} & \dots & a^{(2)}_{nn} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} a^{(3)}_{11} & a^{(3)}_{12} & a^{(3)}_{13} & \dots & a^{(3)}_{1n} \\ 0 & a^{(3)}_{22} & a^{(3)}_{23} & \dots & a^{(2)}_{2n} \\ 0 & 0 & a^{(3)}_{33} & \dots & a^{(3)}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a^{(3)}_{n3} & \dots & a^{(3)}_{nn} \end{bmatrix}, \dots, \begin{bmatrix} a^{(n)}_{11} & a^{(n)}_{12} & a^{(n)}_{13} & \dots & a^{(n)}_{1n} \\ 0 & a^{(n)}_{22} & a^{(n)}_{23} & \dots & a^{(n)}_{2n} \\ 0 & 0 & a^{(n)}_{33} & \dots & a^{(n)}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \right\}$$

In short,

$$\mathbb{T}_n(\mathbf{e}_j) = \left(a^{(j)}_{ik}\right)_{i,k=1}^{n}, \; j = 1, 2, \dots n$$

where

$$a^{(j)}_{i,k} = 0; \; i, j = \overline{k+1, n}, \quad k = \overline{1, n-1} \tag{3.6}$$

Note that, under the constraints given by (3.6), we shall use Theorem 3.1 which reduces the components of vector $V(x)_k$ to

$$V(\mathbf{x})_k = \sum_{l=1}^{k} a^{(l)}_{l,k} x_l^2 + \sum_{l=1}^{k} \sum_{j=l+1}^{n} \left(a^{(j)}_{l,k} + a^{(l)}_{j,k}\right) x_l x_j \text{ where } k = \overline{1, n-1} \tag{3.7}$$

$$V(\mathbf{x})_n = x_n^2 + \sum_{l=1}^{n-1} a^{(l)}_{l,n} x_l^2 + \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \left(a^{(j)}_{l,n} + a^{(l)}_{j,n}\right) x_l x_j \tag{3.8}$$

Theorem 3.6. *Let V be a quadratic operator given by* (3.3) *and let the corresponding collection $\{\mathbb{T}_n(\mathbf{e}_j)\}$ satisfy the constraints* (3.6)*. If*

$$\sum_{l=1}^{n-1} \left(a^{(j)}_{i,l} + a^{(i)}_{j,l}\right) \leq 1 \quad \textit{for every} \quad j = \overline{i+1, n} \quad \textit{where } i = \overline{1, n-1}, \tag{3.9}$$

then V is a b-bistochastic operator.

Proof. Clearly from (3.7) one gets

$$\begin{aligned}
V(\mathbf{x})_1 &= a_{1,1}^{(1)}x_1^2 + \sum_{j=2}^{n}\left(a_{1,1}^{(j)} + a_{j,1}^{(1)}\right)x_1x_j \\
V(\mathbf{x})_2 &= a_{1,2}^{(1)}x_1^2 + a_{2,2}^{(1)}x_2^2 + \sum_{j=2}^{n}\left(a_{1,2}^{(j)} + a_{j,2}^{(1)}\right)x_1x_j + \sum_{j=3}^{n}\left(a_{2,2}^{(j)} + a_{j,2}^{(2)}\right)x_2x_j \\
&\vdots \\
V(\mathbf{x})_k &= a_{1,k}^{(1)}x_1^2 + \cdots + a_{k,k}^{(k)}x_k^2 + \sum_{j=2}^{n}\left(a_{1,k}^{(j)} + a_{j,k}^{(1)}\right)x_1x_j + \cdots + \\
&\quad + \sum_{j=k+1}^{n}\left(a_{k,k}^{(j)} + a_{j,k}^{(k)}\right)x_kx_j
\end{aligned}$$

Therefore, one finds that

$$\begin{aligned}
\sum_{l=1}^{k} V(\mathbf{x})_l &= x_1\left(x_1\left(\sum_{l=1}^{k} a_{1,l}^{(1)}\right) + \sum_{j=2}^{n}\left(\sum_{l=1}^{k}\left(a_{1,l}^{(j)} + a_{j,l}^{(1)}\right)\right)x_j\right) + \\
&\quad x_2\left(x_2\left(\sum_{l=2}^{k} a_{2,l}^{(2)}\right) + \sum_{j=3}^{n}\left(\sum_{l=2}^{k}\left(a_{2,l}^{(j)} + a_{j,l}^{(2)}\right)\right)x_j\right) + \cdots + \\
&\quad x_k\left(x_k\left(a_{k,k}^{(k)}\right) + \sum_{j=k+1}^{n}\left(a_{k,k}^{(j)} + a_{j,k}^{(k)}\right)x_j\right),
\end{aligned}$$

for any $k = 1, 2, \ldots, n-1$.
Due to stochasticity, then we know $\sum_{l=i}^{k} a_{i,l}^{(i)} \leq 1$ for every $i = \overline{1,k}$. Moreover, based on the conditions (3.9), we have

$$x_t\left(x_t\left(\sum_{l=t}^{k} a_{t,l}^{(t)}\right) + \sum_{j=t+1}^{n}\left(\sum_{l=t}^{k}\left(a_{t,l}^{(j)} + a_{j,l}^{(t)}\right)\right)x_j\right) \leq x_t\left(x_t + \sum_{j=2}^{n} x_j\right)$$

for any $t \in \{1, \ldots, k\}$. By virtue of the last inequality, it will imply that

$$\begin{aligned}
\sum_{l=1}^{k} V(\mathbf{x})_l &\leq x_1\left(x_1 + \sum_{j=2}^{n} x_j\right) + x_2\left(x_2 + \sum_{j=3}^{n} x_j\right) + \cdots + x_k\left(x_k + \sum_{j=k+1}^{n} x_j\right) \\
&\leq \sum_{l=1}^{k} x_l
\end{aligned}$$

This completes the proof. □

Remind that, the reverse is not true. For instance, a quadratic operator (again it is a q.s.o.)

$$\left\{\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right\}, \tag{3.10}$$

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which is a $b-$bistochastic operator i.e.

$$V(\mathbf{x})_1 = x_1x_2 \leq x_1 \text{ and } V(\mathbf{x})_1 + V(\mathbf{x})_2 = x_1x_2 + x_1x_2 \leq x_1 + x_2.$$

One can see that if $i = 1$ and $j = 2$, then $2a_{1,1}^{(2)} + 2a_{2,2}^{(1)} = 2 > 1$.

References

[1] Bernstein S 1942 *Ann. Math. Statistics* **13** 53–61
[2] Ganikhodzhaev R N 1993 *Russ. Math. Surv.* **48** 244–246.
[3] Ganikhodzhaev R, Mukhamedov F and Rozikov U 2011 *Infin. Dimens. Anal. Quant. Prob. Relat. Top.* **14** 279–335
[4] Hardy G H, Littlewood JE and Polya G 1952 *Inequalities* (Cambridge: Cambridge Univ. Press)
[5] Hofbauer J and Sigmund K 1998 *Evolutionary Games and Population Dynamics* (Cambridge: Cambridge Univ. Press)
[6] Kesten H 1972 *Adv. Appl. Probab.* **2** 1–82
[7] Lyubich Yu I 1992 *Mathematical structures in population genetics* (Berlin: Springer)
[8] Marshall A W, Olkin I and Arnold B.C 2011 *Inequalities: Theory of Majorization and its Applications* (Berlin: Springer)
[9] Mukhamedov F and Embong A F 2015 *J. Inequal. Appl.* **2015** 226
[10] Mukhamedov F and Ganikhodjaev N 2015 *Lect. Notes Math.* **2133** (Berlin: Springer)
[11] Mukhamedov F, Qaralleh I and Rozali W N F A W 2014 *Sains Malaysiana* **43** 1275–1281
[12] Mukhamedov F, Saburov M and Qaralleh I 2013 *Abst. Appl. Anal.* **2013** 942038
[13] Mukhamedov F and Supar N A 2015 *Bull. Malays. Math. Sci. Soc.* **38** 1281–1296
[14] Mukhamedov F and Taha M H 2015 *AIP Conf. Proc.* **1660** 050057
[15] Narendra S G, Samaresh C M and Elliott W M 1971 *Rev. Mod. Phys.* **43** 231–276
[16] Parker D S and Ram P (1996) *Greed and majorization* (US: University of California)
[17] Plank M and Losert V 1995 *J. Math. Phys.* **36** 3520-3543
[18] Rozikov U A and Zhamilov U U 2009 *Sb. Math.* **200** 1339–1351
[19] Takeuchi Y 1996 *Global dynamicsal properties of Lotka-Volterra systems* (Singapore: World Scientific)
[20] Volterra V 1927 *ssociation Franc. Lyon* **1926** 96-98