

On $HL^2(L, L)$ for semisimple Leibniz algebras

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Abstract. In this paper we present a decomposition of $HL^n(L, L)$ into a direct sum of some subspaces for a finite dimensional complex semisimple Leibniz algebra L . Furthermore, we provide a more specific decomposition in case $n = 2$ into two subspaces. We verify that one of those subspaces annihilates for specific Leibniz algebras with liezation \mathfrak{sl}_2 and some others.

1. Introduction

Leibniz algebras were rediscovered by A. Bloh [3] and later were given another impulse of investigation due to works of J.-L. Loday [7] who discovered that the classical Chevalley–Eilenberg boundary map in the exterior module of a Lie algebra can be lifted to the tensor module. To prove that it becomes a chain complex the only identity needed is a so called Leibniz identity which yields to definition of Leibniz algebra and Leibniz (co)homology.

In this work we study Leibniz algebra cohomology and using an idea from [1] we decompose the cochain space $CL^n(L, L)$ into a direct sub of special subspaces that we call level spaces. Further we concentrate on so called semisimple Leibniz algebras. They are Leibniz algebras with corresponding Lie algebras being semisimple. Semisimple Leibniz algebra L can be naturally consider as a \mathbb{Z}_2 -graded algebra with L_0 being its liezation and $L_1 = I$. We notice that due to structural properties of semisimple Leibniz algebras, coboundary operator preserves the level spaces. Thus we are able to establish in Theorem 4.3 a decomposition of $HL^n(L, L)$ into direct sum of n subspaces, that we label from $-n + 1$ to 0.

Our goal is to check proposed conjecture in [2] that $HL^2(L, L) = 0$ for any finite-dimensional complex semisimple Leibniz algebra L . Authors in [2] validate the claim for simple Leibniz algebra with liezation $L/I \cong \mathfrak{sl}_2$ over \mathbb{C} .

In order to conjecture to hold, we elaborate on each of the subspaces in the



decomposition of $HL^2(L, L)$ to be zero and present equivalent conditions to the conjecture to be true in Proposition 5.6 and Proposition 5.7 in terms of Lie algebra and their representations.

In the last section, we establish that space of level -1 is zero for all finite-dimensional Leibniz algebras with liezation \mathfrak{sl}_2 in Theorem 6.1 and we check analogous statement for another Leibniz algebra with liezation $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ in Theorem 6.2.

2. Preliminaries

Given a Lie algebra and its non-trivial module a different algebraic structure arises on the direct sum.

Example 2.1. ([5]) Let \mathfrak{g} be a Lie algebra and M be a \mathfrak{g} -module. Consider $L = \mathfrak{g} \oplus M$ with a bracket $[(g_1, m_1), (g_2, m_2)] := ([g_1, g_2], -g_2.m_1)$. Then

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \quad (1)$$

holds for any $x, y, z \in L$ and this algebra is not a Lie algebra if the action of \mathfrak{g} on M is not trivial.

Looking to the above identity (1) one notices that the right multiplication operator $[-, z]$ by any element z is a derivation (see [6]) and satisfies a so called Leibniz rule. Although defined earlier by A. Bloh [3] these objects attracted more attention after series of J.-L. Loday and his collaborators' works.

Definition 2.2. An algebra $(L, [-, -])$ over a field \mathbb{K} is called a Leibniz algebra if identity (1) holds.

Leibniz algebra of Example 2.1 is denoted as $\mathfrak{g} \dot{+} M$ and is called hemisemidirect product Leibniz algebra in [5].

Any Lie algebra is a Leibniz algebra but not the converse. Given a Leibniz algebra L its two sided ideal generated by elements $[x, x]$ for all $x \in L$ is very important. In this work we denote this ideal by I . The quotient Leibniz algebra L/I is easily seen to be a Lie algebra, called the liezation of Leibniz algebra L .

Definition 2.3. The set $\text{Ann}_R(L) = \{x \in L \mid [L, x] = 0\}$ of a Leibniz algebra L is called the right annihilator of L .

One can show that $\text{Ann}_R(L)$ is an ideal of L . Note that due to Leibniz identity it follows that $[L, I] = 0$. Thus I is a subset of a right annihilator $\text{Ann}_R(L) = \{x \in L \mid [L, x] = 0\}$. The center is defined as $Z(L) = \{x \in L \mid [x, L] = [L, x] = 0\}$. If $I = L$ then Leibniz algebra is a trivial algebra with all products being zero. If $I = 0$ then L is a Lie algebra. In this work we assume I to be non trivial, eliminating Lie and trivial Leibniz algebras from the study.

One has a short exact sequence $0 \rightarrow I \rightarrow L \xrightarrow{f} \mathfrak{g} \rightarrow 0$. Note that projection f is universal in the sense that a Leibniz map from L to any Lie algebra factors through f .

A representation M of a Leibniz algebra L is introduced in [6].

Definition 2.4. A vector space M is called a representation or bimodule over a Leibniz algebra L if there are two bilinear maps:

$$[-, -]: L \times M \rightarrow M \quad \text{and} \quad [-, -]: M \times L \rightarrow M$$

satisfying the following three axioms

$$\begin{aligned} [m, [x, y]] &= [[m, x], y] - [[m, y], x], \\ [x, [m, y]] &= [[x, m], y] - [[x, y], m], \\ [x, [y, m]] &= [[x, y], m] - [[x, m], y], \end{aligned}$$

for any $m \in M$, $x, y \in L$.

A Leibniz bimodule is called *anti-symmetric* when $[x, m] = 0$ for $x \in L$, $m \in M$. Provided that, the last two required identities vanish. Since ideal I is in the right-annihilator, if $x \in I$ or $y \in I$ the first identity vanishes as well. Therefore, the action of L on anti-symmetric L -bimodule M is determined by the action of complement of I that is required to satisfy only the first axiom.

Now for $g \in \mathfrak{g}$ define $g \circ m := -[m, \phi(g)]$, where ϕ is an inverse map of f . These are well-defined, since if $g = g'$ then $\phi(g - g') \in I$ and I is in the right annihilator. Moreover, the first identity turns into

$$\begin{aligned} [g_1, g_2] \circ m &= -[m, \phi([g_1, g_2])] = -[m, [\phi(g_1), \phi(g_2)]] = \\ &= -[[m, \phi(g_1)], \phi(g_2)] + [[m, \phi(g_2)], \phi(g_1)] = g_2 \circ [m, \phi(g_1)] - g_1 \circ [m, \phi(g_2)] \\ &= g_1 \circ (g_2 \circ m) - g_2 \circ (g_1 \circ m), \end{aligned}$$

i.e. M becomes a left \mathfrak{g} -module (or a right \mathfrak{g} -module with action $m \circ g = [m, \phi(g)]$).

Conversely, if M is a left \mathfrak{g} -module with an action $\circ : \mathfrak{g} \times M \rightarrow M$, then by defining $[m, l] := -f(l) \circ m$ for $l \in L$ implies by the arguments above that M is an anti-symmetric L -bimodule.

Thus, anti-symmetric Leibniz L -bimodule is equivalent to a Lie algebra \mathfrak{g} -module of its liezation \mathfrak{g} . For instance, an ideal generated by squares I can be considered as a Lie algebra module over liezation \mathfrak{g} .

Call Leibniz algebra L *semisimple* if \mathfrak{g} is semisimple. L is called *simple* if the only non-trivial ideal of L is $I \neq [L, L]$. These agree with suggested definition in [1]. T. Pirashvili proved the following statement.

Proposition 2.5. ([10]) Let $f : L \rightarrow \mathfrak{g}$ be an epimorphism from an arbitrary finite dimensional Leibniz algebra L to semisimple Lie algebra \mathfrak{g} . Then f admits a section.

Considering a semisimple Leibniz algebra L its liezation by definition is a semisimple Lie algebra. The fundamental projection onto its liezation admits a section due to the above proposition. This leads to a key fact that is the base of the study on decomposition of cohomology of semisimple Leibniz algebras in the following sections.

Corollary 2.6. *Let L be a finite dimensional semisimple Leibniz algebra with liezation \mathfrak{g} . Then $L \cong \mathfrak{g} \dot{+} I$.*

A cohomology of a Leibniz algebra L with coefficients in the representation M are defined in [8] as follows.

Define the space of n -cochains $CL^n(L, M) = \text{Hom}_{\mathbb{K}}(L^{\otimes n}, M)$ for $n \geq 0$ and a \mathbb{K} -homomorphism $\partial^n : CL^n(L, M) \rightarrow CL^{n+1}(L, M)$ by

$$\begin{aligned} (\partial^n f)(x_1, \dots, x_{n+1}) := & [x_1, f(x_2, \dots, x_{n+1})] + \\ & + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}), x_i] + \\ & + \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x}_j, \dots, x_{n+1}). \end{aligned}$$

This $(CL^*(L, M), \partial)$ is a cochain complex. Its n -th cohomology group is well defined by $HL^n(L, M) := ZL^n(L, M)/BL^n(L, M)$, where the elements $ZL^n(L, M) := \ker \partial^n$ and $BL^n(L, M) := \text{im } \partial^{n-1}$ are called n -cocycles and n -coboundaries, respectively.

Recall that it is noted in [8] that

$$M^L := \{m \in M \mid [l, m] = 0, \forall l \in L\} = HL^0(L, M)$$

which is called *bisubmodule of left invariants* of M .

In Leibniz algebras a derivation is defined as usual.

Definition 2.7. *A linear map $d: L \rightarrow M$ is called a derivation of L in M if*

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for any } x, y \in L.$$

Space of all derivations from L to M is denoted by $\text{Der}(L, M)$.

Moreover, for a given $m \in M$ a map $R_m : L \rightarrow M$ defined by $R_m(l) = [l, m]$ is a derivation. It is called an inner derivation and running through all of the elements of the bimodule we obtain space of inner derivation $\mathcal{I}nnder(L, M)$.

It is known that $HL^1(L, M) = \text{Der}(L, M)/\mathcal{I}nnder(L, M)$.

If M is an anti-symmetric L -bimodule then inner derivations are zero and

$$HL^1(L, M) = \text{Der}(L, M) = \{f : L \rightarrow M \mid f([l_1, l_2]) = [f(l_1), l_2]\}.$$

Further, in [8] considering L^a as the antisymmetric representation whose underlying \mathbb{K} module is L and action $L \times L^a \rightarrow L$ is Leibniz bracket of L , it is claimed that $HL^1(L, L^a) \neq 0$. Indeed, taking an identity map $id|_L$ we see that $id \in HL^1(L, L^a)$. With similar arguments once can obtain that $HL^1(L, I) \neq 0$ where I is an L -module by the algebra bracket action.

Note that when M is symmetric L -bimodule [8] establishes

$$HL^1(L, M) = HL^1(\mathfrak{g}, M) = H^1(\mathfrak{g}, M)$$

for liezation \mathfrak{g} of Leibniz algebra L .

However, similar statement for anti-symmetric M does not hold.

One of the main tools in this work relies on T. Pirashvili's result [10] on Leibniz cohomology of semisimple Lie algebras.

Theorem 2.8. [10] Let \mathfrak{g} be a finite dimensional semisimple Lie algebra and M be a Leibniz representation of \mathfrak{g} . Then $HL^n(\mathfrak{g}, M) = 0$ for all $n \geq 2$.

It was proved by constructing some spectral sequences. The same result was proved by P. Ntolo [9] using Casimir element and constructing the explicit homotopy.

3. Decomposition of $CL^n(L, L)$ for a Leibniz algebra

Let V be a vector space over a field \mathbb{K} and $V = V_0 \oplus V_1$ be a non-trivial decomposition. The goal of this section is to present the decomposition of $\text{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$ that will be heavily used in the next sections.

For a natural number n and an integer k define $\mathcal{I}(n, k)$ to be a set of bijections from $\{1, 2, \dots, n\}$ to a set of k ones and $n - k$ zeros for $0 \leq k \leq n$ and $\mathcal{I}(n, k) = \emptyset$ otherwise.

For a bijection $\pi \in \mathcal{I}(n, k)$ and a number j equal to 0 or 1 let us denote for short $H_{\pi(1)\dots\pi(n)}^j := \text{Hom}_{\mathbb{K}}(V_{\pi(1)} \otimes \dots \otimes V_{\pi(n)}, V_j)$.

Denote by $CL^n(V, V) = \text{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$ and define for $-n \leq k \leq 1$ a subspace $CL^n(V, V)_{(-k)} = \left(\bigoplus_{\pi \in \mathcal{I}(n, k)} H_{\pi(1)\dots\pi(n)}^0 \right) \oplus \left(\bigoplus_{\pi \in \mathcal{I}(n, k+1)} H_{\pi(1)\dots\pi(n)}^1 \right)$. Note that there are $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ summands in the above decomposition. Moreover, for any $H_{\pi(1)\dots\pi(n)}^j$ in this decomposition one has $k = j - (\pi(1) + \dots + \pi(n))$.

Let us call subspace $CL^n(V, V)_{(k)}$ of level k . The following statement presents a decomposition isomorphic to $CL^n(V, V)$ into subspaces of all possible levels $-n, \dots, 0, 1$.

Proposition 3.1. $CL^n(V, V) \cong \bigoplus_{k=-n}^1 CL^n(V, V)_{(-k)}$.

Proof. We have

$$\begin{aligned} CL^n(V, V) &= \text{Hom}(V^{\otimes n}, V) = \text{Hom}((V_0 \oplus V_1)^{\otimes n}, V_0 \oplus V_1) \\ &\cong \text{Hom}((V_0 \oplus V_1)^{\otimes n}, V_0) \oplus \text{Hom}((V_0 \oplus V_1)^{\otimes n}, V_1) \\ &\cong \text{Hom}(V_1 \otimes V_1 \otimes \dots \otimes V_1, V_0) \oplus \\ &\quad \oplus \left(\bigoplus_{i \in \mathcal{I}(n, n-1)} H_{i(1)\dots i(n)}^0 \oplus \bigoplus_{i \in \mathcal{I}(n, n)} H_{i(1)\dots i(n)}^1 \right) \oplus \\ &\quad \vdots \\ &\quad \oplus \left(\bigoplus_{i \in \mathcal{I}(n, 0)} H_{i(1)\dots i(n)}^0 \oplus \bigoplus_{i \in \mathcal{I}(n, 1)} H_{i(1)\dots i(n)}^1 \right) \oplus \\ &\quad \oplus \text{Hom}(V_0 \otimes V_0 \otimes \dots \otimes V_0, V_1), \end{aligned}$$

or in more compact form

$$CL^n(V, V) \cong \bigoplus_{k=-1}^n \left(\bigoplus_{i \in \mathcal{I}(n, k)} H_{i(1)\dots i(n)}^0 \right) \oplus \left(\bigoplus_{i \in \mathcal{I}(n, k+1)} H_{i(1)\dots i(n)}^1 \right).$$

Hence, the claim is proved. □

4. Decomposition of $HL^n(L, L)$ for semisimple Leibniz algebra

Now consider a semi-simple Leibniz algebra L as an *adjoint* representation of L , i.e. L acting on itself by Leibniz algebra bracket. Due to Corollary 2.6 we have $L \cong I \dot{+} \mathfrak{g}$. Setting $L_0 = \mathfrak{g}$ and $L_1 = I$ cochain spaces $CL^n(L, L)$ admits the decomposition of Proposition 3.1. However, due to $[L, I] = 0$ one can obtain more properties of the coboundary operator.

Proposition 4.1. *Coboundary operator ∂ preserves the level spaces.*

Proof. For a non-zero $\varphi \in CL^n(L, L)_{(k)}$ consider $\partial\varphi(x_1, \dots, x_n, x_{n+1})$, where exactly m of x_1, \dots, x_{n+1} belong to I . It is evident that if $m < k$ or $m > k + 1$ then $\partial\varphi(x_1, \dots, x_n, x_{n+1}) = 0$. We will consider the other possible cases.

Let $\varphi \in H_{i(1)\dots i(n)}^0$ for some $i \in \mathcal{I}(n, k)$. If $m = k$ then due to $\text{im } \varphi \in \mathfrak{g}$ it follows that $\partial\varphi(x_1, \dots, x_n, x_{n+1}) \in \mathfrak{g}$. If $m = k + 1$ then $\partial\varphi(x_1, \dots, x_n, x_{n+1})$ belongs to I if $x_1 \in I$ and zero otherwise. Hence,

$$\partial(H_{i(1)\dots i(n)}^0) \subseteq \left(\bigoplus_{j \in \mathcal{I}(n+1, k)} H_{j(1)\dots j(n+1)}^0 \right) \oplus H_{1i(1)\dots i(n)}^1.$$

Now consider $\varphi \in H_{i(1)\dots i(n)}^1$ for some $i \in \mathcal{I}(n, k+1)$. It is obvious that $m = k$ implies $\partial\varphi(x_1, \dots, x_n, x_{n+1}) = 0$. One verifies the last possibility $m = k + 1$ to yield $\partial(H_{i(1)\dots i(n)}^1) \subseteq \bigoplus_{j \in \mathcal{I}(n+1, k+1)} H_{j(1)\dots j(n+1)}^1$.

Thus $\partial(CL^n(L, L)_{(-k)}) \subseteq CL^{n+1}(L, L)_{(-k)}$ for all $-1 \leq k \leq n$. □

Let us denote by $ZL^n(L, L)_{(k)}$ the kernel of restriction of ∂ on $CL^n(L, L)_{(n)}$ and by $BL^n(L, L)_{(k)}$ the image of restriction of ∂ on $CL^{n-1}(L, L)_{(k)}$ for all $-n \leq k \leq 1$.

Proposition 4.2. *Let L be a finite-dimensional semisimple Leibniz algebra. Then $ZL^n(L, L)_{(-n)} = 0$ for all positive integers n .*

Proof. Let $\varphi \in ZL^n(L, L)_{(-n)}$. Then $\varphi : \otimes^n I \rightarrow \mathfrak{g}$ and $\partial\varphi(g, i_1, \dots, i_n) = [g, \varphi(i_1, \dots, i_n)] = 0$ for all $g \in \mathfrak{g}, i_1, \dots, i_n \in I$. This yields $\text{im } \varphi \in Z(\mathfrak{g})$ which is zero since \mathfrak{g} is semi-simple. Therefore, $\varphi = 0$ and we are done. □

Provided with Proposition 4.1 one has $BL^n(L, L)_{(k)} \subseteq ZL^n(L, L)_{(k)}$ and consider subspace $ZL^n(L, L)_{(k)}/BL^n(L, L)_{(k)}$. Let us denote it by $HL^n(L, L)_{(k)}$, for all $-n \leq k \leq 1$. As a consequence on Propositions 3.1 and 4.1 we have the following

Theorem 4.3. *Let L be a finite-dimensional semisimple Leibniz algebra. Then*

$$HL^n(L, L) \cong HL^n(L, L)_{(-n+1)} \oplus \dots \oplus HL^n(L, L)_{(-1)} \oplus HL^n(L, L)_{(0)}$$

for $n \geq 2$ and $HL^1(L, L) \cong HL^1(L, L)_{(0)} \oplus HL^1(\mathfrak{g}, I)$.

Proof. Clearly, we have

$$\begin{aligned} BL^n(L, L) &\cong \bigoplus_{k=-n}^1 BL^n(L, L)_{(-k)}, \\ ZL^n(L, L) &\cong \bigoplus_{k=-n}^1 ZL^n(L, L)_{(-k)}. \end{aligned}$$

Note that by definition $BL^n(L, L)_{(-n)} = 0$ and together with Proposition 4.2 we obtain $HL^n(L, L)_{(-n)} = 0$.

Observe that $HL^n(L, L)_{(1)} = HL^n(\mathfrak{g}, I)$. If $n > 1$ by Theorem 2.8 it is zero. Hence, the result follows. \square

5. $HL^2(L, L)$ for semisimple Leibniz algebra

Let us describe $BL^2(L, L) = BL^2(L, L)_{(-1)} \oplus BL^2(L, L)_{(0)} \oplus BL^2(L, L)_{(1)}$ first.

Proposition 5.1. *A coboundary operator ∂ acts on $CL^1(L, L)$ as follows:*

$$H_1^0 \hookrightarrow H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1, \quad H_1^1 \oplus H_0^0 \rightarrow H_{10}^0 \oplus H_{00}^0, \quad H_0^1 \rightarrow H_{00}^1.$$

Proof. Consider ∂d for $d \in CL^1(L, L)$. We list the only non-zero actions of ∂d on an element $(x, y) \in (L_{i(1)}, L_{i(2)})$ depending on d .

1. Let $d \in CL^1(L, L)_{(-1)} = H_1^0$. Then $\partial(H_1^0) \subseteq H_{11}^1 \oplus H_{01}^0 \oplus H_{10}^0$. Indeed,

$$\begin{aligned} \partial d(x_0, y_1) &= [x_0, d(y_1)] \in L_0, \\ \partial d(x_1, y_0) &= [d(x_1), y_0] - d([x_1, y_0]) \in L_0, \\ \partial d(x_1, y_1) &= [x_1, d(y_1)] \in L_1. \end{aligned}$$

Moreover, if $\partial d = 0$ then from $[x_0, d(y_1)] = 0$ we obtain $d = 0$, i.e. ∂ sends H_1^0 to $H_{11}^1 \oplus H_{01}^0 \oplus H_{10}^0$ injectively. In particular, this implies $\dim BL_{(-1)}^2(L, L) = \dim H_1^0 = \dim \text{Hom}(L_1, L_0) = \dim L_1 \cdot \dim L_0$.

2. Let $d \in CL^1(L, L)_{(0)} = H_1^1 \oplus H_0^0$. Then we analyse separate cases.

2.1 Let $d_1 \in H_1^1$. Then $\partial(H_1^1) \subseteq H_{10}^1$. Indeed,

$$\partial d_1(x_1, y_0) = [d_1(x_1), y_0] - d_1([x_1, y_0]) \in L_1.$$

2.2 Let $d_0 \in H_0^0$. Then $\partial(H_0^0) \subseteq H_{00}^0 \oplus H_{10}^1$. Indeed,

$$\partial d_0(x_0, y_0) = [d_0(x_0), y_0] + [x_0, d_0(y_0)] - d_0([x_0, y_0]) \in L_0,$$

$$\partial d_0(x_1, y_0) = [x_1, d_0(y_0)] \in L_1.$$

3. Let $d \in CL^1(L, L)_{(1)} = H_0^1$. Then $\partial(H_0^1) \subseteq H_{00}^1$. Indeed, $\partial d(x_0, y_0) = [d(x_0), y_0] - d([x_0, y_0]) \subseteq L_1$. \square

Now we will concentrate on $ZL^2(L, L)$. For a $\varphi \in CL^2(L, L)$ we have

$$\begin{aligned} \partial\varphi(x, y, z) &= [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] \\ &\quad + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y) \quad (2) \end{aligned}$$

By Proposition 4.2 there is a decomposition $ZL^2(L, L) = ZL^2(L, L)_{(-1)} \oplus ZL^2(L, L)_{(0)} \oplus ZL^2(L, L)_{(1)}$. We have $ZL^2(L, L)_{(-1)} = \ker \partial|_{H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1}$ and proposition below shows that coboundary operator takes every component of $CL^2(L, L)_{(-1)}$ injectively.

Proposition 5.2. *A coboundary operator ∂ acts on $CL^2(L, L)_{(-1)}$ as follows*

$$\begin{aligned} H_{01}^0 &\hookrightarrow H_{101}^1 \oplus H_{010}^0 \oplus H_{001}^0, \\ H_{10}^0 &\hookrightarrow H_{110}^1 \oplus H_{100}^0 \oplus H_{010}^0, \\ H_{11}^1 &\hookrightarrow H_{101}^1 \oplus H_{110}^1. \end{aligned}$$

Proof. We have $CL^2(L, L)_{(-1)} = H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1$. Now let us examine how ∂ acts on each of the subspaces.

Case 1. Let $\varphi \in H_{01}^0$. Then in the system (2) the only non-zero equalities are

$$\begin{aligned} \partial\varphi(x_1, y_0, z_1) &= [x_1, \varphi(y_0, z_1)] \in L_1, \\ \partial\varphi(x_0, y_0, z_1) &= [x_0, \varphi(y_0, z_1)] + [\varphi(x_0, z_1), y_0] - \varphi([x_0, y_0], z_1) \in L_0, \\ \partial\varphi(x_0, y_1, z_0) &= -[\varphi(x_0, y_1), z_0] + \varphi(x_0, [y_1, z_0]) + \varphi([x_0, z_0], y_1) \in L_0. \end{aligned}$$

Hence, $H_{01}^0 \rightarrow H_{101}^1 \oplus H_{010}^0 \oplus H_{001}^0$. Now if $\varphi \in \ker \partial|_{H_{01}^0}$ then $[x_1, \varphi(y_0, z_1)] = 0$ implies $\varphi(y_0, z_1) \subseteq I = L_1$ while $\text{im } \varphi \subseteq L_0$.

Thus $\varphi = 0$ and $H_{01}^0 \hookrightarrow H_{101}^1 \oplus H_{010}^0 \oplus H_{001}^0$.

Case 2. Let $\varphi \in H_{10}^0$. Similarly, we have the following equalities from system (2):

$$\begin{aligned} \partial\varphi(x_0, y_1, z_0) &= [x_0, \varphi(y_1, z_0)] \in L_0 \\ \partial\varphi(x_1, y_0, z_0) &= -[\varphi(x_1, y_0), z_0] + [\varphi(x_1, z_0), y_0] \\ &\quad + \varphi(x_1, [y_0, z_0]) - \varphi([x_1, y_0], z_0) + \varphi([x_1, z_0], y_0) \in L_0 \\ \partial\varphi(x_1, y_1, z_0) &= [x_1, \varphi(y_1, z_0)] \in L_1 \end{aligned}$$

Hence, $H_{10}^0 \rightarrow H_{110}^1 \oplus H_{100}^0 \oplus H_{010}^0$. Analogously as above, if $\varphi \in \ker \partial|_{H_{10}^0}$ then equality $[x_1, \varphi(y_1, z_0)] = 0$ implies $\varphi = 0$. Therefore, $H_{10}^0 \hookrightarrow H_{110}^1 \oplus H_{100}^0 \oplus H_{010}^0$.

Case 3. Let $\varphi \in H_{11}^1$, then system (2) implies

$$\begin{aligned} \partial\varphi(x_1, y_0, z_1) &= [\varphi(x_1, z_1), y_0] - \varphi([x_1, y_0], z_1) \in L_1, \\ \partial\varphi(x_1, y_1, z_0) &= -[\varphi(x_1, y_1), z_0] + \varphi(x_1, [y_1, z_0]) + \varphi([x_1, z_0], y_1) \in L_1. \end{aligned}$$

Assuming $\varphi \in \ker \partial|_{H_{11}^1}$ from the equalities $\partial\varphi(x_1, z_0, y_1) = \partial\varphi(x_1, y_1, z_0) = 0$ we obtain $\varphi(x_1, [y_1, z_0]) = 0$. Due to $[I, \mathfrak{g}] = I$ it implies $\varphi = 0$. Hence, $H_{11}^1 \hookrightarrow H_{101}^1 \oplus H_{110}^1$. □

We have $\partial(H_{00}^0 \oplus H_{01}^1 \oplus H_{10}^1) \subseteq H_{000}^0 \oplus H_{100}^1 \oplus H_{010}^1 \oplus H_{001}^1$. Now let us examine how ∂ acts on each of the subspaces.

Proposition 5.3. *Coboundary operator acts on $CL^2(L, L)_{(0)}$ as follows*

$$\begin{aligned} \partial : H_{00}^0 &\hookrightarrow H_{100}^1 \oplus H_{000}^0 \\ \partial : H_{10}^1 &\rightarrow H_{100}^1 \\ \partial : H_{01}^1 &\hookrightarrow H_{010}^1 \oplus H_{001}^1 \end{aligned}$$

and $ZL^2(L, L)_{(0)} = \ker \partial|_{H_{00}^0 \oplus H_{10}^1}$.

Proof. Case 1. Assume $\varphi \in H_{00}^0$, by system (2) we have

$$\begin{aligned} \partial\varphi(x_1, y_0, z_0) &= [x_1, \varphi(y_0, z_0)] \in L_1, \\ \partial\varphi(x_0, y_0, z_0) &= [x_0, \varphi(y_0, z_0)] - [\varphi(x_0, y_0), z_0] + [\varphi(x_0, z_0), y_0] \\ &\quad + \varphi(x_0, [y_0, z_0]) - \varphi([x_0, y_0], z_0) + \varphi([x_0, z_0], y_0) \in L_0. \end{aligned}$$

Given that $\varphi \in \ker \partial|_{H_{00}^0}$ then $[x_1, \varphi(y_0, z_0)] = 0$ which implies $\varphi = 0$. Thus, $\partial : H_{00}^0 \hookrightarrow H_{100}^1 \oplus H_{000}^0$.

Case 2. Assuming $\varphi \in H_{10}^1$ non-zero equalities of system (2) yields

$$\begin{aligned} \partial\varphi(x_1, y_0, z_0) &= -[\varphi(x_1, y_0), z_0] + [\varphi(x_1, z_0), y_0] \\ &\quad + \varphi(x_1, [y_0, z_0]) - \varphi([x_1, y_0], z_0) + \varphi([x_1, z_0], y_0) \subseteq L_1. \end{aligned}$$

Hence, $\partial : H_{10}^1 \rightarrow H_{100}^1$.

Case 3. Let $\varphi \in H_{01}^1$. System (2) provides

$$\begin{aligned} \partial\varphi(x_0, y_1, z_0) &= -[\varphi(x_0, y_1), z_0] + \varphi(x_0, [y_1, z_0]) + \varphi([x_0, z_0], y_1) \subseteq L_1; \\ \partial\varphi(x_0, y_0, z_1) &= [\varphi(x_0, z_1), y_0] - \varphi([x_0, y_0], z_1) \in L_1. \end{aligned}$$

Then $\partial : H_{01}^1 \hookrightarrow H_{010}^1 \oplus H_{001}^1$ since assuming $\varphi \in \ker \partial|_{H_{01}^1}$ the last two equalities yield $[x_1, \varphi(y_0, z_0)] = 0$ which implies $\varphi = 0$.

From above it follows that

$$ZL^2(L, L)_{(0)} = \ker \partial|_{H_{00}^0 \oplus H_{01}^1 \oplus H_{10}^1} = \ker \partial|_{H_{00}^0 \oplus H_{10}^1} \oplus \ker \partial|_{H_{01}^1} = \ker \partial|_{H_{00}^0 \oplus H_{10}^1},$$

which finishes the proof. \square

As a consequence of the last two propositions we obtain

Theorem 5.4. *For a finite dimensional semisimple Leibniz algebra L over \mathbb{C} we have an isomorphism*

$$HL^2(L, L) \cong HL^2(L, L)_{(-1)} \oplus HL^2(L, L)_{(0)},$$

where

$$HL^2(L, L)_{(-1)} = \ker \partial|_{H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1} / \partial(H_1^0) \text{ and}$$

$$HL^2(L, L)_{(0)} = \ker \partial|_{H_{00}^0 \oplus H_{10}^1} / \partial(H_1^1 \oplus H_0^0).$$

From this point we will concentrate our study on each of the subspaces of level -1 and 0 separately.

Proposition 5.5. *Any 2-cocycle $\varphi = \varphi_{10}^0 + \varphi_{01}^0 + \varphi_{11}^1 \in \ker \partial|_{H_{10}^0 \oplus H_{01}^0 \oplus H_{11}^1}$ is uniquely determined by a map $\phi \in \text{Hom}(I \otimes \mathfrak{g}) \rightarrow \mathfrak{g}$ that satisfies*

$$\phi(x_1, [y_0, z_0]) = \phi([x_1, y_0], z_0) - \phi([x_1, z_0], y_0).$$

Moreover, φ is determined by the following equalities

$$\begin{cases} \varphi_{01}^0(x_0, [z_1, y_0]) = -[x_0, \phi(z_1, y_0)] \\ \varphi_{11}^1(x_1, [z_1, y_0]) = -[x_1, \phi(z_1, y_0)] \\ \varphi_{10}^0([x_1, y_0], z_0) = \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0] \end{cases} \quad (3)$$

Proof. Let $\varphi = \varphi_{11}^1 + \varphi_{01}^0 + \varphi_{10}^0 \in \ker \partial|_{H_{11}^1 \oplus H_{01}^1 \oplus H_{10}^1} = ZL_{-1}^2$. Then $\partial\varphi = 0$ implies the following:

$$0 = \partial\varphi(x_1, y_0, z_1) = [\varphi_{11}^1(x_1, z_1), y_0] - \varphi_{11}^1([x_1, y_0], z_1) + [x_1, \varphi_{01}^0(y_0, z_1)],$$

$$0 = \partial\varphi(x_0, y_0, z_1) = [x_0, \varphi_{01}^0(y_0, z_1)] + [\varphi_{01}^0(x_0, z_1), y_0] - \varphi_{01}^0([x_0, y_0], z_1),$$

$$0 = \partial\varphi(x_0, z_1, y_0) = -[\varphi_{01}^0(x_0, z_1), y_0] + \varphi_{01}^0(x_0, [z_1, y_0]) + \varphi_{01}^0([x_0, y_0], z_1) + [x_0, \varphi_{10}^0(z_1, y_0)],$$

$$0 = \partial\varphi(x_1, y_0, z_0) = -[\varphi_{10}^0(x_1, y_0), z_0] + [\varphi_{10}^0(x_1, z_0), y_0] + \varphi_{10}^0(x_1, [y_0, z_0]) - \varphi_{10}^0([x_1, y_0], z_0) + \varphi_{10}^0([x_1, z_0], y_0),$$

$$0 = \partial\varphi(x_1, z_1, y_0) = -[\varphi_{11}^1(x_1, z_1), y_0] + \varphi_{11}^1(x_1, [z_1, y_0]) + \varphi_{11}^1([x_1, y_0], z_1) + [x_1, \varphi_{10}^0(z_1, y_0)].$$

Adding the first and the last equalities we obtain

$$\varphi_{11}^1(x_1, [z_1, y_0]) = -[x_1, \varphi_{01}^0(y_0, z_1) + \varphi_{10}^0(z_1, y_0)].$$

Adding the second and the third equalities we obtain

$$\varphi_{01}^0(x_0, [z_1, y_0]) = -[x_0, \varphi_{01}^0(y_0, z_1) + \varphi_{10}^0(z_1, y_0)].$$

Hence, these are defining relations for $\varphi = \varphi_{11}^1 + \varphi_{01}^0 + \varphi_{10}^0 \in ZL_{-1}^2$:

$$\left\{ \begin{array}{l} 0 = [\varphi_{11}^1(x_1, z_1), y_0] - \varphi_{11}^1([x_1, y_0], z_1) + [x_1, \varphi_{01}^0(y_0, z_1)] \\ 0 = [x_0, \varphi_{01}^0(y_0, z_1)] + [\varphi_{01}^0(x_0, z_1), y_0] - \varphi_{01}^0([x_0, y_0], z_1) \\ \varphi_{01}^0(x_0, [z_1, y_0]) = -[x_0, \varphi_{01}^0(y_0, z_1) + \varphi_{10}^0(z_1, y_0)] \\ 0 = -[\varphi_{10}^0(x_1, y_0), z_0] + [\varphi_{10}^0(x_1, z_0), y_0] + \varphi_{10}^0(x_1, [y_0, z_0]) - \varphi_{10}^0([x_1, y_0], z_0) + \varphi_{10}^0([x_1, z_0], y_0) \\ \varphi_{11}^1(x_1, [z_1, y_0]) = -[x_1, \varphi_{01}^0(y_0, z_1) + \varphi_{10}^0(z_1, y_0)]. \end{array} \right.$$

Taking third and last equations one can deduce the first equation using the fact that any element $z_1 \in L_1$ has a decomposition $z_1 = \sum_{l \leq k \leq m} [z_k^1, z_k^0]$ for some

$z_k^i \in L_i$ for $i = 0, 1$ and $1 \leq k \leq m$, $m \in \mathbb{N}$. Indeed,

$$\begin{aligned} & [\varphi_{11}^1(x_1, z_1), y_0] - \varphi_{11}^1([x_1, y_0], z_1) + [x_1, \varphi_{01}^0(y_0, z_1)] = \\ & = - \left[[x_1, \sum_{l \leq k \leq m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)], y_0 \right] \\ & \quad + \left[[x_1, y_0], \sum_{l \leq k \leq m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0) \right] \\ & \quad - \left[x_1, [y_0, \sum_{l \leq k \leq m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)] \right] = 0. \end{aligned}$$

Similarly, we can check that third equality implies the second:

$$\begin{aligned} & [x_0, \varphi_{01}^0(y_0, z_1)] + [\varphi_{01}^0(x_0, z_1), y_0] - \varphi_{01}^0([x_0, y_0], z_1) \\ &= -[x_0, [y_0, \sum_{l \leq k \leq m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0)]] \\ & \quad - \left[\left[x_0, \sum_{l \leq k \leq m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0) \right], y_0 \right] \\ & \quad + \left[[x_0, y_0], \sum_{l \leq k \leq m} \varphi_{01}^0(z_k^0, z_k^1) + \varphi_{10}^0(z_k^1, z_k^0) \right] = 0. \end{aligned}$$

Define $\phi \in \text{Hom}(L_1 \otimes L_0, L_0)$ by $\phi(i, g) = \varphi_{01}^0(g, i) + \varphi_{10}^0(i, g)$. Then substituting $\varphi_{10}^0(i, g) = \phi(i, g) - \varphi_{01}^0(g, i)$ into the fourth equation we obtain the following:

$$\begin{aligned} 0 &= -[\phi(x_1, y_0) - \varphi_{01}^0(y_0, x_1), z_0] + [\phi(x_1, z_0) - \varphi_{01}^0(z_0, x_1), y_0] + \\ & \quad + (\phi(x_1, [y_0, z_0]) - \varphi_{01}^0([y_0, z_0], x_1)) - (\phi([x_1, y_0], z_0) - \varphi_{01}^0(z_0, [x_1, y_0])) \\ & \quad + (\phi([x_1, z_0], y_0) - \varphi_{01}^0(y_0, [x_1, z_0])) = \\ &= -[\phi(x_1, y_0), z_0] + [\phi(x_1, z_0), y_0] + \phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) \\ & \quad + \phi([x_1, z_0], y_0) + [\varphi_{01}^0(y_0, x_1), z_0] - [\varphi_{01}^0(z_0, x_1), y_0] - \varphi_{01}^0([y_0, z_0], x_1) \\ & \quad + \varphi_{01}^0(z_0, [x_1, y_0]) - \varphi_{01}^0(y_0, [x_1, z_0]) = \\ &= (-[\phi(x_1, y_0), z_0] + \varphi_{01}^0(z_0, [x_1, y_0])) + ([\phi(x_1, z_0), y_0] - \varphi_{01}^0(y_0, [x_1, z_0])) \\ & \quad + ([\varphi_{01}^0(y_0, x_1), z_0] - [\varphi_{01}^0(z_0, x_1), y_0] - \varphi_{01}^0([y_0, z_0], x_1)) \\ & \quad + (\phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) + \phi([x_1, z_0], y_0)) = \\ & \quad \text{(since the bilinear forms involved here map to the Lie algebra, we can} \\ & \quad \text{use antisymmetry)} \\ &= ([z_0, \phi(x_1, y_0)] + \varphi_{01}^0(z_0, [x_1, y_0])) - ([y_0, \phi(x_1, z_0)] + \varphi_{01}^0(y_0, [x_1, z_0])) \\ & \quad + ([y_0, \varphi_{01}^0(z_0, x_1)] + [\varphi_{01}^0(y_0, x_1), z_0] - \varphi_{01}^0([y_0, z_0], x_1)) \\ & \quad + (\phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) + \phi([x_1, z_0], y_0)). \end{aligned}$$

Now the first two expressions in parentheses are zero due to third equality. The third expression in parentheses is also zero due to second equality. Hence, we obtain

$$\phi(x_1, [y_0, z_0]) - \phi([x_1, y_0], z_0) + \phi([x_1, z_0], y_0) = 0.$$

Since any element $z_1 \in L_1$ admits a decomposition $z_1 = \sum_{l \leq k \leq m} [z_k^1, z_k^0]$ for some $z_k^i \in L_i$ for $i = 0, 1$ and $1 \leq k \leq m$, $m \in \mathbb{N}$, we have

$$\varphi_{10}^0(z_1, x_0) = \phi(z_1, x_0) - \varphi_{01}^0(x_0, z_1) = \phi(z_1, x_0) - \sum_{l \leq k \leq m} \varphi_{01}^0(x_0, [z_k^1, z_k^0])$$

$$= \phi(z_1, x_0) + [x_0, \sum_{l \leq k \leq m} \phi(z_k^1, z_k^0)].$$

In particular,

$$\varphi_{10}^0([x_1, y_0], z_0) = \phi([x_1, y_0], z_0) + [z_0, \phi(x_1, y_0)] = \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0].$$

Summarizing, we can determine all components of $\varphi = \varphi_{11}^1 + \varphi_{01}^0 + \varphi_{10}^0 \in ZL_{-1}^2$ in terms of $\phi \in \text{Hom}(L_1 \otimes L_0) \rightarrow L_0$

$$\begin{cases} \varphi_{01}^0(x_0, [z_1, y_0]) = -[x_0, \phi(z_1, y_0)] \\ \varphi_{11}^1(x_1, [z_1, y_0]) = -[x_1, \phi(z_1, y_0)] \\ \varphi_{10}^0([x_1, y_0], z_0) = \phi([x_1, y_0], z_0) - [\phi(x_1, y_0), z_0] \end{cases}$$

where ϕ satisfies

$$\phi(x_1, [y_0, z_0]) = \phi([x_1, y_0], z_0) - \phi([x_1, z_0], y_0).$$

For the sake of convenience, let us re-denote ϕ by $\phi(i, g) = -\varphi_{01}^0(g, i) - \varphi_{10}^0(i, g)$. Then

$$\begin{cases} \varphi_{01}^0(x_0, [z_1, y_0]) = [x_0, \phi(z_1, y_0)] \\ \varphi_{11}^1(x_1, [z_1, y_0]) = [x_1, \phi(z_1, y_0)] \\ \varphi_{10}^0([x_1, y_0], z_0) = [\phi(x_1, y_0), z_0] - \phi([x_1, y_0], z_0) \end{cases}$$

which finishes the proof of the proposition. \square

It is conjectured in [2] that $HL^2(L, L) = 0$ for any semisimple Leibniz algebra L . Authors in [2] validate the claim for simple Leibniz algebra with $L/I \cong \mathfrak{sl}_2$. Armed with the proposition above in order to check if $HL^2(L, L)_{(-1)} = 0$ we arrive into an equivalent statement.

Let \mathfrak{g} be a finite dimensional semisimple complex Lie algebra and I be a finite dimensional \mathfrak{g} -module. Denote by $i.g$ the action of $g \in \mathfrak{g}$ on $i \in I$.

Proposition 5.6. *Let $\phi \in \text{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$. Then $HL^2(L, L)_{(-1)} = 0$ if and only if*

$$\phi(i, [g_1, g_2]) = \phi(i.g_1, g_2) - \phi(i.g_2, g_1) \tag{*}$$

for all $g_1, g_2 \in \mathfrak{g}, i \in I$ yields existence of a $d \in \text{Hom}(I, \mathfrak{g})$ such that $\phi(i, g) = d(i.g)$.

Proof. Recall that $BL^2(L, L)_{(-1)}$ consists of $\partial d =: \psi = \psi_{01}^0 + \psi_{11}^1 + \psi_{10}^0$, where $d \in \text{Hom}(L_1, L_0)$ and $\psi_{ij}^k \in H_{ij}^k$ are given by

$$\begin{cases} \psi_{01}^0(x_0, y_1) = [x_0, d(y_1)] \\ \psi_{11}^1(x_1, y_1) = [x_1, d(y_1)] \\ \psi_{10}^0(x_1, y_0) = [d(x_1), y_0] - d([x_1, y_0]) \end{cases}$$

Consider $\varphi \in ZL^2(L, L)_{(-1)}$. By Proposition 5.5 its components $\varphi = \varphi_{11}^1 + \varphi_{01}^0 + \varphi_{10}^0$ are completely determined by system of equations 3 and a map $\phi \in \text{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$ that satisfies (*).

Our conjecture holds, if and only $\varphi = \psi$. Consider $\varphi_{01}^0 = \psi_{01}^0$. Then $\varphi_{01}^0(g_0, [i, g]) = [g_0, \phi(i, g)] = [g_0, d([i, g])]$ which yields $[g_0, \phi(i, g) - d([i, g])] = 0$ for any $g_0, g \in \mathfrak{g}, i \in I$. Due to semi-simplicity it follows that $\phi(i, g) = d([i, g])$. Note that by construction $[i, g] = i.g$ Conversely, if $\phi(i, g) = d([i, g])$ holds then one can check easily that $\varphi = \psi$. \square

Let us consider $HL^2(L, L)_{(0)}$ space. Verification if the later one is zero is not known. However, we present an equivalent conjecture in the next proposition.

Proposition 5.7. *Let L be a semisimple Leibniz algebra over \mathbb{C} . Statement $HL^2(L, L)_{(0)} = 0$ is valid if and only if for any map $\psi \in \text{Hom}(I \otimes \mathfrak{g}, I)$ that satisfies*

$$[\psi(i, g_1), g_2] - [\psi(i, g_2), g_1] - \psi(i, [g_1, g_2]) + \psi(i.g_1, g_2) - \psi(i.g_2, g_1) = 0,$$

it follows that there exist $g_0 \in \mathfrak{g}$ and $d \in \text{Hom}(I, I)$ such that

$$\psi(i, g) = i.[g_0, g] + d(i).g - d(i.g).$$

Proof. By Theorem 5.4 the space $ZL^2(L, L)_{(0)}$ consists of $\varphi = \varphi_{00}^0 + \varphi_{10}^1$, where $\varphi_{00}^0 \in H_{00}^0$ and $\varphi_{10}^1 \in H_{10}^1$. We have the following defining equalities for φ :

$$\partial\varphi_{00}^0(x_1, y_0, z_0) = [x_1, \varphi_{00}^0(y_0, z_0)];$$

$$\partial\varphi_{00}^0(x_0, y_0, z_0) = [x_0, \partial\varphi_{00}^0(y_0, z_0)] - [\partial\varphi_{00}^0(x_0, y_0), z_0] + [\partial\varphi_{00}^0(x_0, z_0), y_0] + \partial\varphi_{00}^0(x_0, [y_0, z_0]) - \partial\varphi_{00}^0([x_0, y_0], z_0) + \partial\varphi_{00}^0([x_0, z_0], y_0);$$

$$\partial\varphi_{10}^1(x_1, y_0, z_0) = -[\varphi_{10}^1(x_1, y_0), z_0] + [\varphi_{10}^1(x_1, z_0), y_0] + \varphi_{10}^1(x_1, [y_0, z_0]) - \varphi_{10}^1([x_1, y_0], z_0) + \varphi_{10}^1([x_1, z_0], y_0),$$

where elements with subscripts equal to 1 belong to $L_1 := I$, and $L_0 := \mathfrak{g}$ otherwise.

Therefore, for $\varphi = \varphi_{00}^0 + \varphi_{10}^1 \in ZL^2(L, L)_{(0)}$ it is necessary and sufficient that

$$(1) [x_1, \varphi_{00}^0(y_0, z_0)] - [\varphi_{10}^1(x_1, y_0), z_0] + [\varphi_{10}^1(x_1, z_0), y_0] + \varphi_{10}^1(x_1, [y_0, z_0]) - \varphi_{10}^1([x_1, y_0], z_0) + \varphi_{10}^1([x_1, z_0], y_0) = 0,$$

$$(2) \varphi_{00}^0|_{H_{000}^0} \text{ is a Lie 2-cocycle.}$$

Recall, that for $d = d_1 + d_0 \in H_1^1 \oplus H_0^0$ where $d_0 \in H_0^0$, $d_1 \in H_1^1$ we have $\partial(H_0^0) \subseteq H_{00}^0 \oplus H_{10}^1$ and $\partial(H_1^1) \subseteq H_{10}^1$ with

$$\partial d(x_1, y_0) = \partial d_0(x_1, y_0) + \partial d_1(x_1, y_0) = [x_1, d_0(y_0)] + [d_1(x_1), y_0] - d_1([x_1, y_0]),$$

$$\partial d(x_0, y_0) = [d_0(x_0), y_0] + [x_0, d_0(y_0)] - d_0([x_0, y_0]).$$

Since $H^2(L_0, L_0) = 0$ we have $\varphi_{00}^0 = \partial d_0|_{L_0 \otimes L_0}$ for some $d_0 \in H_0^0$.

Using Leibniz identity one can check the following equality

$$[x_1, \varphi_{00}^0(y_0, z_0)] = -\partial d(x_1, [y_0, z_0]) + \partial d([x_1, y_0], z_0) \\ - \partial d([x_1, z_0], y_0) + [\partial d([x_1, y_0]), z_0] - [\partial d([x_1, z_0]), y_0],$$

where $d = d_0 + d_1$ and $d_1 \in \text{Hom}(L_1, L_1)$ is arbitrary.

Let us denote by $\psi = \varphi_{10}^1 - \partial d|_{L_1 \otimes L_0}$. Then $\psi \in \text{Hom}(L_1 \otimes L_0, L_1)$ and condition (1) is equivalent to

$$[\psi(x_1, y_0), z_0] - [\psi(x_1, z_0), y_0] - \psi(x_1, [y_0, z_0]) + \psi([x_1, y_0], z_0) - \psi([x_1, z_0], y_0) = 0.$$

Observe that $\varphi_{00}^0 + \varphi_{10}^1 = \partial d_0 + \partial d_1 + \psi$ and we are done if there exist $\bar{d}_0 \in H_0^0$ and $\bar{d}_1 \in H_1^1$ such that

$$\partial \bar{d}_0|_{L_0 \otimes L_0} + \partial(\bar{d}_0 + \bar{d}_1)|_{L_1 \otimes L_0} = \partial d_0|_{L_0 \otimes L_0} + \partial(d_0 + d_1)|_{L_1 \otimes L_0} + \psi.$$

Therefore, for $\varphi_{00}^0 + \varphi_{10}^1 \in ZL^2(L, L)_{(0)}$ to be a 2-coboundary it is necessary and sufficient to the following conditions to take place:

- (i) $\partial \bar{d}_0|_{L_0 \otimes L_0} = \partial d_0|_{L_0 \otimes L_0}$.
- (ii) $\partial(\bar{d}_0 + \bar{d}_1)|_{L_1 \otimes L_0} = \partial(d_0 + d_1)|_{L_1 \otimes L_0} + \psi$.

Now (i) implies that $\bar{d}_0 - d_0 \in Z^1(\mathfrak{g}, \mathfrak{g}) = \text{Der}(\mathfrak{g})$. Since \mathfrak{g} is semisimple it is known that $\text{Der}(\mathfrak{g}) = \text{Innder}(\mathfrak{g})$. Therefore, there exists $g_0 \in \mathfrak{g}$ such that $(\bar{d}_0 - d_0)(g) = [g_0, g]$. Putting this into (ii) yields

$$\psi(i, g) = [i, [g_0, g]] + [(\bar{d}_1 - d_1)(i), g] - (\bar{d}_1 - d_1)([i, g]).$$

Re-denoting $d = \bar{d}_1 - d_1$ finishes the proof. □

Summarizing, results of this section we have the following conjectures for semi-simple finite-dimensional Lie algebra \mathfrak{g} and its right module I that are equivalent to $HL^2(L, L)_{(-1)} = 0$ and $HL^2(L, L)_{(0)} = 0$, correspondingly.

Conjecture 1. Let $\phi \in \text{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$ satisfy

$$\phi(i, [g_1, g_2]) = \phi(i.g_1, g_2) - \phi(i.g_2, g_1).$$

Then there exists $d \in \text{Hom}(I, \mathfrak{g})$ such that $\phi(i, g) = d(i.g)$.

Conjecture 2. Let $\psi \in \text{Hom}(I \otimes \mathfrak{g}, I)$ satisfy

$$[\psi(i, g_1), g_2] - [\psi(i, g_2), g_1] - \psi(i, [g_1, g_2]) + \psi(i.g_1, g_2) - \psi(i.g_2, g_1) = 0.$$

Then there exist $g_0 \in \mathfrak{g}$ and $d \in \text{Hom}(I, I)$ such that

$$\psi(i, g) = i.[g_0, g] + d(i).g - d(i.g).$$

6. Verification of $HL^2(L, L)_{(-1)} = 0$ for some algebras

Let \mathfrak{g} be a finite dimensional semisimple complex Lie algebra and I be finite dimensional right \mathfrak{g} -module. Denote by $i.g$ the action of $g \in \mathfrak{g}$ on $i \in I$.

For $\phi \in \text{Hom}(I \otimes \mathfrak{g}, \mathfrak{g})$ let us introduce a map $\Phi_\phi \in \text{Hom}(I \otimes \mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ defined by

$$\Phi_\phi(i, g_1, g_2) = \phi(i, [g_1, g_2]) - \phi(i.g_1, g_2) + \phi(i.g_2, g_1).$$

Note that $\Phi_\phi(i, g_1, g_2) = -\Phi_\phi(i, g_2, g_1)$ and $\Phi_\phi(i, g, g) = 0$.

Propositions 5.6 and 5.7 claim:

- $HL^2(L, L)_{(-1)} = 0 \iff [\Phi_\phi = 0 \text{ implies } \phi(i, g) = d(i.g) \text{ for some } d \in \text{Hom}(I, \mathfrak{g});]$
- $HL^2(L, L)_{(0)} = 0 \iff [\Phi_\phi = [\psi(i, g_1), g_2] - [\psi(i, g_2), g_1] \text{ implies that there exist } g_0 \in \mathfrak{g} \text{ and } d \in \text{Hom}(I, I) \text{ such that } \psi(i, g) = i.[g_0, g] + d(i).g - d(i.g).]$

In [2] it was verified that $HL^2(L, L)_{(-1)} = 0$ for simple Leibniz algebra with liezation \mathfrak{sl}_2 . Below we present more general result when L is not necessarily simple Leibniz algebra with liezation \mathfrak{sl}_2 .

Let $I = \{x_0, x_1, \dots, x_m\}$ be an irreducible right \mathfrak{sl}_2 -module. The action is very well-known to be as follows:

$$\begin{aligned} x_k.e &= -k(m+1-k)x_{k-1}, & k &= 1, \dots, m. \\ x_k.f &= x_{k+1}, & k &= 0, \dots, m-1, \\ x_k.h &= (m-2k)x_k & k &= 0, \dots, m, \end{aligned}$$

and Lie algebra multiplication on $\mathfrak{sl}_2 = \{e, f, h\}$ to be

$$\begin{aligned} [e, h] &= 2e, & [h, f] &= 2f, & [e, f] &= h, \\ [h, e] &= -2e & [f, h] &= -2f, & [f, e] &= -h. \end{aligned}$$

Theorem 6.1. *Let L be a finite dimensional Leibniz algebra with liezation \mathfrak{sl}_2 . Then $HL^2(L, L)_{(-1)} = 0$.*

Proof. First assume that I is irreducible \mathfrak{sl}_2 -module. As mentioned above, we have the basis $\{x_0, x_1, \dots, x_m\}$ of I .

Define a map $d : I \rightarrow \mathfrak{g}$ by $d(x_k) = \phi(x_{k-1}, f)$ for all $1 \leq k \leq m$ and $d(x_0) = \frac{1}{m}\phi(x_0, h)$.

From $\Phi_\phi(x_m, f, h) = 0$ one obtains $\phi(x_m, f) = 0$ which is in accordance with $d(x_m.f) = 0$. Hence, we have $\phi(x_k, f) = d(x_k.f)$ for all $0 \leq k \leq m$.

Condition $\Phi_\phi(x_k, f, h) = 0$ for $0 \leq k \leq m-1$ simplifies to

$$-2\phi(x_k, f) = \phi(x_k, [f, h]) = \phi(x_k.f, h) - \phi(x_k.h, f) = \phi(x_{k+1}, h) - (m-2k)\phi(x_k, f)$$

which yields

$$\phi(x_{k+1}, h) = (m-2(k+1))\phi(x_k, f) = (m-2(k+1))d(x_{k+1}) = d(x_{k+1}.h).$$

Together with $\phi(x_0, h) = md(x_0) = d(x_0.h)$ we obtain $\phi(x_k, h) = d(x_k.h)$ for all $0 \leq k \leq m$.

Using $\Phi_\phi(x_k, e, f) = 0$ for $1 \leq k \leq m - 1$ we have the following chain of equalities

$$\begin{aligned} (m - 2k)d(x_k) &= d(x_k \cdot h) = \phi(x_k, h) = \phi(x_k, [e, f]) \\ &= \phi(x_k \cdot e, f) - \phi(x_k \cdot f, e) = \\ &= -k(m + 1 - k)\phi(x_{k-1}, f) - \phi(x_{k+1}, e) \\ &= -k(m + 1 - k)d(x_k) - \phi(x_{k+1}, e). \end{aligned}$$

This results in $\phi(x_{k+1}, e) = -(k+1)(m-k)d(x_k) = d(x_{k+1} \cdot e)$ for $1 \leq k \leq m-1$.

Now $\Phi_\phi(x_0, e, f) = 0$ yields $\phi(x_1, e) = -\phi(x_0, h) = -md(x_0) = d(x_1 \cdot e)$ and $\Phi_\phi(x_0, e, h) = 0$ results in $\phi(x_0, e) = 0$ which is the same as $d(x_0 \cdot e) = 0$. Thus we have $\phi(x_k, e) = d(x_k \cdot e)$ for all $0 \leq k \leq m$.

Thus $\phi(i, g) = d(i \cdot g)$ for all $i \in I, g \in \mathfrak{g}$ and by Theorem 5.6 in this case we obtain $HL^2(L, L)_{(-1)} = 0$.

Now assume that I is a finite dimensional \mathfrak{sl}_2 -module. Then it is completely reducible. Let J be an irreducible submodule of I . Then for restriction of ϕ on $\text{Hom}(J \otimes \mathfrak{g}, \mathfrak{g})$ by the previous construction we have a map $d_J \in \text{Hom}(J, \mathfrak{g})$ such that $\phi(j, g) = d(j \cdot g)$. Defining $d \in \text{Hom}(I, \mathfrak{g})$ as a direct sum of d_J of all irreducible submodules we obtain the desired map. \square

Consider a Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and let $I = I_1 \oplus I_2$ be a sum of two irreducible \mathfrak{sl}_2 -modules of the same dimension (hence, isomorphic). As shown in work [4] the action of \mathfrak{g} on I is as follows:

$$\begin{aligned} x_k \cdot e_1 &= -k(m + 1 - k)x_{k-1}, & k = 1, \dots, m, \\ x_k \cdot f_1 &= y_{k+1}, & k = 0, \dots, m - 1, \\ x_k \cdot h_1 &= (m - 2k)y_k, & k = 0, \dots, m, \\ y_k \cdot e_1 &= -k(m + 1 - k)y_{k-1}, & k = 1, \dots, m, \\ y_k \cdot f_1 &= y_{k+1}, & k = 0, \dots, m - 1, \\ y_k \cdot h_1 &= (m - 2k)y_k, & k = 0, \dots, m, \\ \\ x_j \cdot e_2 &= y_j \cdot h_2 = y_j & j = 0, \dots, m \\ x_j \cdot h_2 &= y_j \cdot f_2 = -x_j & j = 0, \dots, m \end{aligned}$$

where $I_1 = \text{Span}\{x_0, x_1, \dots, x_m\}$, $I_2 = \text{Span}\{y_0, y_1, \dots, y_m\}$ and $\mathfrak{sl}_2^i = \langle e_i, f_i, h_i \rangle$ for $i = 1, 2$.

Let $L_1 = I_1 \oplus I_2 \oplus \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2$ be a Leibniz algebra with table of multiplication as above.

Proposition 6.2. *For Leibniz algebra L_1 we have $HL^2(L_1, L_1)_{(-1)} = 0$.*

Proof. Define a map $d : I \rightarrow \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2$ by $d(x_k) = \phi(x_{k-1}, f_1)$, $d(y_k) = \phi(y_{k-1}, f_1)$ for all $1 \leq k \leq m$ and $d(x_0) = \frac{1}{m}\phi(x_0, h_1)$, $d(y_0) = \frac{1}{m}\phi(y_0, h_1)$. Then by the proof of Theorem 6.1 we have $\phi(i, g) = d(i \cdot g)$ for all $i \in I_1 \oplus I_2$ and $g \in \mathfrak{sl}_2^1$. Hence, we only need to show the desired equality when $g \in \mathfrak{sl}_2^2$.

Conditions $\Phi_\phi(x_k, f_2, h_2) = 0$ and $\Phi_\phi(y_k, e_2, h_2) = 0$ yield

$$\phi(x_k, f_2) = 0 = d(x_k \cdot f_2), \quad \phi(y_k, e_2) = 0 = d(y_k \cdot e_2),$$

respectively, for $0 \leq k \leq m$.

Next conditions $\Phi_\phi(x_k, e_2, f_2) = 0$ and $\Phi_\phi(x_k, e_2, h_2) = 0$ gives us $\phi(x_k, h_2) = \phi(y_k, f_2)$ and $\phi(x_k, e_2) = \phi(y_k, h_2)$ for $0 \leq k \leq m$, respectively. Recall $x_k \cdot h_2 = y_k \cdot f_2$ and $x_k \cdot e_2 = y_k \cdot h_2$.

Using $\Phi_\phi(x_k, e_2, f_1) = 0$ one obtain

$$\phi(x_{k+1}, e_2) = \phi(y_k, f_1) = d(y_k \cdot f_1) = d(y_{k+1}) = d(x_{k+1} \cdot e_2)$$

for $0 \leq k \leq m - 1$.

One derives equality $\phi(x_0, e_2) = \frac{1}{m}\phi(y_0, h_1) = d(y_0) = d(x_0 \cdot e_2)$ using $\Phi_\phi(x_0, e_2, h_1) = 0$.

Similarly, using $\Phi_\phi(x_k, f_1, h_2) = 0$ one obtain

$$\phi(x_{k+1}, h_2) = -\phi(x_k, f_1) = -d(x_k \cdot f_1) = -d(x_{k+1}) = d(x_{k+1} \cdot h_2)$$

for $0 \leq k \leq m - 1$. As for the only undefined missing part one verifies $\phi(x_0, h_2) = -\frac{1}{m}\phi(x_0, h_1) = -d(x_0) = d(x_0 \cdot h_2)$ using $\Phi_\phi(x_0, h_1, h_2) = 0$.

Hence, we have proved that $\phi(i, g) = d(i \cdot g)$ for all $i \in I_1 \oplus I_2, g \in \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^2$ and by Theorem 5.6 in this case we obtain $HL^2(L_1, L_1)_{(-1)} = 0$. \square

Acknowledgments

Authors were supported by Ministerio de Economía y Competitividad (Spain), grant MTM2013-43687-P (European FEDER support included). The third author was also supported by Xunta de Galicia, grant GRC2013-045 (European FEDER support included).

References

- [1] Abdulkassymova A S, Dzhumadil'daev A S 2000 Simple Leibniz algebras of rank 1 Abstract presented to the IX International Conference of the Representation Theory of Algebras pp 17–18
- [2] Adashev J Q, Ladra M, Omirov B A 2015 On the second cohomology group of simple Leibniz algebras arXiv:1502.00609 pp 1–11
- [3] Bloh A 1965 On a generalization of the concept of Lie algebra *Dokl. Akad. Nauk SSSR* **165** pp 471–473
- [4] Camacho L M, Gómez-Vidal S, Omirov B A, Karimjanov I A 2014 Leibniz algebras whose semisimple part is related to sl_2 , *Bull. Malays. Math. Sci. Soc.* (to appear)
- [5] Kinyon M K, Weinstein A, 2001 Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces, *Amer. J. Math.* **123**(3) pp 525–550
- [6] Loday J-L 1992 Cyclic homology *Grundle. Math. Wiss. Bd.* **301** Springer-Verlag, Berlin
- [7] Loday J-L 1993 Une version non commutative des algèbres de Lie: les algèbres de Leibniz *Enseign. Math. (2)* **39** pp 269–293
- [8] Loday J-L, Pirashvili T 1993 Universal enveloping algebras of Leibniz algebras and (co)homology *Math. Ann.* **296** pp 139–158
- [9] Ntolo P 2005 Homologie de Leibniz d'algèbres de Lie semi-simples, *C. R. Acad. Sci. Paris Sér. I Math.* **318**(8) pp 707–710
- [10] Pirashvili T 1994 On Leibniz homology *Annales de l'Institut Fourier.* **44**(2) pp 401–411
- [11] Rakhimov I S, Masutova K K, Omirov B A 2014 On derivations of semisimple Leibniz algebras arXiv:1412.4367 pp 1–9
- [12] Turdibaev R M 2015 Cohomologies of semisimple Leibniz algebras with coefficients in the adjoint representation *Proc. of conference "Modern methods of mathematical physics and their application"* Tashkent pp 1–2