Leibniz algebras associated with some finite-dimensional representation of Diamond Lie algebra

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Abstract. In this paper we classify Leibniz algebras whose associated Lie algebra is four-dimensional Diamond Lie algebra $\mathfrak{D}$ and the ideal generated by squares of elements is represented by one of the finite-dimensional indecomposable $\mathfrak{D}$-modules U_n^1, U_n^2, W_n^1 or W_n^2.

1. Introduction

Leibniz algebras were discovered by A. Bloh in 1965 who called them D-algebras [1]. They attracted interest after Jean-Louis Loday [2] noticed that the classical Chevalley-Eilenberg boundary map in the exterior module of a Lie algebra can be lifted to the tensor module which yields a new chain complex. In fact this complex is well defined for any Leibniz algebra. This motivated him to introduce the notion of right (equivalently, left) Leibniz algebra, which is a nonassociative algebra with the right (equivalently, left) multiplication operator being a derivation. Thus a Leibniz algebra satisfies all defining properties of a Lie algebra except the antisymmetry of its product.

During the last 20 years the theory of Leibniz algebras has been actively investigated and numerous papers devoted to the study of these algebras. In fact, many results of theory of Lie algebras have been extended to Leibniz algebras case. For instance, the classical results on Cartan subalgebras [3, 4], Engel's theorem [5], Levi's decomposition [6], properties of solvable algebras with given nilradical [7] and others from the theory of Lie algebras are also true for Leibniz algebras.

Namely, the analogue of Levi's decomposition for Leibniz algebras asserts that any Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra. Therefore, the main problem of the description of finite-dimensional Leibniz algebras consists of the study of solvable Leibniz algebras.

In fact, each non-Lie Leibniz algebra L contains a non-trivial ideal (further denoted by I), which is the subspace spanned by the squares of elements of the algebra L. Moreover, it is readily to see that this ideal belongs to the right annihilator of L, that is $[L, I] = 0$. Note also that the ideal I is the minimal ideal with the property that the quotient algebra L/I is a Lie algebra (the quotient algebra is said to be associated Lie algebra to a Leibniz algebra L).

One of the approaches to the investigation of Leibniz algebras is a description of such algebras whose quotient algebra with respect to the ideal I is a given Lie algebra [8, 9, 10, 11].

The map $I \times L/I \to I$ defined as $(i, \overline{x}) \mapsto [i, x]$ endows I with a structure of L/I-module. Considering the direct sum of vector spaces $Q(L) = L/I \oplus I$, then the operation $(-,-)$ defines the Leibniz algebra structure on $Q(L)$ with multiplication

$$[\overline{x}, \overline{y}] = \overline{[x, y]}, \quad [\overline{x}, i] = [x, i], \quad [i, \overline{x}] = 0, \quad [i, j] = 0, \qquad x, y \in L, \ i, j \in I.$$

Therefore, for given a Lie algebra G and a G-module M, we can construct a Leibniz algebra $L = G \oplus M$ by the above construction.

In the paper [12] the notion of irreducible Leibniz representation is introduced and it is shown that there are only two kinds irreducible representations. Namely, one of them coincides with Lie representation (it said to be Lie representation) and another one has trivial action on the left side, while the action on the right side is irreducible (it said to be Leibniz representation). Following these concepts we shall say Leibniz representation (Leibniz module) for that representation (respectively, module) which has trivial action on the left side.

In this paper we study Leibniz algebras whose associated Lie algebra is the four-dimensional Diamond Lie algebra $\mathfrak{D}$ and the ideal I is one of its finite-dimensional indecomposable Leibniz representations of $\mathfrak{D}$ which are described in [13].

Actually, for a Leibniz algebra L with associated Diamond Lie algebra $\overline{\mathfrak{D}} = L/I$ we could decompose it into direct sum of vector spaces $L = \mathfrak{D} \oplus I$, where $\mathfrak{D}$ is the preimage of $\overline{\mathfrak{D}}$ under the natural homomorphism $\varphi \colon L \to \overline{\mathfrak{D}}$. Clearly, the ideal I can be considered as Leibniz $\overline{\mathfrak{D}}$-module. Taking into account that the ideal I is contained in right annihilator of the algebra L, the multiplications in L are determined from the products $[\mathfrak{D}, \mathfrak{D}]$ and $[I, \overline{\mathfrak{D}}]$. Since I is a Leibniz module over the Lie algebra $\overline{\mathfrak{D}}$, then the product $[I, \overline{\mathfrak{D}}]$ corresponds to a chosen right Lie $\overline{\mathfrak{D}}$-module. Thus, the main problem of the description of Leibniz algebras with associated Lie algebra $\overline{\mathfrak{D}}$ and with the ideal I chosen by specific right $\overline{\mathfrak{D}}$-module consists to identifying the product $[\mathfrak{D}, \mathfrak{D}]$.

Throughout the paper all vector spaces and algebras are finite-dimensional over the field of the complex numbers.

2. Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 2.1. *An algebra $(L, [-,-])$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x, y, z \in L$, the so-called Leibniz identity*

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - \big[[x, z], y\big]$$

holds.

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The real Diamond Lie algebra $\mathfrak{D}$ is a four-dimensional Lie algebra with basis $\{J, P_1, P_2, T\}$ and non-zero relations

$$[J, P_1] = P_2, \qquad [J, P_2] = -P_1, \qquad [P_1, P_2] = T.$$

The complexification of the Diamond Lie algebra: $\mathfrak{D} \otimes_{\mathbb{R}} \mathbb{C}$ displays the following complex basis:

$$\{P_+ = P_1 - iP_2, \quad P_- = P_1 + iP_2, \quad T, \quad J\},$$

where i is the imaginary unit, whose nonzero commutators are

$$[J, P_+] = iP_+, \qquad [J, P_-] = -iP_-, \qquad [P_+, P_-] = 2iT.$$

Since throughout the paper we shall consider only complex Diamond algebra, the same notation $\mathfrak{D}$ does not lead to confusion.

In the paper [13] the authors construct, for any $n \in \mathbb{N}$, a $(3n+3)$-dimensional Lie module V_n over the algebra $\mathfrak{D}$, which is endowed with a basis $\{v_k^j\}_{k=0,\cdots,n}^{j=0,1,2}$. In the paper they use the action $\mathfrak{D} \cdot V_n$ (the action $V_n \cdot \mathfrak{D}$ evidently is defined by the antisymmetric law).

In the Leibniz algebra $L = \mathfrak{D} \oplus V_n$ we shall identify the action $V_n \cdot \mathfrak{D}$ with the product $[V_n, D]$. In order to have compatibility instead of the action $\mathfrak{D} \cdot V_n$ we will use $V_n \cdot \mathfrak{D}$ as follows:

$$\begin{array}{llllll}
v_k^j \cdot J = -\frac{i}{2}(n-2k)v_k^j, & & & k = 0, \dots, n, & j = 0, 1, 2, \\
v_k^j \cdot P_+ = -(n-k+1)v_{k-1}^{j+1}, & v_0^j \cdot P_+ = 0, & v_k^2 \cdot P_+ = 0, & k = 1, \dots, n, & j = 0, 1, \\
v_k^j \cdot P_- = -(k+1)v_{k+1}^{j+1}, & v_n^j \cdot P_- = 0, & v_k^2 \cdot P_- = 0, & k = 0, \dots, n-1, & j = 0, 1, \\
v_k^0 \cdot T = \frac{i}{2}(n-2k)v_k^2, & v_k^j \cdot T = 0, & & k = 0, \dots, n, & j = 1, 2.
\end{array} \tag{1}$$

Remark 2.2. *Here we changed* $T \cdot v_k^0 = 2i(n-2k)v_k^2$*, misprint of [13], to the correct expression* $T \cdot v_k^0 = -\frac{i}{2}(n-2k)v_k^2$.

In order to have the compatibility with the above representations, in the law of Diamond algebra we use the following table of multiplication (we make a change of basis $P'_+ = P_-$, $P'_- = P_+$):

$$[J, P_+] = -iP_+, \quad [J, P_-] = iP_-, \quad [P_+, P_-] = -2iT.$$

In the next proposition two different decompositions of module V_n are presented.

Proposition 2.3 ([13])**.** *Let* V_n *be the* $\mathfrak{D}$*-module constructed above and let us denote by* $\lfloor x \rfloor$ *the integer part of* x*, then*

- *if* $n = 0$*, then* V_n *decomposes in the direct sum of three trivial one-dimensional modules.*
- *If* $n = 2j$*,* $j \in \mathbb{N}, j \geq 1$*, then* V_n *decomposes into the direct sum* $V_n = U_n^1 \oplus U_n^2$ *of two modules, respectively, of dimension* $3(n/2) + 2$ *and* $3(n/2) + 1$*, given by* $U_n^1 = \operatorname{span}\{v_0^0, v_{2k}^0, v_{2k-1}^1, v_0^2, v_{2k}^2\}_{k=1,\dots,\lfloor n/2 \rfloor}$ *and* $U_n^2 = \operatorname{span}\{v_{2k-1}^0, v_0^1, v_{2k}^1, v_{2k-1}^2\}_{k=1,\dots,\lfloor n/2 \rfloor}$.
- *If* $n = 2j + 1$*,* $j \in \mathbb{N}, j \geq 0$*, then* V_n *decomposes into the direct sum* $V_n = W_n^1 \oplus W_n^2$ *of two modules of equal dimension* $3n/2$*, respectively, given by* $W_n^1 = \operatorname{span}\{v_{2k}^0, v_{2k+1}^1, v_{2k}^2\}_{k=0,\dots,\lfloor n/2 \rfloor}$ *and* $W_n^2 = \operatorname{span}\{v_{2k+1}^0, v_{2k}^1, v_{2k+1}^2\}_{k=0,\dots,\lfloor n/2 \rfloor}$.

In the paper [13] it is proved that the terms of the above decompositions are indecomposable $\mathfrak{D}$-modules.

Theorem 2.4. *The modules $U_n^1,\ U_n^2,\ W_n^1$ and W_n^2 are indecomposable $\mathfrak{D}$-modules.*

Further, we shall need the result that specifies the simplest case, that is $[\mathfrak{D},\mathfrak{D}]\subseteq\mathfrak{D}$, with some conditions on the ideal I.

Lemma 2.5 ([11])**.** *Let L be a Leibniz algebra such that $L/I\cong\overline{\mathfrak{D}}$ and I a Leibniz $\overline{\mathfrak{D}}$-module. If there exists a basis $\{X_1,X_2,\dots,X_n\}$ of I such that $[X_i,J]=\alpha_iX_i,\ \alpha_i\notin\{-2;0;2\}$, where $1\le i\le n$, then $[\mathfrak{D},\mathfrak{D}]\subseteq\mathfrak{D}$.*

3. Main result

This section is devoted to the study of Leibniz algebras with associated Diamond Lie algebra and with the condition that the ideal I is one of indecomposable $\mathfrak{D}$-modules $U_n^1,\ U_n^2,\ W_n^1$ or W_n^2.

3.1. Leibniz algebras whose ideal I is the $\mathfrak{D}$-module U_n^1.

Let $\{v_0^0,\ v_{2k}^0,\ v_{2k-1}^1,\ v_0^2,\ v_{2k}^2\}_{k=1,\dots,n/2}$ be the basis of module U_n^1 chosen in Proposition 2.3 with even n. Then from Proposition 2.3 and action 1 we have the products in the Leibniz algebra L:

$$\begin{cases}
[v_{2k}^0,J]=\frac{i}{2}(n-4k)v_{2k}^0, & k=0,\dots,\frac{n}{2},\\
[v_{2k-1}^1,J]=\frac{i}{2}(n-4k+2)v_{2k-1}^1, & k=1,\dots,\frac{n}{2},\\
[v_{2k}^2,J]=\frac{i}{2}(n-4k)v_{2k}^2, & k=0,\dots,\frac{n}{2},\\
[v_{2k}^0,P_+]=(n-2k+1)v_{2k-1}^1, & k=1,\dots,\frac{n}{2},\\
[v_{2k-1}^1,P_+]=(n-2k+2)v_{2k-2}^2, & k=1,\dots,\frac{n}{2},\\
[v_{2k}^0,P_-]=(2k+1)v_{2k+1}^1, & k=0,\dots,\frac{n}{2}-1,\\
[v_{2k-1}^1,P_-]=2kv_{2k}^2, & k=1,\dots,\frac{n}{2},\\
[v_{2k}^0,T]=-\frac{i}{2}(n-4k)v_{2k}^2, & k=0,\dots,\frac{n}{2}.
\end{cases}\tag{2}$$

Theorem 3.1. *An arbitrary Leibniz algebra with corresponding Lie algebra $\overline{\mathfrak{D}}$ and I associated with $\overline{\mathfrak{D}}$-module defined by (2) admits a basis $\{J,P_+,P_-,T,v_0^0,v_{2k}^0,v_{2k-1}^1,v_0^2,v_{2k}^2\}_{k=1,\dots,n/2}$, where n is even and the table of multiplication $[\mathfrak{D},\mathfrak{D}]$ has the following form:*

- $n=4s$

$$\begin{cases}
[J,P_+]=-iP_+, & [J,P_-]=iP_-, & [P_+,P_-]=-2iT,\\
[P_+,J]=iP_+, & [P_-,J]=-iP_-, & [P_-,P_+]=2iT+2\alpha_1v_{2s}^2,\\
[J,T]=\alpha_1v_{2s}^2 & [J,J]=\alpha_2v_{2s}^2, & [P_+,P_+]=\alpha_3v_{2s-2}^2,\\
[P_-,P_-]=\alpha_4v_{2s+2}^2.
\end{cases}$$

- $n=4s-2$

$$\begin{cases}
[J,P_+]=-iP_+, & [P_+,J]=iP_++2is\beta_1v_{2s-2}^2,\\
[J,P_-]=iP_-, & [P_-,J]=-iP_--2is\beta_1v_{2s}^2,\\
[P_+,P_-]=-2iT, & [P_-,P_+]=2iT+2\beta_2v_{2s-1}^1,\\
[J,J]=\beta_1v_{2s-1}^1, & [J,T]=\beta_2v_{2s-1}^1,\\
[P_+,P_+]=\beta_3v_{2s-3}^1, & [P_-,P_-]=\beta_4v_{2s+1}^1,\\
[P_+,T]=2is\beta_2v_{2s-2}^2, & [T,P_+]=-i(2s\beta_2-(s-1)\beta_3)v_{2s-2}^2,\\
[P_-,T]=-2is\beta_2v_{2s}^2, & [T,P_-]=i(4s\beta_2-(s-1)\beta_4)v_{2s}^2.
\end{cases}$$

where $\alpha_i,\ \beta_i \in \mathbb{C},\ 1 \le i \le 4.$

Proof. We will consider two cases $n = 4s$ and $n = 4s-2$.

Let us introduce notation

$$[J,J] = a_0^0 v_0^0 + \sum_{k=1}^{n/2} a_{2k}^0 v_{2k}^0 + \sum_{k=1}^{n/2} a_{2k-1}^1 v_{2k-1}^1 + a_0^2 v_0^2 + \sum_{k=1}^{n/2} a_{2k}^2 v_{2k}^2.$$

Case 1. Let $n = 4s$. Taking the following change of basis:

$$J' = J + \frac{ia_0^0}{2s} v_0^0 + \sum_{k=1}^{s-1} \frac{ia_{2k}^0}{2s-2k} v_{2k}^0 + \sum_{k=s+1}^{2s} \frac{ia_{2k}^0}{2s-2k} v_{2k}^0 + \sum_{k=1}^{2s} \frac{ia_{2k-1}^1}{2s-2k+1} v_{2k-1}^1$$
$$+ \frac{ia_0^2}{2s} v_0^2 + \sum_{k=1}^{s-1} \frac{ia_{2k}^2}{2s-2k} v_{2k}^2 + \sum_{k=s+1}^{2s} \frac{ia_{2k}^2}{2s-2k} v_{2k}^2,$$

we can assume that $[J,J] = a_{2s}^0 v_{2s}^0 + a_{2s}^2 v_{2s}^2$.

Lifting from the quotient Lie algebra $\overline{\mathfrak{D}}$ to the Leibniz algebra L we have

$$[J,P_+] = -iP_+ + b_0^0 v_0^0 + \sum_{k=1}^{2s} b_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} b_{2k-1}^1 v_{2k-1}^1 + b_0^2 v_0^2 + \sum_{k=1}^{2s} b_{2k}^2 v_{2k}^2,$$
$$[J,P_-] = iP_- + c_0^0 v_0^0 + \sum_{k=1}^{2s} c_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} c_{2k-1}^1 v_{2k-1}^1 + c_0^2 v_0^2 + \sum_{k=1}^{2s} c_{2k}^2 v_{2k}^2,$$
$$[P_+,P_-] = -2iT + d_0^0 v_0^0 + \sum_{k=1}^{2s} d_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} d_{2k-1}^1 v_{2k-1}^1 + d_0^2 v_0^2 + \sum_{k=1}^{2s} d_{2k}^2 v_{2k}^2.$$

Making the change of basis elements as follows:

$$P'_+ = P_+ + ib_0^0 v_0^0 + \sum_{k=1}^{2s} ib_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s} ib_{2k-1}^1 v_{2k-1}^1 + ib_0^2 v_0^2 + \sum_{k=1}^{2s} ib_{2k}^2 v_{2k}^2,$$
$$P'_- = P_- - ic_0^0 v_0^0 - \sum_{k=1}^{2s} ic_{2k}^0 v_{2k}^0 - \sum_{k=1}^{2s} ic_{2k-1}^1 v_{2k-1}^1 - ic_0^2 v_0^2 - \sum_{k=1}^{2s} ic_{2k}^2 v_{2k}^2,$$
$$T' = T + \tfrac{i}{2}(d_0^0 v_0^0 + \sum_{k=1}^{2s} d_{2k}^0 v_{2k}^0 + (d_1^1 + ib_0^0) v_1^1 + \sum_{k=2}^{2s} (d_{2k-1}^1 + i(2k-1) b_{2k-2}^0) v_{2k-1}^1 + d_0^2 v_0^2$$
$$+ \sum_{k=1}^{2s} (d_{2k}^2 + 2ikb_{2k-1}^1) v_{2k}^2),$$

we derive the products

$$[J,P_+] = -iP_+, \quad [J,P_-] = iP_-, \quad [P_+,P_-] = -2iT.$$

Considering the chain of equalities

$$P_+ = i[J,P_+] = [J,[P_+,J]] = [[J,P_+],J] - [[J,J],P_+],$$
$$P_- = -i[J,P_-] = [J,[P_-,J]] = [[J,P_-],J] - [[J,J],P_-],$$

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we conclude $[P_+, J] = iP_+ + ia^0_{2s}(2s+1)v^1_{2s-1}$ and $[P_-, J] = -iP_- - ia^0_{2s}(2s+1)v^1_{2s+1}$, respectively.

We set

$$[P_+, P_+] = q^0_0 v^0_0 + \sum_{k=1}^{2s} q^0_{2k} v^0_{2k} + \sum_{k=1}^{2s} q^1_{2k-1} v^1_{2k-1} + q^2_0 v^2_0 + \sum_{k=1}^{2s} q^2_{2k} v^2_{2k},$$

$$[P_-, P_-] = l^0_0 v^0_0 + \sum_{k=1}^{2s} l^0_{2k} v^0_{2k} + \sum_{k=1}^{2s} l^1_{2k-1} v^1_{2k-1} + l^2_0 v^2_0 + \sum_{k=1}^{2s} l^2_{2k} v^2_{2k},$$

$$[J, T] = r^0_0 v^0_0 + \sum_{k=1}^{2s} r^0_{2k} v^0_{2k} + \sum_{k=1}^{2s} r^1_{2k-1} v^1_{2k-1} + r^2_0 v^2_0 + \sum_{k=1}^{2s} r^2_{2k} v^2_{2k}.$$

Applying the Leibniz identity to the following triples we get further constraints on the products.

Leibniz identity		Constraint
$\{P_+, J, P_+\}$	$\Rightarrow$	$a^0_{2s} = q^0_0 = q^2_0 = 0,\ \ q^1_{2k-1} = 0,\ 1 \le k \le 2s,\ \ q^0_{2k} = q^2_{2k} = 0,\ k \ne s-1,\ s > 1$
$\{P_-, J, P_-\}$	$\Rightarrow$	$l^0_0 = l^2_0 = 0,\ \ l^1_{2k-1} = 0,\ 1 \le k \le 2s,\ \ l^0_{2k} = l^2_{2k} = 0 = 0,\ k \ne s+1.$

Thus, we obtain

$$[J, J] = a^2_{2s} v^2_{2s}, \qquad [P_+, J] = iP_+, \qquad [P_-, J] = -iP_-,$$
$$[P_+, P_+] = q^0_{2s-2} v^0_{2s-2} + q^2_{2s-2} v^2_{2s-2}, \quad [P_-, P_-] = l^0_{2s+2} v^0_{2s+2} + l^2_{2s+2} v^2_{2s+2}.$$

Note that for $s = 1$, we have $[P_+, P_+] = q^0_0 v^0_0 + q^2_0 v^2_0$, which agrees with the case $s > 1$.

Considering the Leibniz identity to the following triples implies the following constraints:

Leibniz identity		Constraint
$\{J, T, J\}$	$\Longrightarrow$	$[J, T] = r^0_{2s} v^0_{2s} + r^2_{2s} v^2_{2s}$,
$\{P_+, J, P_-\}$	$\Longrightarrow$	$[T, J] = 0$,
$\{J, P_+, T\}$	$\Longrightarrow$	$[P_+, T] = i(2s+1) r^0_{2s} v^1_{2s-1}$,
$\{J, P_-, T\}$	$\Longrightarrow$	$[P_-, T] = -i(2s+1) r^0_{2s} v^1_{2s+1}$.

Taking into account that $[P_+, P_-] = -2iT$ in the following chain of equalities

$$-2i[J, T] = [J, [P_+, P_-]] = [[J, P_+], P_-] - [[J, P_-], P_+] = -i[P_+, P_-] - i[P_-, P_+],$$

we deduce $[P_-, P_+] = 2iT + 2r^0_{2s} v^0_{2s} + 2r^2_{2s} v^2_{2s}$.

In order to identify the products $[T, P_+]$ and $[T, P_-]$, we introduce the notations:

$$[T, P_+] = m^0_0 v^0_0 + \sum_{k=1}^{2s} m^0_{2k} v^0_{2k} + \sum_{k=1}^{2s} m^1_{2k-1} v^1_{2k-1} + m^2_0 v^2_0 + \sum_{k=1}^{2s} m^2_{2k} v^2_{2k},$$

$$[T, P_-] = t^0_0 v^0_0 + \sum_{k=1}^{2s} t^0_{2k} v^0_{2k} + \sum_{k=1}^{2s} t^1_{2k-1} v^1_{2k-1} + t^2_0 v^2_0 + \sum_{k=1}^{2s} t^2_{2k} v^2_{2k}.$$

In a similar way as above we obtain

Leibniz identity		Constraint
$\{T, P_+, J\}$	$\Longrightarrow$	$[T, P_+] = m^1_{2s-1}v^1_{2s-1}$,
$\{T, P_-, J\}$	$\Longrightarrow$	$[T, P_-] = t^1_{2s+1}v^1_{2s+1}$,
$\{P_+, P_-, T\}$	$\Longrightarrow$	$[T, T] = -s(2s+1)r^0_{2s}v^2_{2s}$,
$\{P_+, P_+, T\}$	$\Longrightarrow$	$q^0_{2s-2} = -(s+1)(2s+1)r^0_{2s}$,
$\{P_+, P_+, P_-\}$	$\Longrightarrow$	$m^1_{2s-1} = -i/2(2s+1)(2s^2+s+1)r^0_{2s}$,
$\{P_-, P_-, T\}$	$\Longrightarrow$	$l^0_{2s+2} = -(2s+1)(s+1)r^0_{2s}$,
$\{T, P_+, P_-\}$	$\Longrightarrow$	$t^1_{2s+1} = -i/2(2s+1)(2s^2+s+3)r^0_{2s}$.

From the restrictions, we derive

$$\begin{aligned}
[P_+, P_+] &= -(s+1)(2s+1)r^0_{2s}v^0_{2s-2} + q^2_{2s-2}v^2_{2s-2},\\
[T, P_+] &= -i/2(2s+1)(2s^2+s+1)r^0_{2s}v^1_{2s-1},\\
[P_-, P_-] &= -(2s+1)(s+1)r^0_{2s}v^0_{2s+2} + l^2_{2s+2}v^2_{2s+2},\\
[T, P_-] &= -i/2(2s+1)(2s^2+s+3)r^0_{2s}v^1_{2s+1}.
\end{aligned}$$

Finally, if we apply the Leibniz identity to the triple of elements $\{P_-, P_+, P_-\}$, we obtain $r^0_{2s} = 0$. Thus, by assuming $(r^2_{2s}, a^2_{2s}, q^2_{2s-2}, l^2_{2s+2}) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, we get the first family of the theorem.

Case 2. Let $n = 4s-2$. Taking the change of basis in the following form:

$$\begin{aligned}
J' = J &+ \frac{ia^0_0}{2s-1}v^0_0 + \sum_{k-1}^{2s-1}\frac{ia^0_{2k}}{2s-2k-1}v^0_{2k} + \sum_{k=1}^{s-1}\frac{ia^1_{2k-1}}{2s-2k}v^1_{2k-1}\\
&+ \sum_{k=s+1}^{2s-1}\frac{ia^1_{2k-1}}{2s-2k}v^1_{2k-1} + \frac{ia^2_0}{2s-1}v^2_0 + \sum_{k=1}^{2s-1}\frac{ia^2_{2k}}{2s-2k-1}v^2_{2k},
\end{aligned}$$

we can assume that $[J, J] = a^1_{2s-1}v^1_{2s-1}$.

Analogously to the previous case, we can get the products

$$[J, P_+] = -iP_+, \qquad [J, P_-] = iP_-, \qquad [P_+, P_-] = -2iT.$$

Verifying the Leibniz identity on triples we have the following restrictions:

Leibniz identity		Constraint
$\{J, J, P_+\}$	$\Longrightarrow$	$[P_+, J] = iP_+ + 2isa^1_{2s-1}v^2_{2s-2}$,
$\{J, J, P_-\}$	$\Longrightarrow$	$[P_-, J] = -iP_- - 2isa^1_{2s-1}v^2_{2s}$.

We put

$$
\begin{aligned}
[P_+, P_+] &= q_0^0 v_0^0 + \sum_{k=1}^{2s-1} q_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s-1} q_{2k-1}^1 v_{2k-1}^1 + q_0^2 v_0^2 + \sum_{k=1}^{2s-1} q_{2k}^2 v_{2k}^2,\\
[P_-, P_-] &= l_0^0 v_0^0 + \sum_{k=1}^{2s-1} l_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s-1} l_{2k-1}^1 v_{2k-1}^1 + l_0^2 v_0^2 + \sum_{k=1}^{2s-1} l_{2k}^2 v_{2k}^2,\\
[J, T] &= r_0^0 v_0^0 + \sum_{k=1}^{2s-1} r_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s-1} r_{2k-1}^1 v_{2k-1}^1 + r_0^2 v_0^2 + \sum_{k=1}^{2s-1} r_{2k}^2 v_{2k}^2.
\end{aligned}
$$

From the Leibniz identity, we have

Leibniz identity		Constraint
$\{P_+, J, P_+\}$	$\Longrightarrow$	$[P_+, P_+] = q_{2s-3}^1 v_{2s-3}^1,$
$\{P_-, J, P_-\}$	$\Longrightarrow$	$[P_-, P_-] = l_{2s+1}^1 v_{2s+1}^1,$
$\{J, J, T\}$	$\Longrightarrow$	$[J, T] = r_{2s-1}^1 v_{2s-1}^1,$
$\{P_+, J, P_-\}$	$\Longrightarrow$	$[T, J] = 0,$
$\{J, P_+, T\}$	$\Longrightarrow$	$[P_+, T] = 2isr_{2s-1}^1 v_{2s-2}^2,$
$\{J, P_-, T\}$	$\Longrightarrow$	$[P_-, T] = -2isr_{2s-1}^1 v_{2s}^2,$
$\{J, P_+, P_-\}$	$\Longrightarrow$	$[P_-, P_+] = 2iT + 2r_{2s-1}^1 v_{2s-1}^1.$

Setting

$$
\begin{aligned}
[T, P_+] &= m_0^0 v_0^0 + \sum_{k=1}^{2s-1} m_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s-1} m_{2k-1}^1 v_{2k-1}^1 + m_0^2 v_0^2 + \sum_{k=1}^{2s-1} m_{2k}^2 v_{2k}^2,\\
[T, P_-] &= t_0^0 v_0^0 + \sum_{k=1}^{2s-1} t_{2k}^0 v_{2k}^0 + \sum_{k=1}^{2s-1} t_{2k-1}^1 v_{2k-1}^1 + t_0^2 v_0^2 + \sum_{k=1}^{2s-1} t_{2k}^2 v_{2k}^2,
\end{aligned}
$$

and applying the Leibniz identity to the following triples of elements:

$$\{T, P_+, J\},\ \{T, P_-, J\},\ \{P_+, P_-, T\},\ \{P_+, P_+, P_-\},\ \{T, P_+, P_-\},\ \{P_-, P_+, P_-\},$$

we derive

$$
\begin{aligned}
[T, P_+] &= (i(s-1)q_{2s-3}^1 - 2isr_{2s-1}^1) v_{2s-2}^2,\\
[T, P_-] &= (4isr_{2s-1}^1 - i(s-1)l_{2s-2}^1) v_{2s}^2,\\
[T, T] &= 0.
\end{aligned}
$$

Finally, by denoting $(a_{2s-1}^1, r_{2s-1}^1, q_{2s-3}^1, l_{2s+1}^1) = (\beta_1, \beta_2, \beta_3, \beta_4)$, we have the second family. $\square$

3.2. Leibniz algebras whose ideal I is the $\mathfrak{D}_C$-module U_n^2.
Suppose that the ideal I is defined as a Leibniz $\mathfrak{D}$-module by the irreducible representation U_n^2 and $\{v^0_{2k-1}, v^1_0, v^1_{2k}, v^2_{2k-1}\}_{k=1,\dots,n/2}$ for even n is the basis of I chosen as in Proposition 2.3. Then the products $[I,\mathfrak{D}]$ have the form:

$$\begin{cases} [v^0_{2k-1},J]=\frac{i}{2}(n-4k+2)v^0_{2k-1}, & k=1,\dots,\frac{n}{2},\\ [v^1_{2k},J]=\frac{i}{2}(n-4k)v^1_{2k}, & k=0,\dots,\frac{n}{2},\\ [v^2_{2k-1},J]=\frac{i}{2}(n-4k+2)v^2_{2k-1}, & k=1,\dots,\frac{n}{2},\\ [v^0_{2k-1},P_+]=(n-2k+2)v^1_{2k-2}, & k=1,\dots,\frac{n}{2},\\ [v^1_{2k},P_+]=(n-2k+1)v^2_{2k-1}, & k=1,\dots,\frac{n}{2},\\ [v^0_{2k-1},P_-]=2kv^1_{2k}, & k=1,\dots,\frac{n}{2},\\ [v^1_{2k},P_-]=(2k+1)v^2_{2k+1}, & k=0,\dots,\frac{n}{2}-1,\\ [v^0_{2k-1},T]=-i/2(n-4k+2)v^2_{2k-1}, & k=1,\dots,\frac{n}{2}. \end{cases}$$

Theorem 3.2. *An arbitrary Leibniz algebra with corresponding Lie algebra $\overline{\mathfrak{D}}$ and with the ideal I defined as a Leibniz $\overline{\mathfrak{D}}$-module U_n^2 admits a basis $\{J,P_+,P_-,T,v^0_{2k-1},v^1_0,v^1_{2k},v^2_{2k-1}\}_{k=1,\dots,n/2}$, where n is even such that the table of multiplication $[\mathfrak{D},\mathfrak{D}]$ has the following form:*

- $n=4s$

$$\begin{cases} [J,P_+]=-iP_+, & [P_+,J]=iP_++i(2s+1)\gamma_1v^2_{2s-1},\\ [J,P_-]=iP_-, & [P_-,J]=-iP_--i(2s+1)\gamma_1v^2_{2s+1},\\ [P_+,P_-]=-2iT, & [P_-,P_+]=2iT+2\gamma_2v^1_{2s},\\ [J,J]=\gamma_1v^1_{2s}, & [J,T]=\gamma_2v^1_{2s},\\ [P_+,P_+]=\gamma_3v^1_{2s-2}, & [P_-,P_-]=\gamma_4v^1_{2s+2},\\ [P_+,T]=i(2s+1)\gamma_2v^2_{2s-1}, & [T,P_+]=-i((2s+1)\gamma_2-\frac{(2s-1)\gamma_3}{2})v^2_{2s-1},\\ [P_-,T]=-i(2s+1)\gamma_2v^2_{2s+1}, & [T,P_-]=i(2(2s+1)\gamma_2-\frac{(2s-1)\gamma_4}{2})v^2_{2s+1}, \end{cases}$$

- $n=4s-2$

$$\begin{cases} [J,P_+]=-iP_+, & [J,P_-]=iP_-, & [P_+,P_-]=-2iT,\\ [P_+,J]=iP_+, & [P_-,J]=-iP_-, & [P_-,P_+]=2iT+2\delta_1v^2_{2s-1},\\ [J,T]=\delta_1v^2_{2s-1} & [J,J]=\delta_2v^2_{2s-1}, & [P_+,P_+]=\delta_3v^2_{2s-3},\\ [P_-,P_-]=\delta_4v^2_{2s+1}, \end{cases}$$

where $\gamma_i,\ \delta_i\in\mathbb{C},\ 1\le i\le 4$.

Proof. Let us denote

$$[J,J]=\sum_{k=1}^{n/2}a^0_{2k-1}v^0_{2k-1}+a^1_0v^1_0+\sum_{k=1}^{n/2}a^1_{2k}v^1_{2k}+\sum_{k=1}^{n/2}a^2_{2k-1}v^2_{2k-1}.$$

In a similar way to the proof of Theorem 3.1 we will consider the cases $n=4s$ and $n=4s-2$.

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Case 1. Let $n = 4s$. Taking the change of basis elements as follows:

$$J' = J + \sum_{k=1}^{2s} \frac{ia_{2k-1}^0}{2s-2k+1} v_{2k-1}^0 + \frac{ia_0^1}{2s} v_0^1 + \sum_{k=1}^{s-1} \frac{ia_{2k}^1}{2s-2k} v_{2k}^1 + \sum_{k=s+1}^{2s} \frac{ia_{2k}^1}{2s-2k} v_{2k}^1 + \sum_{k=1}^{2s} \frac{ia_{2k-1}^2}{2s-2k+1} v_{2k-1}^2,$$

we can assume that $[J, J] = a_{2s}^1 v_{2s}^1$.

Applying similar arguments as in the proof of Theorem 3.1, we derive

$$[J, P_+] = -iP_+, \quad [J, P_-] = iP_-, \quad [P_+, P_-] = -2iT.$$

We set

$$[P_+, P_+] = \sum_{k=1}^{2s} q_{2k-1}^0 v_{2k-1}^0 + q_0^1 v_0^1 + \sum_{k=1}^{2s} q_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} q_{2k-1}^2 v_{2k-1}^2,$$

$$[P_-, P_-] = \sum_{k=1}^{2s} l_{2k-1}^0 v_{2k-1}^0 + l_0^1 v_0^1 + \sum_{k=1}^{2s} l_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} l_{2k-1}^2 v_{2k-1}^2,$$

$$[J, T] = \sum_{k=1}^{2s} r_{2k-1}^0 v_{2k-1}^0 + r_0^1 v_0^1 + \sum_{k=1}^{2s} r_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} r_{2k-1}^2 v_{2k-1}^2.$$

Applying the Leibniz identity to the following triples of elements:

$$\{J, P_+, J\},\ \{J, P_-, J\},\ \{P_+, J, P_+\},\ \{P_-, J, P_-\},\ \{J, J, T\},\ \{P_+, J, P_-\},$$

we deduce restrictions which imply the following expressions for the products

$$[P_+, J] = iP_+ + ia_{2s}^1(2s+1)v_{2s-1}^2, \quad [P_-, J] = -iP_- - ia_{2s}^1(2s+1)v_{2s+1}^2,$$
$$[P_+, P_+] = q_{2s-2}^1 v_{2s-2}^1, \quad [P_-, P_-] = l_{2s+2}^1 v_{2s+2}^1, \quad [J, T] = r_{2s}^1 v_{2s}^1, \quad [T, J] = 0.$$

Moreover, we have

Leibniz identity		Constraint
$\{J, P_+, T\}$	$\Longrightarrow$	$[P_+, T] = ir_{2s}^1(2s+1)v_{2s-1}^2,$
$\{J, P_-, T\}$	$\Longrightarrow$	$[P_-, T] = -ir_{2s}^1(2s+1)v_{2s+1}^2,$
$\{J, P_+, P_-\}$	$\Longrightarrow$	$[P_-, P_+] = 2iT + 2r_{2s}^1 v_{2s}^1.$

We also denote

$$[T, P_+] = \sum_{k=1}^{2s} m_{2k-1}^0 v_{2k-1}^0 + m_0^1 v_0^1 + \sum_{k=1}^{2s} m_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} m_{2k-1}^2 v_{2k-1}^2,$$

$$[T, P_-] = \sum_{k=1}^{2s} t_{2k-1}^0 v_{2k-1}^0 + t_0^1 v_0^1 + \sum_{k=1}^{2s} t_{2k}^1 v_{2k}^1 + \sum_{k=1}^{2s} t_{2k-1}^2 v_{2k-1}^2.$$

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Applying the Leibniz identity to the elements $\{T, P_+, J\}$, $\{T, P_-, J\}$, $\{P_+, P_-, P_-\}$, we get

$$\begin{aligned}
[T, P_+] &= m^0_{2s-1} v^0_{2s-1} + m^2_{2s-1} v^2_{2s-1},\\
[T, P_-] &= t^0_{2s+1} v^0_{2s+1} + t^2_{2s+1} v^2_{2s+1},\\
[T, T] &= 0.
\end{aligned}$$

Finally, we have

Leibniz identity		Constraint
$\{P_+, P_+, P_-\}$	$\Longrightarrow$	$m^0_{2s-1} = 0, \quad m^2_{2s-1} = 1/2i(2s-1)q^1_{2s-2} - i(2s+1)r^1_{2s}$,
$\{T, P_+, P_-\}$	$\Longrightarrow$	$t^0_{2s+1} = 0$,
$\{P_-, P_+, P_-\}$	$\Longrightarrow$	$t^2_{2s+1} = 2i(2s+1)r^1_{2s} - 1/2i(2s-1)l^1_{2s+2}$.

Denoting the parameters $(a^1_{2s}, r^1_{2s}, q^1_{2s-2}, l^1_{2s+2})$ by $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, we obtain the first family of the theorem.

Case 2. Let $n = 4s-2$. The second family of the theorem is obtained by applying similar arguments as in the previous case. $\square$

3.3. Leibniz algebras whose ideal I is the Leibniz $\mathfrak{D}$-module either W^1_n or W^2_n.

Let L be a Leibniz algebra such that the ideal I is defined as a Leibniz $\mathfrak{D}$-module by indecomposable Lie representation W^1_n of the algebra $\mathfrak{D}$ [13]. Then one can assume that $I = \operatorname{span}\{v^0_{2k}, v^1_{2k+1}, v^2_{2k}\}_{k=0,\dots,\lfloor n/2\rfloor}$ where n is odd and

$$\begin{cases}
[v^0_{2k}, J] = \frac{i}{2}(n-4k)v^0_{2k}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^1_{2k+1}, J] = \frac{i}{2}(n-4k-2)v^1_{2k-1}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^2_{2k}, J] = \frac{i}{2}(n-4k)v^2_{2k}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^0_{2k}, P_+] = (n-2k+1)v^1_{2k-1}, & k = 1, \dots, \lfloor n/2\rfloor,\\
[v^1_{2k+1}, P_+] = (n-2k)v^2_{2k}, & k = 1, \dots, \lfloor n/2\rfloor,\\
[v^0_{2k}, P_-] = (2k+1)v^1_{2k+1}, & k = 0, \dots, \lfloor (n-1)/2\rfloor,\\
[v^1_{2k+1}, P_-] = (2k+2)v^2_{2k+2}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^0_{2k}, T] = -i/2(n-4k)v^2_{2k}, & k = 0, \dots, \lfloor n/2\rfloor
\end{cases} \tag{3}$$

Let now the ideal I defined as a Leibniz $\mathfrak{D}$-module by the indecomposable Lie representation W^2_n of the algebra $\mathfrak{D}$ [13]. Then one can assume $I = \operatorname{span}\{v^0_{2k+1}, v^1_{2k}, v^2_{2k+1}\}_{k=0,\dots,\lfloor n/2\rfloor}$, where n is odd and

$$\begin{cases}
[v^0_{2k+1}, J] = \frac{i}{2}(n-4k-2)v^0_{2k+1}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^1_{2k}, J] = \frac{i}{2}(n-4k)v^1_{2k}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^2_{2k+1}, J] = \frac{i}{2}(n-4k-2)v^2_{2k+1}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^0_{2k+1}, P_+] = (n-2k)v^1_{2k}, & k = 1, \dots, \lfloor n/2\rfloor,\\
[v^1_{2k}, P_+] = (n-2k+1)v^2_{2k-1}, & k = 1, \dots, \lfloor n/2\rfloor,\\
[v^0_{2k+1}, P_-] = (2k+2)v^1_{2k+2}, & k = 0, \dots, \lfloor (n-1)/2\rfloor,\\
[v^1_{2k}, P_-] = (2k+1)v^2_{2k+1}, & k = 0, \dots, \lfloor n/2\rfloor,\\
[v^0_{2k+1}, T] = -i/2(n-4k-2)v^2_{2k+1}, & k = 0, \dots, \lfloor n/2\rfloor.
\end{cases} \tag{4}$$

Theorem 3.3. *Let L be a Leibniz algebra with associated Diamond Lie algebra $\overline{\mathfrak{D}}$ and the ideal I is defined as a Leibniz $\mathfrak{D}$-module by indecomposable Lie representation either W_n^1 or W_n^2. Then*

$$[\mathfrak{D}, \mathfrak{D}] \subseteq \mathfrak{D}.$$

Proof. The proof of the theorem follows from the products (3)–(4) and Lemma 2.5. □

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