

On a generalized uniform zero-two law for positive contractions of non-commutative L_1 -spaces

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Abstract. In this paper we prove a non-commutative version of the uniform "zero-two" law for positive contractions of L_1 -spaces associated with von Neumann algebras.

1. Introduction

Let (X, \mathcal{F}, μ) be a measure space with a positive σ -additive measure μ and let $L_1(X, \mathcal{F}, \mu)$ be the usual associated real L_1 -space. A linear operator $T : L_1(X, \mathcal{F}, \mu) \rightarrow L_1(X, \mathcal{F}, \mu)$ is called a *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$.

In [14] it was proved that if T is a positive contraction of L_1 -space then $\sup_{\|f\| \leq 1} \lim_{n \rightarrow \infty} \|T^{n+1} - T^n\|$ is 0 or 2, and this was called "zero-two" law. Using the method of [14], Foguel [4] proved that $\lim_{n \rightarrow \infty} \|T^{n+1} - T^n\|$ is 0 or 2. In [16] the last result was restated as follows:

Theorem 1.1. [16] *Let $T : L_1 \rightarrow L_1$ be a positive contraction. If there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - T^m\| < 2$ then*

$$\lim_{n \rightarrow \infty} \|T^{n+1} - T^n\| = 0.$$

To prove this theorem Zahoropol [16] reduced it to the following theorem.

Theorem 1.2. [16] *Let $T : L_1 \rightarrow L_1$ be a positive contraction. Then for the following statements:*

- (i) *there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - T^m\| < 2$;*
- (ii) *there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - (T^{m+1} \wedge T^m)\| < 1$;*
- (iii) *one has*

$$\lim_{n \rightarrow \infty} \|T^{n+1} - T^n\| = 0.$$

the implications hold: (i) \Rightarrow (ii) \Rightarrow (iii).

To establish the implication (ii) \Rightarrow (iii) the following auxiliary fact is needed.

Theorem 1.3. [16] *Let $T, S : L_1 \rightarrow L_1$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$ then $\|S^n - T^n\| < 1$ for all $n \in \mathbb{N}$.*



We note that a "zero-two" law for Markov processes was proved in [2], which allowed to study random walks on locally compact groups. Other extensions and generalizations of the formulated law have been investigated by many authors [3, 5, 8, 17]. The aim of this paper is to prove a non-commutative version of the "zero-two" law for positive contractions of L_1 -spaces associated with von Neumann algebras. We emphasize that Theorem 1.1 is included in the main result as a particular case.

2. Preliminaries

Throughout the paper M would be a von Neumann algebra with the unit $\mathbf{1}$ and let τ be a faithful normal semi finite trace on M . Therefore we omit this condition from the formulation of theorems. Recall that an element $x \in M$ is called *self-adjoint* if $x = x^*$. The set of all self-adjoint elements is denoted by M_{sa} . By M_* we denote a pre-dual space to M (see for definitions [1],[15]).

Let $\mathfrak{N} = \{x \in M : \tau(|x|) < \infty\}$, here $|x|$ denotes the modules of an element x , i.e. $|x| = \sqrt{x^*x}$. Define the map $\|\cdot\|_1 : \mathfrak{N} \rightarrow [0, \infty)$ defined by the formula $\|x\|_1 = \tau(|x|)$ is a norm (see [12]). The completion of \mathfrak{N} with respect to the norm $\|\cdot\|_1$ is denoted by $L_1(M, \tau)$. It is known [12] that the spaces $L_1(M, \tau)$ and M_* are isometrically isomorphic, therefore they can be identified. Further we will use this fact without noting.

Theorem 2.1. [12] *The space $L_1(M, \tau)$ coincides with the set*

$$L^1 = \{x = \int_{-\infty}^{\infty} \lambda de_{\lambda} : \int_{-\infty}^{\infty} |\lambda| d\tau(e_{\lambda}) < \infty\}.$$

Moreover,

$$\|x\|_1 = \int_{-\infty}^{\infty} |\lambda| d\tau(e_{\lambda}).$$

It is known [12] that the equality

$$L_1(M, \tau) = L_1(M_{sa}, \tau) + iL_1(M_{sa}, \tau) \quad (2.1)$$

is valid. Note that $L_1(M_{sa}, \tau)$ is a pre-dual to M_{sa} .

Let $T : L_1(M, \tau) \rightarrow L_1(M, \tau)$ be any bounded linear operator, by \tilde{T} we denote its restriction to $L_1(M_{sa}, \tau)$. Then due to (2.1) we have $T(x + iy) = \tilde{T}(x) + i\tilde{T}(y)$, where $x, y \in L_1(M_{sa}, \tau)$. This means that any linear bounded operator is uniquely defined by its restriction to $L_1(M_{sa}, \tau)$. Therefore, in what follows, we only consider linear operators on $L_1(M_{sa}, \tau)$ over real numbers.

Recall that a linear operator T is called *positive* if $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a *contraction* if $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in L_1(M_{sa}, \tau)$. Denote

$$\|T\| = \sup\{\|Tx\|_1 : \|x\|_1 = 1, x \in L_1(M_{sa}, \tau)\}.$$

Let $T, S : L_1 \rightarrow L_1$ be two positive contractions. In what follows, we write $T \leq S$ if $S - T$ is a positive operator.

The following auxiliary facts are well known (see for example [11]).

Lemma 2.2. *Let $T : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be a positive operator. Then*

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, x \geq 0} \|Tx\|.$$

Lemma 2.3. *Let $T, S : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be two positive contraction such that $T \leq S$. Then for every $x \in L_1(M_{sa}, \tau)$, $x \geq 0$ the equality holds*

$$\|Sx - Tx\| = \|Sx\| - \|Tx\|.$$

3. Dominant operators

Using the methods of [11, 9] one can prove the following result which is related to dominant operators.

Theorem 3.1. *Let $Z, T, S : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be positive contractions such that $T \leq S$ and $ZS = SZ$. If there is an $n_0 \in \mathbb{N}$ such that $\|Z(S^{n_0} - T^{n_0})\| < 1$. Then $\|Z(S^n - T^n)\| < 1$ for every $n \geq n_0$.*

We note that the last theorem extends a main result of the paper [11]. Namely, we have the following corollary.

Corollary 3.2. *Let $T, S : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be positive contractions such that $T \leq S$. If there is an $n_0 \in \mathbb{N}$ such that $\|S^{n_0} - T^{n_0}\| < 1$. Then $\|S^n - T^n\| < 1$ for every $n \geq n_0$.*

The proof immediately follows if one takes $Z = Id$. Note that if $n_0 = 1$ and M is a commutative von Neumann algebra, then from Corollary 3.2 we immediately get the Zaharopol's result (see Theorem 1.3).

4. A generalized uniform zero-two law

In this section we are going to prove a non-commutative version of the uniform zero-two law for positive contractions on L^1 .

Let $T : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be a positive contraction. Then its conjugate T^* acts on M_{sa} and it is also positive and enjoys $T^*\mathbf{1} \leq \mathbf{1}$. If one has $T^*\mathbf{1} = \mathbf{1}$, then T is called *unital positive contraction*.

Let us formulate our main result.

Theorem 4.1. *Let $Z : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be a unital positive contraction. Assume that $T : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ be unital positive contractions such that $ZT = TZ$. If there are $m, k \in \mathbb{N}_0$ and a positive contraction $S : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau)$ such that $SZ = ZS$ with*

$$ZT^{m+k} \geq ZS, \quad ZT^m \geq ZS \quad \text{with} \quad (4.1)$$

$$\|Z(T^{m+k} - S)\| < 1, \quad \|Z(T^m - S)\| < 1. \quad (4.2)$$

then for any $\varepsilon > 0$ there are $M \in \mathbb{N}$ and $n_0 \in \mathbb{N}_0$ such that

$$\|Z^M(T^{n+k} - T^n)\| < \varepsilon \quad \text{for all } n \geq n_0.$$

Proof. First we note that for any positive contraction T on L_1 -spaces [?, p. 310] there is $\gamma > 0$ such that

$$\left\| \left(\frac{I + T^k}{2} \right)^\ell - T^k \left(\frac{I + T^k}{2} \right)^\ell \right\| \leq \frac{\gamma}{\sqrt{\ell}}. \quad (4.3)$$

Now take any $\varepsilon > 0$ and fix $\ell_\varepsilon \in \mathbb{N}$ such that $\gamma/\sqrt{\ell_\varepsilon} < \varepsilon/2$.

Define

$$Q_1 = \frac{1}{2}(T^{m+k} - S) + \frac{1}{2}T^k(T^m - S).$$

It then follows from (4.1), (4.2) that ZQ_1 is positive and $\|ZQ_1\| < 1$. Moreover, one has

$$T^{m+k} = \left(\frac{I + T^k}{2} \right) S + Q_1$$

where I stands for the identity mapping.

For each $\ell \in \mathbb{N}$ let us define

$$Q_{\ell+1} = \left(\frac{I + T^k}{2} \right)^\ell Q_1 S^\ell + T^{m+k} Q_\ell, \quad \ell \in \mathbb{N}.$$

Taking into account the positivity of S and Q_1 , one can see that Q_ℓ is a positive operator on $L_1(M_{sa}, \tau)$ and $ZQ_\ell = Q_\ell Z$. Moreover, one has

$$T^{\ell(m+k)} = \left(\frac{I + T^k}{2} \right)^\ell S^\ell + Q_\ell, \quad \ell \in \mathbb{N}. \quad (4.4)$$

Now let us put $V_\ell^{(1)} = S^\ell$, and

$$V_\ell^{(d+1)} = T^{\ell(m+k)} V_\ell^{(d)} + V_\ell^{(1)} Q_\ell^d, \quad d \in \mathbb{N}.$$

One can see that for every $d, \ell \in \mathbb{N}$ the operator $ZV_\ell^{(d)}$ is positive, since Z and S are commuting. Moreover, one has

$$T^{d\ell(m+k)} = \left(\frac{I + T^k}{2} \right)^\ell V_\ell^{(d)} + Q_\ell^d, \quad d, \ell \in \mathbb{N}. \quad (4.5)$$

From $Z^*(\mathbf{1}) = \mathbf{T}^*(\mathbf{1}) = \mathbf{1}$, it follows from (4.5) that

$$V_\ell^{(d)*}(\mathbf{1}) + Q_\ell^{*d}(\mathbf{1}) = \mathbf{1}.$$

Now the positivity of $ZV_\ell^{(d)}$ and ZQ_ℓ imply that $\|ZV_\ell^{(d)}\| \leq 1$ and $\|ZQ_\ell\| \leq 1$.

From (4.1) and (4.2), due to Theorem 3.1, one finds that $\|Z(T^{\ell m} - S^\ell)\| < 1$ for all $\ell \in \mathbb{N}$. Using this inequality with $T^*(\mathbf{1}) = \mathbf{1}$ and the positivity of $Z(T^{\ell m} - S^\ell)$ we find that

$$\|Z(T^{\ell m} - S^\ell)\| = \|((T^*)^{\ell m} - S^{*\ell})Z^*\| = \|\mathbf{1} - S^{*\ell}(\mathbf{1})\| < 1. \quad (4.6)$$

The equality (4.4) yields that

$$Q_\ell^*(\mathbf{1}) = \mathbf{1} - S^{*\ell}(\mathbf{1}).$$

Hence, from (4.6) with the positivity of ZQ_ℓ we obtain

$$\|ZQ_\ell\| = \|Q_\ell^*(\mathbf{1})\| = \|\mathbf{1} - S^{*\ell}(\mathbf{1})\| < 1$$

for all $\ell \in \mathbb{N}$.

Therefore, there is a number $d_\varepsilon \in \mathbb{N}$ such that $\|(ZQ_{\ell_\varepsilon})^{d_\varepsilon}\| < \frac{\varepsilon}{4}$. From the commutativity of Z and Q_ℓ one finds

$$\|Z^{d_\varepsilon} Q_{\ell_\varepsilon}^{d_\varepsilon}\| < \frac{\varepsilon}{4}. \quad (4.7)$$

Now putting $n_0 = d_\varepsilon \ell_\varepsilon (m + k)$, from (4.5) with (4.7) we obtain

$$\begin{aligned} \|Z^{d_\varepsilon}(T^{n_0+k} - T^{n_0})\| &\leq \left\| Z^{d_\varepsilon} \left(T^k \left(\frac{I + T^k}{2} \right)^{\ell_\varepsilon} - \left(\frac{I + T^k}{2} \right)^{\ell_\varepsilon} \right) V_{\ell_\varepsilon}^{(d_\varepsilon)} \right\| \\ &\quad + \|Z^{d_\varepsilon} Q_{\ell_\varepsilon}^{d_\varepsilon} (T^k - I)\| \\ &\leq \left\| T^k \left(\frac{I + T^k}{2} \right)^{\ell_\varepsilon} - \left(\frac{I + T^k}{2} \right)^{\ell_\varepsilon} \right\| + 2\|Z^{d_\varepsilon} Q_{\ell_\varepsilon}^{d_\varepsilon}\| \\ &\leq \frac{\gamma}{\sqrt{\ell_\varepsilon}} + 2 \cdot \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Take any $n \geq n_0$ then from the last inequality one gets

$$\|Z^{d_\varepsilon}(T^{n+k} - T^n)\| = \|Z^{d_\varepsilon}T^{n-n_0}(T^{n_0+k} - T^{n_0})\| \leq \|T^{n_0+k} - T^{n_0}\| < \varepsilon$$

which completes the proof. \square

Remark 4.2. *We note the followings:*

- (a) *If $Z = I$ then a similar kind of result has been proved in [13] for positive contractions of C^* -algebras.*
- (b) *If the algebra M is commutative, then Theorem 4.1 covers the main result of [9, 10]. Note that vector-valued versions of the zero-two law have been proved in [6, 7].*
- (c) *Since the dual of $L_1(M_{sa}, \tau)$ is M_{sa} then due to the duality theory the proved Theorem 4.1 holds true if one replaces L_1 -space with M_{sa} .*

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