

The Connected and Disjoint Union of Semi Jahangir Graphs Admit a Cycle-Super (a, d) -Atimagic Total Labeling

Dafik^{1,2}, I.H. Agustin^{1,3}, D. Hardiyantik³

¹CGANT - University of Jember

²Mathematics Education Department - University of Jember

³Mathematics Department - University of Jember

E-mail: d.dafik@unej.ac.id; hestyarin@gmail.com

Abstract. We assume that all graphs in this paper are finite, undirected and no loop and multiple edges. Given a graph G of order p and size q . Let H', H be subgraphs of G . By H' -covering, we mean every edge in $E(G)$ belongs to at least one subgraph of G isomorphic to a given graph H . A graph G is said to be an (a, d) - H -antimagic total labeling if there exist a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ such that for all subgraphs H' isomorphic to H , the total H -weights $w(H) = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ form an arithmetic sequence $\{a, a+d, a+2d, \dots, a+(s-1)d\}$, where a and d are positive integers and s is the number of all subgraphs H' isomorphic to H . Such a labeling is called super if $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$. In this paper, we will discuss a cycle-super (a, d) -atimagicness of a connected and disjoint union of semi jahangir graphs. The results show that those graphs admit a cycle-super (a, d) -atimagic total labeling for some feasible $d \in \{0, 1, 2, 4, 6, 7, 10, 13, 14\}$.

We use a handbook of graph theory written by Gross *et. al* [4] to define all basic definitions of graph in this paper. For p and q are respectively the order and size of graph, by a labeling of a graph, we mean any mapping that sends some set of graph elements to a set of positive integers. The labelings are called vertex labelings or edge labelings If the domain is respectively a vertex-set $V(G)$ or a edge-set $E(G)$. Moreover, the labelings are called *total* labelings if the domain is $V(G) \cup E(G)$. Simanjuntak *et al.* in [13] introduced an (a, d) -edge-antimagic total labeling of G of order p and size q . It is a one-to-one mapping f taking the vertices and edges of G onto $\{1, 2, \dots, p+q\}$ such that the edge-weights $W_f(uv) = f(u) + f(v) + f(uv)$, $uv \in E(G)$ form an arithmetic sequence $\{a, a+d, \dots, a+(q-1)d\}$, where the first term a is $a > 0$ and the common difference d is $d \geq 0$. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

Gutiérrez, and Lladó in [3, 8] expanded the edge-magic total labeling into a magic total covering. They defined that a graph G admits an H' -magic covering, where H' is subgraph of G isomorphic to a given graph H , if the total H -weights $w(H) = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = \lambda(H)$ is a constant magic sum and $\lambda(H)$ is a constant supermagic sum of H if $f : V(G) \rightarrow \{1, 2, \dots, p\}$. Some relevant results can be found in [7, 9, 10, 12]. Recently Feňovčíková *et. al* [2] proved that wheels are cycle antimagic.

Motivated by these two previous labelings, Inayah *et al.* [5] introduced the $(a, d) - H$ - antimagic total labeling. A graph G is said to be an (a, d) - H -antimagic total labeling if there exist a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ such that for all subgraphs H' isomorphic to H , the total H -weights $w(H) = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ form an arithmetic sequence



$\{a, a + d, a + 2d, \dots, a + (s - 1)d\}$, where a and d are positive integers and s is the number of all subgraphs H' isomorphic to H . Similarly, such a labeling is called super if $f : V(G) \rightarrow \{1, 2, \dots, p\}$. Inayah *et. al* [6] proved that, $shack(H, k)$ which contains exactly k subgraphs isomorphic to H is H -super antimagic, for H is a non-trivial connected graph and $k \geq 2$ is an integer.

We will discuss the existence of a cycle-super (a, d) -antimagicness of a connected and disjoint union of semi jahangir graphs. For H -supermagic graphs, we have found some results. For example Rizvi, *et.al*. [11] proved the disjoint union of isomorphic copies of fans, triangular ladders, ladders, wheels, and graphs obtained by joining a star $K_{1,n}$ with K_1 , and also disjoint union of non-isomorphic copies of ladders and fans are cycle-supermagic labelings, but for super antimagic labelings, it remains widely open to explore.

The Results

Prior to present the main results, we repropose a lemma proved by Dafik *et.al* in [1], it will be useful to find the existence of H -super antimagic graphs. This lemma showed a least upper bound for feasible value of d for a graph to be super (a, d) - H -antimagic total labeling.

Lemma 1. [1] Let G be a simple graph of order p and size q . If G is super (a, d) - H -antimagic total labeling then $d \leq \frac{(p_G - p_{H'})p_{H'} + (q_G - q_{H'})q_{H'}}{s-1}$, for H'_j are subgraphs isomorphic to H , $p_G = |V(G)|$, $q_G = |E(G)|$, $p_{H'} = |V(H')|$, $q_{H'} = |E(H')|$, and $s = |H'_j|$.

Proof: Assume that a (p, q) -graph has a super (a, d) - H -antimagic total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, p_G + q_G\}$ and the total H -weights $w(H) = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = \{a, a + d, a + 2d, \dots, a + (s - 1)d\}$. The minimum possible total H -weight in the labeling f is at least $1 + 2 + \dots + p_{H'} + (p_G + 1) + (p_G + 2) + \dots + (p_G + q_{H'}) = \frac{p_{H'}}{2} + \frac{p_{H'}^2}{2} + q_{H'}p_G + \frac{q_{H'}}{2} + \frac{q_{H'}^2}{2}$. Thus, $a \geq \frac{p_{H'}}{2} + \frac{p_{H'}^2}{2} + q_{H'}p_G + \frac{q_{H'}}{2} + \frac{q_{H'}^2}{2}$. On the other hand, the maximum possible total H -weight is at most $p_G + p_G - 1 + p_G - 2 + \dots + (p_G - (p_{H'} - 1)) + (p_G + q_G) + (p_G + q_G - 1) + (p_G + q_G - 2) + \dots + (p_G + q_G - (q_{H'} - 1)) = p_{H'}p_G - \frac{p_{H'}-1}{2}(p_{H'}) + q_{H'}p_G + q_{H'}q_G - \frac{q_{H'}-1}{2}(q_{H'})$. So we obtain $a + (s - 1)d \leq p_{H'}p_G - \frac{p_{H'}-1}{2}(p_{H'}) + q_{H'}p_G + q_{H'}q_G - \frac{q_{H'}-1}{2}(q_{H'})$. Simplifying the inequality then we will have the desired upper bound of d . \square

From now on we will introduce our terminology of connected semi Jahangir and disjoint union of semi Jahangir graphs.

A semi Jahangir graph, denoted by SJ_n , is a connected graph with vertex set $V(SJ_n) = \{p, x_i, y_k; \text{for } 1 \leq i \leq n + 1, 1 \leq k \leq n\}$ and edge set $E(SJ_n) = \{px_i; 1 \leq i \leq n + 1\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{y_i x_{i+1}; 1 \leq i \leq n\}$. Since we study a super (a, d) - H -antimagic total labeling for $H' = C_4$ isomorphic to H , thus $p_G = |V(SJ_n)| = 2n + 2$, $q_G = |E(SJ_n)| = 3n + 1$, $p_{H'} = |V(C_4)| = 4$, $q_{H'} = |E(C_4)| = 4$, $s = |H'_j| = |C_4| = n$. If semi Jahangir graph SJ_n has a super (a, d) - C_4 -antimagic total labeling then it follows from Lemma 1 the upper bound of $d \leq 20$.

A disjoint union of semi Jahangir graph, denoted by mSJ_n , is a disconnected graph with vertex set $V(mSJ_n) = \{p^j, x_i^j, y_k^j; \text{for } 1 \leq i \leq n + 1, 1 \leq k \leq n, 1 \leq j \leq m\}$ and edge set $E(mSJ_n) = \{p^j x_i^j; 1 \leq i \leq n + 1, 1 \leq j \leq m\} \cup \{x_i^j y_i^j; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i^j x_{i+1}^j; 1 \leq i \leq n, 1 \leq j \leq m\}$. Since we study a super (a, d) - H -antimagic total labeling for $H' = C_4$ isomorphic to H , thus $p_G = |V(mSJ_n)| = 2mn + 2m$, $q_G = |E(mSJ_n)| = 3mn + m$, $p_{H'} = |V(C_4)| = 4$, $q_{H'} = |E(C_4)| = 4$, $s = |H'_j| = |C_4| = nm$. If disjoint union of semi Jahangir graph mSJ_n has a super (a, d) - F_n -antimagic total labeling then it follows from Lemma 1 the upper bound of $d \leq 25$.

Now we start to describe the result of the super (a, d) - C_4 -antimagic total labeling of semi Jahangir graph, denoted by SJ_n , in the following theorems.

Theorem 1. For $n \geq 2$, the graph SJ_n admits a super $(15n + 21, 1) - C_4$ antimagic total labeling.

Proof. Define the vertex and edge labeling f_1 as follows

$$\begin{aligned} f_1(p) &= 1 \\ f_1(x_i) &= i + 1, \text{ for } 1 \leq i \leq n + 1 \\ f_1(y_i) &= n + i + 2, \text{ for } 1 \leq i \leq n \\ f_1(px_i) &= 2n + i + 2, \text{ for } 1 \leq i \leq n + 1 \\ f_1(x_iy_i) &= 5n - 2i + 4, \text{ for } 1 \leq i \leq n \\ f_1(y_ix_{i+1}) &= 5n - 2i + 5, \text{ for } 1 \leq i \leq n \end{aligned}$$

The vertex and edge labelings f_1 are a bijective function $f_1 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \dots, 5n + 3\}$. The H -weights of SJ_n , for $1 \leq i \leq n$ under the labeling f_1 , constitute the following sets $w_{f_1} = f_1(p) + f_1(x_i) + f_1(x_{i+1}) + f_1(y_i) = (1) + (i + 1) + (i + 1 + 1) + (n + i + 2) = n + 3i + 6$, and the total H -weights of SJ_n constitute the following sets $W_{f_1} = w_{f_1} + f_1(px_i) + f_1(px_{i+1}) + f_1(x_iy_i) + f_1(y_ix_{i+1}) = (n + 3i + 6) + (2n + i + 2) + (2n + i + 1 + 2) + (5n - 2i + 4) + (5n - 2i + 5) = 15n + i + 20$. It is easy to see that the set $W_{f_1} = \{15n + 21, 15n + 22, \dots, 16n + 20\}$. Therefore, the graph SJ_n admits a super $(15n + 21, 1)$ - C_4 antimagic total labeling, for $n \geq 2$. \square

Theorem 2. For $n \geq 2$, the graph SJ_n admits a super $(14n + 22, 7) - C_4$ antimagic total labeling.

Proof. Define the vertex labeling f_2 as $f_2(p) = f_1(p)$, $f_2(x_i) = f_1(x_i)$, $f_2(y_i) = f_1(y_i)$ and edge labeling f_2 as follows

$$\begin{aligned} f_2(px_i) &= 4n + i + 2, \text{ for } 1 \leq i \leq n + 1 \\ f_2(x_iy_i) &= 2n + i + 2, \text{ for } 1 \leq i \leq n \\ f_2(y_ix_{i+1}) &= 3n + i + 2, \text{ for } 1 \leq i \leq n \end{aligned}$$

The vertex and edge labelings f_2 are a bijective function $f_2 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \dots, 5n + 3\}$. The H -weights of SJ_n , for $1 \leq i \leq n$ under the labeling f_2 , constitute the following sets $w_{f_2} = w_{f_1}$, and the total H -weights of SJ_n constitute the following sets $W_{f_2} = w_{f_2} + f_2(px_i) + f_2(px_{i+1}) + f_2(x_iy_i) + f_2(y_ix_{i+1}) = (n + 3i + 6) + (4n + i + 2) + (4n + i + 1 + 2) + (2n + i + 2) + (3n + i + 2) = 14n + 7i + 15$. It is easy to see that the set $W_{f_2} = \{14n + 22, 14n + 29, \dots, 21n + 15\}$. Therefore, the graph SJ_n admits a super $(14n + 22, 7)$ - C_4 antimagic total labeling, for $n \geq 2$. \square

Theorem 3. For $n \geq 2$, the graph SJ_n admits a super $(13n + 23, 10) - C_4$ antimagic total labeling.

Proof. Define the vertex and edge labeling f_3 as follows

$$\begin{aligned} f_3(p) &= 1 \\ f_3(x_i) &= 2i, \text{ for } 1 \leq i \leq n + 1 \\ f_3(y_i) &= 2i + 1, \text{ for } 1 \leq i \leq n \\ f_3(px_i) &= f_2(px_i) \\ f_3(x_iy_i) &= f_2(x_iy_i) \\ f_3(y_ix_{i+1}) &= f_2(y_ix_{i+1}) \end{aligned}$$

The vertex and edge labelings f_3 are a bijective function $f_3 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \dots, 5n + 3\}$. The H -weights of SJ_n , for $1 \leq i \leq n$ under the labeling f_3 , constitute the following sets $w_{f_3} = f_3(p) + f_3(x_i) + f_3(x_{i+1}) + f_3(y_i) = (1) + (2i) + (2(i + 1)) + (2i + 1) = 6i + 4$, and the total H -weights of SJ_n constitute the following sets $W_{f_3} = w_{f_3} + f_3(px_i) + f_3(px_{i+1}) + f_3(x_iy_i) + f_3(y_ix_{i+1}) = (6i + 4) + (4n + i + 2) + (4n + i + 1 + 2) + (2n + i + 2) + (3n + i + 2) = 13n + 10i + 13$. It is easy to see that the set $W_{f_3} = \{13n + 23, 13n + 33, \dots, 23n + 13\}$. Therefore, the graph SJ_n admits a super $(13n + 23, 10)$ - C_4 antimagic total labeling, for $n \geq 2$. \square

Theorem 4. For $n \geq 2$, the graph SJ_n admits a super $(11n + 25, 13) - C_4$ antimagic total labeling.

Proof. Define the vertex and edge labeling f_4 as follows

$$\begin{aligned} f_4(p) &= 1 \\ f_4(x_i) &= n + i + 1, \text{ for } 1 \leq i \leq n + 1 \\ f_4(y_i) &= n - i + 2, \text{ for } 1 \leq i \leq n \\ f_4(px_i) &= 2n + 3i, \text{ for } 1 \leq i \leq n + 1 \\ f_4(x_i y_i) &= 2n + 3i + 1, \text{ for } 1 \leq i \leq n \\ f_4(y_i x_{i+1}) &= 2n + 3i + 2, \text{ for } 1 \leq i \leq n \end{aligned}$$

The vertex and edge labelings f_4 are a bijective function $f_4 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \dots, 5n + 3\}$. The H -weights of SJ_n , for $1 \leq i \leq n$ under the labeling f_4 , constitute the following sets $w_{f_4} = f_4(p) + f_4(x_i) + f_4(x_{i+1}) + f_4(y_i) = (1) + (n + i + 1) + (n + i + 1 + 1) + (n - i + 2) = 3n + i + 6$, and the total H -weights of SJ_n constitute the following sets $W_{f_4} = w_{f_4} + f_4(px_i) + f_4(px_{i+1}) + f_4(x_i y_i) + f_4(y_i x_{i+1}) = (3n + i + 6) + (2n + 3i) + (2n + 3(i + 1)) + (2n + 3i + 1) + (2n + 3i + 2) = 11n + 13i + 12$. It is easy to see that the set $W_{f_4} = \{11n + 25, 11n + 38, \dots, 24n + 12\}$. Therefore, the graph SJ_n admits a super $(11n + 25, 13) - C_4$ antimagic total labeling, for $n \geq 2$. \square

Theorem 5. For $n \geq 2$, the graph SJ_n admits a super $(\frac{19n+54}{2}, 14) - C_4$ antimagic total labeling for n is even, and for $n \geq 2$, the graph SJ_n admits a super $(\frac{19n+53}{2}, 14) - C_4$ antimagic total labeling for n is odd.

Proof. Define the vertex and edge labeling f_5 as follows

$$\begin{aligned} f_5(p) &= 1 \\ f_5(x_i) &= \begin{cases} \frac{i+3}{2}, & \text{for } 1 \leq i \leq n + 1 ; i \text{ is odd} \\ \frac{n+i+4}{2}, & \text{for } 1 < i < n + 1 ; i \text{ is even, } n \text{ is even} \\ \frac{n+i+3}{2}, & \text{for } 1 < i \leq n + 1 ; i \text{ is even, } n \text{ is odd} \end{cases} \\ f_5(y_i) &= n + i + 2, \text{ for } 1 \leq i \leq n \\ f_5(px_i) &= f_4(px_i) \\ f_5(x_i y_i) &= f_4(x_i y_i) \\ f_5(y_i x_{i+1}) &= f_4(y_i x_{i+1}) \end{aligned}$$

The vertex and edge labelings f_5 are a bijective function $f_5 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \dots, 5n + 3\}$. The H -weights of SJ_n , for $1 \leq i \leq n$ under the labeling f_5 , constitute the following sets $w_{f_5} = f_5(p) + f_5(x_i) + f_5(x_{i+1}) + f_5(y_i) = 1 + (\frac{i+3}{2}) + (\frac{n+i+1+4}{2}) + (2n + 2i + 4) = \frac{3n+4i+14}{2}$ for even n , $w_{f_5} = f_5(p) + f_5(x_i) + f_5(x_{i+1}) + f_5(y_i) = 1 + (\frac{i+3}{2}) + (\frac{n+i+1+3}{2}) + (2n + 2i + 4) = \frac{3n+4i+14}{2}$ for odd n and the total H -weights of SJ_n constitute the following sets $W_{f_5} = w_{f_5} + f_5(px_i) + f_5(px_{i+1}) + f_5(x_i y_i) + f_5(y_i x_{i+1}) = (\frac{3n+4i+14}{2}) + (2n + 3i) + (2n + 3(i + 1)) + (2n + 3i + 1) + (2n + 3i + 2) = \frac{19n+28i+26}{2}$ for even n and $W_{f_5} = w_{f_5} + f_5(px_i) + f_5(px_{i+1}) + f_5(x_i y_i) + f_5(y_i x_{i+1}) = (\frac{3n+4i+13}{2}) + (2n + 3i) + (2n + 3(i + 1)) + (2n + 3i + 1) + (2n + 3i + 2) = \frac{19n+28i+25}{2}$ for odd n . It is easy to see that the set $W_{f_5} = \{\frac{19n+54}{2}, \frac{19n+82}{2}, \dots, \frac{47n+26}{2}\}$ for even n and $W_{f_5} = \{\frac{19n+53}{2}, \frac{19n+81}{2}, \dots, \frac{47n+25}{2}\}$ for odd n . Therefore, the graph SJ_n admits a super $(\frac{19n+54}{2}, 14) - C_4$ antimagic total labeling for $n \geq 2$ with even n . And the graph SJ_n admits a super $(\frac{19n+53}{2}, 14) - C_4$ antimagic total labeling for $n \geq 2$ with odd n . \square

We continue to show the result of the super $(a, d) - C_4$ -antimagic total labeling of disjoint union of semi Jahangir graph, SJ_n , in the following theorems.

Theorem 6. For $m, n \geq 2$, the graph mSJ_n admits a super $(18mn + 14m + 4, 0)$ - C_4 antimagic total labeling.

Proof. For $1 \leq j \leq m$, define the vertex and edge labeling g_1 as follows

$$\begin{aligned} g_1(p^j) &= j, \quad 1 \leq j \leq m \\ g_1(x_i^j) &= 2mi + j - m, \text{ for } 1 \leq i \leq n+1, \quad 1 \leq j \leq m \\ g_1(y_i^j) &= 2mn - 2mi + 3m - j + 1, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \\ g_1(p^j x_i^j) &= 4mn + mi + m + j, \text{ for } 1 \leq i \leq n+1, \quad 1 \leq j \leq m \\ g_1(x_i^j y_i^j) &= 4mn - 2mi + 4m - 2j + 2, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \\ g_1(y_i^j x_{i+1}^j) &= 4mn - 2mi + 4m - 2j + 1, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \end{aligned}$$

The vertex and edge labelings g_1 are a bijective function $g_1 : V(mSJ_n) \cup E(mSJ_n) \rightarrow \{1, 2, 3, \dots, 5mn + 3m\}$. The H -weights of mSJ_n , for $1 \leq i \leq n$ and $1 \leq j \leq m$ under the labeling g_1 , constitute the following sets $w_{g_1} = g_1(p^j) + g_1(x_i^j) + g_1(x_{i+1}^j) + g_1(y_i^j) = (j) + (2mi + j - m) + (2m(i+1) + j - m) + (2mn - 2mi + 3m - j + 1) = 2mn + 2mi + 3m + 2j + 1$, and the total H -weights of mSJ_n constitute the following sets $W_{g_1} = w_{g_1} + g_1(p^j x_i^j) + g_1(p^j x_{i+1}^j) + g_1(x_i^j y_i^j) + g_1(y_i^j x_{i+1}^j) = (2mn + 2mi + 3m + 2j + 1) + (4mn + mi + m + j) + (4mn + m(i+1) + m + j) + (4mn - 2mi + 4m - 2j + 2) + (4mn - 2mi + 4m - 2j + 1) = 18mn + 14m + 4$. It is easy to see that the set $W_{g_1} = \{18mn + 14m + 4, 18mn + 14m + 4, \dots, 18mn + 14m + 4\}$. Therefore, the graph mSJ_n admits a super $(18mn + 14m + 4, 0)$ - C_4 antimagic total labeling, for $m, n \geq 2$. \square

Theorem 7. For $m, n \geq 2$, the graph mSJ_n admits a super $(17mn + 14m + 5, 2)$ - C_4 antimagic total labeling.

Proof. For $1 \leq j \leq m$, define the vertex labeling g_2 as $g_2(p^j) = g_1(p^j)$, $g_2(x_i^j) = g_1(x_i^j)$, $g_2(y_i^j) = g_1(y_i^j)$ and edge labeling g_2 as follows

$$\begin{aligned} g_2(p^j x_i^j) &= 4mn + mi + 2m - j + 1, \text{ for } 1 \leq i \leq n+1, \quad 1 \leq j \leq m \\ g_2(x_i^j y_i^j) &= 3mn - mi + 2m + j, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \\ g_2(y_i^j x_{i+1}^j) &= 4mn - mi + 2m + j, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \end{aligned}$$

The vertex and edge labelings g_2 are a bijective function $g_2 : V(mSJ_n) \cup E(mSJ_n) \rightarrow \{1, 2, 3, \dots, 5mn + 3m\}$. The H -weights of mSJ_n , for $1 \leq i \leq n$ and $1 \leq j \leq m$ under the labeling g_2 , constitute the following sets $w_{g_2} = w_{g_1}$, and the total H -weights of mSJ_n constitute the following sets $W_{g_2} = w_{g_2} + g_2(p^j x_i^j) + g_2(p^j x_{i+1}^j) + g_2(x_i^j y_i^j) + g_2(y_i^j x_{i+1}^j) = (2mn + 2mi + 3m + 2j + 1) + (4mn + mi + 2m - j + 1) + (4mn + m(i+1) + 2m - j + 1) + (3mn - mi + 2m + j) + (4mn - mi + 2m + j) = 17mn + 2mi + 12m + 2j + 3$. It is easy to see that the set $W_{g_2} = \{17mn + 14m + 5, 17mn + 14m + 7, \dots, 19mn + 14m + 3\}$. Therefore, the graph mSJ_n admits a super $(17mn + 14m + 5, 2)$ - C_4 antimagic total labeling, for $m, n \geq 2$. \square

Theorem 8. For $m, n \geq 2$, the graph mSJ_n admits a super $(16mn + 14m + 6, 4)$ - C_4 antimagic total labeling.

Proof. For $1 \leq j \leq m$, define the vertex labeling g_3 as $g_3(p^j) = g_1(p^j)$, $g_3(x_i^j) = g_1(x_i^j)$, $g_3(y_i^j) =$

$g_1(y_i^j)$ and edge labeling g_3 as follows

$$\begin{aligned} g_3(p^j x_i^j) &= 4mn + mi + m + j; \quad 1 \leq i \leq n+1, \quad 1 \leq j \leq m, \text{ dan } i \text{ ganjil} \\ g_3(p^j x_i^j) &= 4mn + mi + 2m - j + 1; \quad 1 \leq i \leq n+1, \quad 1 \leq j \leq m, \text{ dan } i \text{ genap} \\ g_3(x_i^j y_i^j) &= 2mn + mi + m + j, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \\ g_3(y_i^j x_{i+1}^j) &= 4mn - mi + 2m + j, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \end{aligned}$$

The vertex and edge labelings g_3 are a bijective function $g_3 : V(mSJ_n) \cup E(mSJ_n) \rightarrow \{1, 2, 3, \dots, 5mn + 3m\}$. The H -weights of mSJ_n , for $1 \leq i \leq n$ and $1 \leq j \leq m$ under the labeling g_3 , constitute the following sets $w_{g_3} = w_{g_1}$, and the total H -weights of mSJ_n constitute the following sets $W_{g_3} = w_{g_3} + g_3(p^j x_i^j) + g_3(p^j x_{i+1}^j) + g_3(x_i^j y_i^j) + g_3(y_i^j x_{i+1}^j) = (2mn + 2mi + 3m + 2j + 1) + (4mn + mi + m + j) + (4mn + m(i+1) + 2m - j + 1) + (2mn + mi + m + j) + (4mn - mi + 2m + j) = 16mn + 4mi + 10m + 4j + 2$. It is easy to see that the set $W_{g_3} = \{16mn + 14m + 6, 16mn + 14m + 10, \dots, 20mn + 14m + 2\}$. Therefore, the graph mSJ_n admits a super $(16mn + 14m + 6, 4)$ - C_4 antimagic total labeling, for $m, n \geq 2$. \square

Theorem 9. For $m, n \geq 2$, the graph mSJ_n admits a super $(15mn + 14m + 7, 6)$ - C_4 antimagic total labeling.

Proof. For $1 \leq j \leq m$, define the vertex labeling g_4 as $g_4(p^j) = g_1(p^j)$, $g_4(x_i^j) = g_1(x_i^j)$, $g_4(y_i^j) = g_1(y_i^j)$ and edge labeling g_4 as follows

$$\begin{aligned} g_4(p^j x_i^j) &= 4mn + mi + m + j, \text{ for } 1 \leq i \leq n+1, \quad 1 \leq j \leq m \\ g_4(x_i^j y_i^j) &= 2mn + mi + m + j, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \\ g_4(y_i^j x_{i+1}^j) &= 3mn + mi + m + j, \text{ for } 1 \leq i \leq n, \quad 1 \leq j \leq m \end{aligned}$$

The vertex and edge labelings g_4 are a bijective function $g_4 : V(mSJ_n) \cup E(mSJ_n) \rightarrow \{1, 2, 3, \dots, 5mn + 3m\}$. The H -weights of mSJ_n , for $1 \leq i \leq n$ and $1 \leq j \leq m$ under the labeling g_4 , constitute the following sets $w_{g_4} = w_{g_1}$, and the total H -weights of mSJ_n constitute the following sets $W_{g_4} = w_{g_4} + g_4(p^j x_i^j) + g_4(p^j x_{i+1}^j) + g_4(x_i^j y_i^j) + g_4(y_i^j x_{i+1}^j) = (2mn + 2mi + 3m + 2j + 1) + (4mn + mi + m + j) + (4mn + m(i+1) + m + j) + (2mn + mi + m + j) + (3mn + mi + m + j) = 15mn + 6mi + 8m + 6j + 1$. It is easy to see that the set $W_{g_4} = \{15mn + 14m + 7, 15mn + 14m + 13, \dots, 21mn + 14m + 1\}$. Therefore, the graph mSJ_n admits a super $(15mn + 14m + 7, 6)$ - C_4 antimagic total labeling, for $m, n \geq 2$. \square

Concluding Remarks

A least upper bound of difference d for connected and disjoint union of graphs are respectively $d \leq 20$ and $d \leq 25$. Apart from obtained d above, we haven't found any result yet, so we propose the following open problem:

Open Problem 1. Apart from $d \in \{1, 7, 10, 13, 14\}$, determine a super (a, d) - C_4 -antimagic total labeling of connected SJ_n , for $d \leq 20$ and $n \geq 2$.

Open Problem 2. Apart from $d \in \{0, 2, 4, 6\}$, determine a super (a, d) - C_4 -antimagic total labeling of disjoint union of m copies of SJ_n , for $d \leq 25$ and $m, n \geq 2$.

Acknowledgement. We gratefully acknowledge the support from DP2M Fundamental Research Grant DIKTI 2015 and CGANT - University of Jember.

References

- [1] Dafik, Slamin, Wuria Novitasari, Super (a,d) - H - antimagic total covering of shackle graph, *Indonesian Journal of Combinatorics* (2015), submitted
- [2] A. S. Feňovčíková, M. Baca, M. Lascsáková, M. Miller, J. Ryan, Wheels are Cycle-Antimagic, *Electronic Notes in Discrete Mathematics* **48** (2015), 1118
- [3] A. Gutiérrez, and A. Lladó, Magic Coverings, *J. Combin. Math. Combin. Comput* **55** (2005), 43-46.
- [4] J.L. Gross, J. Yellen and P. Zhang, *Handbook of Graph Theory*, Second Edition, CRC Press, Taylor and Francis Group, 2014
- [5] N. Inayah, A.N.M. Salman and R. Simanjuntak, On $(a, d) - H$ -antimagic coverings of graphs, *J. Combin. Math. Combin. Comput.* **71** (2009), 273281.
- [6] N. Inayah, R. Simanjuntak, A. N. M. Salman, Super $(a, d) - H$ -antimagic total labelings for shackles of a connected graph H , *The Australasian Journal of Combinatorics*, **57** (2013), 127138.
- [7] P. Jeyanthi, P. Selvagopal, More classes of H -supermagic Graphs, *Intern. J. of Algorithms, Computing and Mathematics* **3(1)** (2010), 93-108.
- [8] A. Lladó and J. Moragas, Cycle-magic graphs, *Discrete Math.* **307** (2007), 2925 2933.
- [9] T.K. Maryati, A. N. M. Salman, E.T. Baskoro, J. Ryan, M. Miller, On H - supermagic labelings for certain shackles and amalgamations of a connected graph, *Utilitas Mathematica*, **83** (2010), 333-342.
- [10] A. A. G. Ngurah, A. N. M. Salman, L. Susilowati, H -supermagic labeling of graphs, *Discrete Math.*, **310** (2010), 1293-1300.
- [11] S.T.R. Rizvi, K. Ali, M. Hussain, Cycle-supermagic labelings of the disjoint union of graphs, *Utilitas Mathematica*, (2014), in press.
- [12] M. Roswitha, E.T. Baskoro, H -magic covering on some classes of graphs, *American Institute of Physics Conference Proceedings* **1450** (2012), 135-138.
- [13] R. Simanjuntak, M. Miller and F. Bertault, Two new (a, d) -antimagic graph labelings, *Proc. Eleventh Australas. Workshop Combin. Alg. (AWOCA)* (2000), 179189.