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Stability of the fixed points of the complex Swift-Hohenberg equation

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Abstract. We performed an investigation of the stability of fixed points in the complex Swift-Hohenberg equation using a variational formulation. The analysis is based on fixed points Euler-Lagrange equations and analytically showed that the Jacobian eigenvalues touched the imaginary axis and in general, Hopf bifurcation arises. The eigenvalues undergo a stability criterion in order to have Hopf's stability. Trial functions and linear loss dispersion parameter ε are responsible for the existence of stable pulse solutions in this system. We study behavior of the stable soliton-like solutions as we vary a bifurcation ε.

1. Introduction

J. Swift and P. C. Hohenberg were proposed Swift-Hohenberg equation in 1977 along with the arising interest in stable spatially localized states [1-3]. This equation is a model equation for a large class of higher-order parabolic model equations.

The real Swift-Hohenberg equation (RSHE) has proved to be an invaluable model equation for system undergoing pattern formation in a two-parameter family of moving fronts [4]. Originally, this equation was suggested as a model of infinite Prandtl number convection [5] but finds application in its simplest form in the theory of buckling [6], nonlinear optics [7,8], and phase transitions [9]. The equation is commonly found in systems exhibiting bistability between two states (homogenous and heterogenous) that is particularly useful in understanding localized structures [4].

However, there are many systems that are described by the Swift-Hohenberg equation for a complex order parameter. This equation models pattern formation arising from an oscillatory instability with a finite wave number at onset [10,11]. Therefore, in nonlinear optics frequently used the complex Swift-Hohenberg equation (CSHE) as their model. The CSHE also describes photorefractive oscillators [12], nondegenerate optical parametric oscillators (OPOs) [1,13,14], semiconductor laser [15], and passively mode-locked lasers that allow the generation of self-shaped ultra-short pulses in a laser system [16]. In general, the resulting system of Swift-Hohenberg equation has complex coefficients and hence time-dependent solutions. The CSHE also has a much richer system (the real and imaginary parts of the order parameter are coupled by the nonlinear terms) and surely the stationary CSHE solutions have nontrivial spatial structure in both amplitude and phase, although we know that the CSHE admits all the time-independent solutions familiar from RSHE. It is clear that this equation can mainly be analyzed only using computer simulations [17], despite the fact some families of exact solutions of the CSHE can be obtained analytically [18].

In this paper, our main purpose is to investigate stability of the CSHE using a variational formulation. Given the complex dynamics of the dissipative solutions, this formulation is predicted on

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the variation of the soliton amplitude, inverse width, and the phase of the pulses with the CSHE parameter. This formulation is useful to study the ground state since it depends on a trial function and a good set of parameters. We will discuss about it generally as our other interest is to establish the occurrence of stable pulses in the system that supposedly the formulation should reveal the expectation of plain pulsating solitons as it exhibits nonhyperbolic fixed points of differential equations where a small periodic orbit encircle the fixed points. However, the understanding on how the system is competent to explore the phase space continuously shows that the stable pulses exist when the linear dissipation coefficient has to correspond to loss.

The rest of the paper is organized as follows. The mathematical model is introduced in Section 2. Investigation of Euler-Lagrange using the variational formulation is performed in Section 3, whose stability is analyzed in Section 4. The numerical solution is obtained in Section 5. Finally, we summarize with our conclusions in Section 6.

2. Mathematical model

In this paper, we study the CSHE in one dimension [19],

$$\begin{aligned}\partial_z A(x,z) = \varepsilon A(x,z) + (b_1 + ic_1)\partial_x^2 A(x,z) - (b_3 - ic_3)\left|A(x,z)\right|^2 A(x,z) - (b_5 - ic_5)\left|A(x,z)\right|^4 A(x,z) \\ + h\partial_x^4 A(x,z)\end{aligned} \tag{1}$$

Where z is propagation distance or cavity round trip number (treated as continuously variable), x is the retarded time in a frame of reference moving with the pulse, and A represents the complex amplitude of the transverse electric field, for example, inside a cavity.

The equation above is fully parametrized by eight real parameters ε, b_1, c_1, b_3, c_3, b_5, c_5, and h. ε is a linear dissipation loss parameter and values of the parameters b_1, c_1, b_3, c_3, b_5, c_5, and h are fixed. We remark that in many applications [19,20,21] the parameters c_1 and c_3 are 0.5 and 1, respectively.

3. The variational formulation for CSHE

The starting variational formulation of nonlinear dissipative system is relatively explored by Kaup and Malomed [22]. Then, a novel formulation of the variational approach for complex Ginzburg Landau equation (CGLE) has been employed by Mancas and Choudhury [20].

The Lagrangian for the complex Swift-Hohenberg equation may be written as

$$\begin{aligned}L_C = r(x,z)^*[\partial_z A - \varepsilon A - (b_1 + ic_1)\partial_z^2 A + (b_3 - ic_3)\left|A\right|^2 A + (b_5 - ic_5)\left|A\right|^4 A - h\partial_x^4 A] \\ + r(x,z)[\partial_z A^* - \varepsilon A^* - (b_1 - ic_1)\partial_x^2 A^* + (b_3 + ic_3)\left|A\right|^2 A^* + (b_5 + ic_5)\left|A\right|^4 A^* - h\partial_x^4 A^*]\end{aligned} \tag{2}$$

Here, $r(x,z)$ is an auxiliary field employed in [20,22]. The trial functions $A(x,z)$ and $r(x,z)$ assume to be generalized considerably over conventional ones to keep the shape relatively simple and integrable. Other than that, it satisfies a perturbative evolution equation dual to the CGLE with all non-Hamiltonian terms reversed in sign [20].

Thus, the chosen trial functions

$$A(x,z) = A_0(z)e^{-\sigma(z)^2 x^2} e^{i\alpha(z)} \tag{3}$$

$$r(x,z) = e^{-\sigma(z)^2 x^2} \tag{4}$$

Here, $A_0(z)$ is the amplitude, $\alpha(z)$ is the phase of the solitons, and $\sigma(z)$ is the inverse width. Substituting the trial functions into equation (2) and the averaged Lagrangian is

$$\begin{aligned}\overline{L}_C &= \int_{-\infty}^{+\infty} L_C dx \\ &= \frac{\sqrt{\pi/2}(-2A_0\sin(\alpha)\sigma\alpha' + \cos(\alpha)(2\sigma A_0' - A_0\sigma'))}{\sigma^2} - \frac{\sqrt{2\pi}\varepsilon A_0\cos(\alpha)}{\sigma} \\ &+ b_1\sqrt{2\pi}A_0\cos(\alpha)\sigma - 3h\sqrt{2\pi}A_0\cos(\alpha)\sigma^3 - c_1\sqrt{2\pi}A_0\sin(\alpha)\sigma \\ &+ \frac{b_3\sqrt{\pi}A_0^3\cos(\alpha)}{\sigma} + \frac{c_3\sqrt{\pi}A_0^3\sin(\alpha)}{\sigma} + \frac{b_5\sqrt{2\pi/3}A_0^5\cos(\alpha)}{\sigma} + \frac{c_5\sqrt{2\pi/3}A_0^5\sin(\alpha)}{\sigma}\end{aligned} \tag{5}$$

To convince this paper could give solutions and behavior of this system of CSHE, equation (5) is similar with the simplified effective Lagrangian in CGLE [20]. However, the only difference lies in the numerical values of the coefficients of the terms.

We vary the new averaged Lagrangian in equation (5) with respect to $A_0(z)$, $\alpha(z)$ and $\sigma(z)$, and we obtain the following Euler-Lagrange equations

$$\frac{\partial \overline{L}_C}{\partial \Omega(z)} - \frac{d}{dz}\left(\frac{\partial \overline{L}_C}{\partial \Omega'(z)}\right) = 0 \tag{6}$$

where $\Omega(z)$ refers to $A_0(z)$, $\alpha(z)$ and $\sigma(z)$. Solving for $\Omega'(z)$ as a system of three ordinary differential equations (ODEs),

$$\begin{aligned}A_0'(z) = \frac{1}{6\sigma^3}\pi[&2\sigma(-3\sqrt{\pi}\varepsilon\cos(\alpha) + 9A_0^2(b_3\cos(\alpha) + c_3\sin(\alpha)) \\ &+5\sqrt{6}A_0^4(b_5\cos(\alpha) + c_5\sin(\alpha))) + 6\sqrt{2}\sigma^3(b_1\cos(\alpha) - c_1\sin(\alpha)) \\ &-18\sqrt{2}h\cos(\alpha)\sigma^5 + 3\sqrt{2}\sigma'\cos(\alpha)]\end{aligned} \tag{7}$$

$$\begin{aligned}\alpha'(z) = \frac{1}{6\sigma^3}\pi A_0[&6A_0^2\sigma(c_3\cos(\alpha) - b_3\sin(\alpha)) \\ &+2\sqrt{6}A_0^4\sigma(c_5\cos(\alpha) - b_5\sin(\alpha)) \\ &+3\sqrt{2}(-2c_1\cos(\alpha)\sigma^3 + \sin(\alpha)(2\varepsilon\sigma - 2b_1\sigma^3 + 6h\sigma^5 - \sigma'))]\end{aligned} \tag{8}$$

$$\begin{aligned}\sigma'(z) = \frac{1}{6\sigma^3}\sqrt{\pi}[&-6A_0^3(b_3\cos(\alpha) + c_3\sin(\alpha)) \\ &-2\sqrt{6}A_0^5(b_5\cos(\alpha) + c_5\sin(\alpha)) - 3\sqrt{2}\cos(\alpha)A_0' \\ &+3\sqrt{2}A_0(2\cos(\alpha)(\varepsilon + b_1\sigma^2 - 9h\sigma^4) + \sin(\alpha)(-2c_1\sigma^2 + \alpha'))]\end{aligned} \tag{9}$$

Equations (7)-(9) are the fixed points of the CSHE. We use the linearization to construct Jacobian matrix of the fixed point of Euler-Lagrange equations as in equations (7)-(9). Mancas and Choudhury [20] evaluated the fixed points using the formulation of Multiple Scales. However, since we are dealing with a highly complicated equation, the best alternative approach to analyze our Lagrangian is by linearization.

We choose the parameter ε as the control bifurcation parameter. The reason for ε being chosen as a variable and an important parameter rather than any other is that it is responsible for the background to be stable in this dissipative system [23]. To get the eigenvalues, we numerically evaluate the three differential equations (7)-(9) in MATHEMATICA.

This system can be written in the characteristic polynomial

$$
\begin{aligned}
S = \frac{1}{108\sigma^7}&(18\sqrt{\pi}\sigma^5(-6A_0^3(c_3cos(\alpha)-b_3sin(\alpha)))-2\sqrt{6}A_0^5(c_5cos(\alpha)-b_5sin(\alpha)) \\
&+cos(\alpha)(-2c_1\sigma^2+\alpha')2\sqrt{6\pi}A_0^3(-3\lambda\sigma)(6\sqrt{\pi}A_0^3(6\lambda\sigma^2)) \\
&(-6\sqrt{\pi}A_0^3\sigma(b_3cos(\alpha)+c_3sin(\alpha)) \\
&-2\sqrt{6\pi}A_0^5\sigma(b_5cos(\alpha)+c_5sin(\alpha))-6\lambda\sigma^2) \\
&+3\sqrt{2\pi}A_0\sigma^3(-2b_1cos(\alpha)+2c_1sin(\alpha)) \\
&+6hcos(\alpha)\sigma^5-cos(\alpha)\sigma'-\sqrt{\pi}A_0(z)(3A_0^2(c_3cos(\alpha)-b_3sin(\alpha)))\sigma \\
&+\sqrt{6}A_0^4(c_5cos(\alpha)-b_5\sigma sin(\alpha)) \\
&+3\sqrt{2}(\epsilon\sigma sin(\alpha)+(c_1cos(\alpha)+b_1sin(\alpha))\sigma^3-9hsin(\alpha)\sigma^5) \\
&+3\sqrt{6}(3\lambda\sigma^3-3\sqrt{2\pi}cos(\alpha)A_0')\sigma^4+3\sqrt{2}(2cos(\alpha) \\
&(\epsilon+b_1\sigma^2-9h\sigma^4+sin(\alpha)(-2c_1\sigma^2+\alpha'))) \\
&((\lambda-\frac{2\sqrt{\pi}A_0(9(b_3cos(\alpha)+c_3sin(\alpha))+\sqrt{6}A_0^2(b_5cos(\alpha)+c_5sin(\alpha)))}{3\sigma}) \\
&(\lambda-\frac{3\sqrt{2}\sqrt{\pi}A_0cos(\alpha)(2\epsilon\sigma-2b_1\sigma^3+6h\sigma^5-\sigma')}{6\sigma^2}))
\end{aligned}
\tag{10}
$$

4. Stability criterion

The stability of the fixed points is determined from the analysis of the eigenvalues λ of the Jacobian matrix of the CSHE. Since all the parameters and the fixed points are extremely involved in the solutions, the Jacobian matrix of the fixed points can be expressed as

$$
J = \begin{bmatrix}
\frac{\partial A_0'}{\partial A_0} & \frac{\partial A_0'}{\partial \alpha} & \frac{\partial A_0'}{\partial \sigma} \\
\frac{\partial \alpha'}{\partial A_0} & \frac{\partial \alpha'}{\partial \alpha} & \frac{\partial \alpha'}{\partial \sigma} \\
\frac{\partial \sigma'}{\partial A_0} & \frac{\partial \sigma'}{\partial \alpha} & \frac{\partial \sigma'}{\partial \sigma}
\end{bmatrix}
\tag{11}
$$

It is shown that our eigenvalues are nonzero purely imaginary solutions. Suppose Hopf bifurcation for a system of the fixed points (7)-(9) occurs when our eigenvalues reach to the imaginary axis. Furthermore, it is corresponds to be periodic solutions when a periodic orbit bifurcates off from the fixed points. Thus, the solution can belong to a wide class of another soliton solution, i.e pulsating soliton. It will show the perfect periodicity and demonstrate the rich dynamics possible in this system.

Variationally obtained Euler-Lagrange equations are the starting point in order to establish a stability criterion using the theorem of Hopf bifurcations [24,25]. Note that the fixed points of equations (7)-(9) are given by a system of transcendental equations.

Following Hopf, to be a stable fixed point within the linearized analysis, the real part of the eigenvalues λ of cubic equation

$$\lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3 = 0 \tag{12}$$

are negative. In order to have Hopf's stability, Routh-Hurwitz's conditions are

$$\beta_1 > 0, \beta_3 > 0, \beta_1\beta_2 - \beta_3 > 0 \tag{13}$$

must be fulfilled. The stability criterion for variationally obtained fixed points of CSHE, equations (7)-(9), is explicitly expressed as follows:

$$\begin{aligned}\beta_2 = &\frac{1}{\sigma^7}[-2\sqrt{\pi}A_0{}^3(b_3\cos(\alpha)+c_3\sin(\alpha)) - \frac{2\sqrt{6\pi}A_0(b_5\cos(\alpha)+c_5\sin(\alpha))}{3} \\ &+\sqrt{2\pi}A_0(2\varepsilon\cos(\alpha)+18h\cos(\alpha)\sigma^4+\sin(\alpha)\alpha')] \\ &<< 0\end{aligned} \tag{14}$$

$$\begin{aligned}\beta_3 = &\frac{\sqrt{\pi}}{\sigma^2}[-6A_0{}^3(c_3\cos(\alpha)-b_3\sin(\alpha)) - 2\sqrt{6}A_0{}^5(c_5\cos(\alpha)-b_5\sin(\alpha)) \\ &+3\sqrt{2}A_0(-2\sin(\alpha)(\varepsilon+b_1\sigma^2-9h\sigma^4)] \\ &<< 0\end{aligned} \tag{15}$$

and

$$\beta_4 = \beta_1\beta_2 - \beta_3 > 0 \tag{16}$$

$$\beta_1 = 3\sqrt{2}A_0(2\varepsilon\cos(\alpha)+18h\cos(\alpha)\sigma^4+\sin(\alpha)\alpha') \tag{17}$$

The coefficient β_3 is everywhere negative on the Euler-Lagrange solutions. Therefore, the equations (14)-(17) as stability criterion implies that the fixed points of Euler-Lagrange equations are unstable, which a pair of eigenvalues crossing the imaginary axis. Hopf bifurcation arises in this system, but does not occur to exhibit stable periodic oscillations.

5. Parameter space in which stable pulse exists

In this section we study the behavior of stable pulse solutions when the linear loss dispersion parameter, ε of the CSHE changes. It would be a confusing and complicated task to try to change several parameters at once. However, we restricted our study to vary ε and fixed all parameters where the particular choice of parameters is not crucial. We find a stable pulse in a certain range and it is necessary to compare it with experimental that is close to an instability threshold.

Let us first introduce some limits in the parameter space in which to look for stable pulse. We should check the validity of the parameter range ε of the CSHE (analytically and experimental results). The experiment for CSHE has been carried out by Pedrosa *et al*. in 2008 [26]. They performed numerical integrations of the Maxwell Bloch (MB) equations and the corresponding CSHE, for a class C laser. They numerically check the validity of the CSHE in terms of its different asymptotic order as an approximation of MB equations behavior when $\varepsilon \to 0$. figure 1 shows the average relative error, d/ε from Maxwell Bloch equations. In figure 2, they want the confirmation of the linear behavior $(d/\varepsilon)_s \sim \varepsilon$ and the slope is an important figure for the detuning and distance to threshold in the CSHE. The approximate distance to threshold range for which the CSHE holds is estimated between 0.003 and 0.0006 for the parameter values analyzed: $\alpha = 0.75$, $w = 0.5$ and $\alpha = 0.5$, $w = 2$ and a small parameter $0 < \varepsilon << 1$.

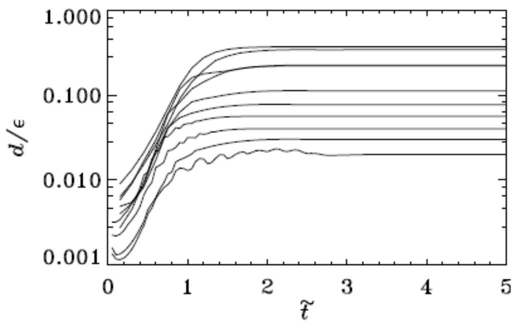


Figure 1. Average relative error d/ε in log scale against time $\tilde{t}$ for different values of ε. Parameters: $\alpha = 0.5$, $w = 2$ [26: p. 528].

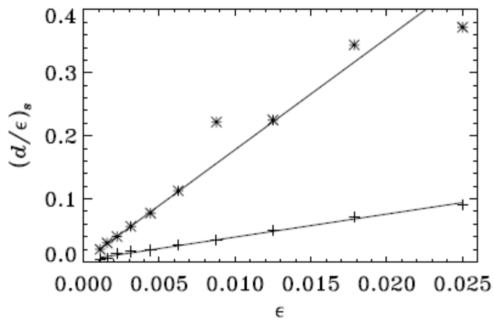


Figure 2. Stationary relative error $(d/\varepsilon)_s$ against ε. Parameters: $\alpha = 0.5$, $w = 2$ and $\alpha = 0.75$, $w = 0.5$ [26: p. 529].

However, Mancas and Choudhury [20] integrate numerically the system of three ordinary differential equations of Complex Ginzburg Landau equation (CGLE). They elucidated the Hopf bifurcation mechanism that responsible for the various pulsating solitary waves by using a trial function as in equation (3). They also chose bifurcation parameters ε and b_3 as they varied ε slowly away from the Hopf curve, $\varepsilon > -0.34581$. When passing through the Hopf curve value $b_3 = -0.216825$ on Hopf bifurcation curve, they found the parameter range for the existence of the pulsating solution as the function of b_3 was $[-0.2531943, -0.1424]$.

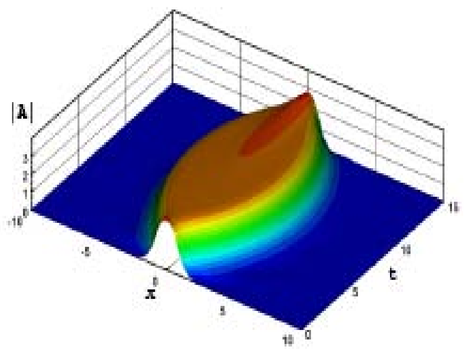


Figure 3. Plain pulsating soliton for $\varepsilon = -0.1$.

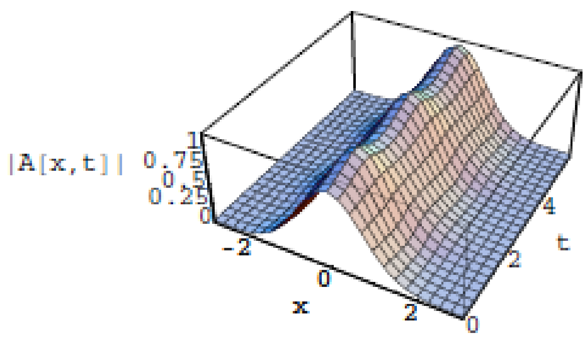


Figure 4. Plain pulsating soliton for $\varepsilon = -0.345481$ and $b_3 = -0.2531943$.

In this paper, we also interested in the behavior of the solutions when we change one of the parameters of the CSHE. To show the dynamics of the spatial-temporal of the stable pulse exist in trial function (3), we integrate the trial functions (3), (4) numerically in MATHEMATICA for different sets of the parameter ε within the regime of stable pulses. The resulting of A_0, α, and σ are then simply inserted into equation (3) where $|A(x,z)|$ is plotted in figure 5. Note that the shape of the initial pulse appears of minor importance as we observe its evolution upon propagation. If the trial functions converge to a stable pulse, the chosen values of the parameter ε belong to the class of soliton solution.

We performed numerical simulations using an Explicit Runge Kutta formulation with step sizes x and z was 0.1. The corresponding values of the coefficients are $b_1 = 0.08$, $h = 0.05$, $b_5 = 0.1$, $c_1 = 0.5$, $c_3 = 1$, $c_5 = -0.1$, and $b_3 = -0.216825$.

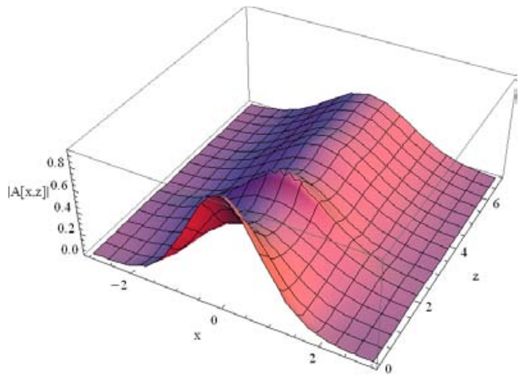
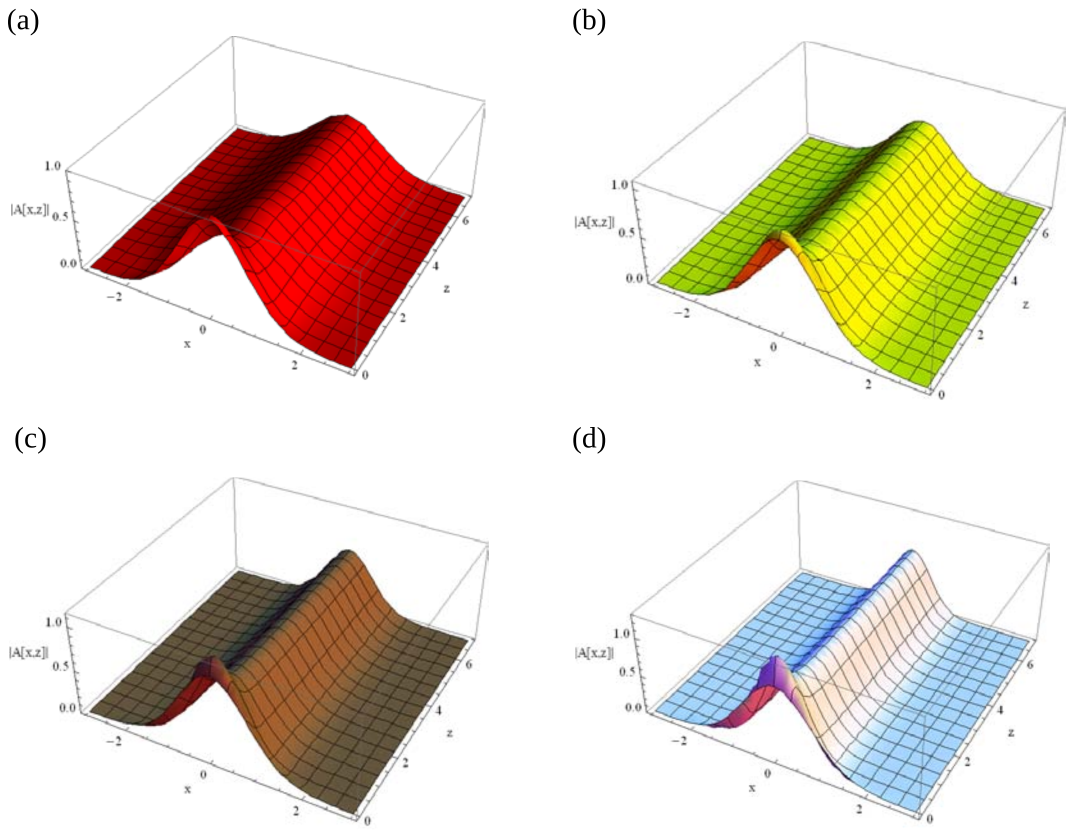


Figure 5. Evolution of solution for $\varepsilon = -0.2$. This stationary solution is initially perturbed as indicated by equation (3).

Next, we consider the detailed effects of varying the parameter ε while the other parameters are fixed. figure 6 shows the result of propagating the trial function (3) with the linear loss dispersion parameter is low enough.



(e)

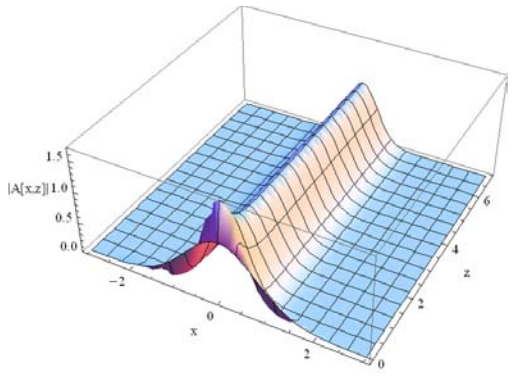


Figure 6. Amplitude $|A(x,z)|$ against time x and propagation distance z for different values of ε; (a) $\varepsilon = -0.28459$, (b) $\varepsilon = -0.40105$, (c) $\varepsilon = -0.61493$, (d) $\varepsilon = -1.00$, (e) $\varepsilon = -2.00$.

Our stable pulses are shown in figure 6. Following Soto-Crespo *et al*. [27], feature above is similar to stationary solution of quintic CGLE. The initial state of the system leads to be a generation of pulses since the parameter ε has been chosen correctly. It reveals that the peak output of amplitude $|A|$ at its highest peak when the value $\varepsilon = -2.00$. Furthermore, the pulse width decreases in space as the value of ε is increasing. The first observation of strictly stable pulselike solutions, figure 5 is like erupting soliton that has been reported by Soto-Crespo *et al*. [28].

Then the process slowly evolves to a stable solution since it perturbed from the trial function (3). We had an evolution when ε was slightly changed and repeating these calculations numerically for other values of ε, we obtained similar soliton solutions where stable propagation of bounded solution is possible.

We have now obtained a stable pulse for fixed points of CSHE and that chosen values of the parameter belong to the class of those that permit the existence of solitons. Thus, setting $\varepsilon = -0.2$ while keeping the rest of the parameters fixed in this case which produced stable pulse-circulation regimes, where the linear loss dissipation term does not destroy the pulse and necessary condition for the existence of stable pulse. The trial functions are important in this system because they generally allow the system to produce the stable pulse for ε in the interval $[-2.00, -0.28459]$. However, it can be seen that the trial functions (3) and (4) do not tend to converge the pulse if ε has smaller values.

6. Conclusion

We have investigated the stability of fixed points of the modified CSHE using a variational formulation. The solutions of Euler-Lagrange equations are considered as fixed points and the eigenvalues of the Jacobian matrix touched the imaginary axis. We have calculated the corresponding stability by using Hopf bifurcation. It becomes unstable and a small periodic orbit unfolds the fixed points. Other than that, the solution in the system cannot periodically change its shape into another wide class of soliton solution. We have also numerically found the range of ε in the parameter space in which stable pulse propagation takes place. The shape of the solution continuously changes when we change ε.

These results suggest the stationary pulses must undergo balance between losses and gain which linear loss dispersion parameter must be recompensed by third-order nonlinear gain of the CSHE. This is beyond the scope of this paper as it can be a future study. Thus, by exploring the phase space continuously shows that the stable pulses exist when the linear dissipation coefficient has corresponded to loss.

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