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Legendre Wavelet Operational Matrix of fractional Derivative through wavelet-polynomial transformation and its Applications in Solving Fractional Order Brusselator system

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Abstract. In this paper we propose the wavelet operational method based on shifted Legendre polynomial to obtain the numerical solutions of nonlinear fractional-order chaotic system known by fractional-order Brusselator system. The operational matrices of fractional derivative and collocation method turn the nonlinear fractional-order Brusselator system to a system of algebraic equations. Two illustrative examples are given in order to demonstrate the accuracy and simplicity of the proposed techniques.

1. Introduction

Fractional calculus has gained an increasing popularity due to its wide range of applications in the fields of engineering, chemistry, finance, physics, aerodynamics, electrodynamics, polymer rheology, economics, biophysics, control theory and so on.[1]-[6]. The development of fractional calculus is being investigated by numerous researchers in different ways through modelling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives which naturally lead to the formulation of fractional differential equations (FDEs). In most cases, the solution of the FDEs does not exist in terms of a finite number of elementary functions, it is therefore fundamental to device numerical methods in order to practically evaluate approximated solutions by means of different schemes and approaches, as such several methods for solving FDEs are available in open literature, Adomian decomposition method [6], variational iteration method [7], homotopy perturbation method [8], predictor-corrector method [9] are some of the few examples. Recently, the idea of approximating the solution of FDEs by orthogonal family of basis functions have been widely used and the most frequently used orthogonal function are sine-cosine functions, block pulse functions, Legendre polynomials, Chebyshev polynomials and Laguerre polynomials. The main idea of using an orthogonal basis is that the problem under consideration reduces to a system of linear or nonlinear algebraic equations [10]. This can be done by truncated series of orthogonal basis function for the solution of the problem

and using the operational matrices [11]
Wavelets are localized functions, which form the basis for $L^2(\mathbb{R})$, so that localized pulse problems can easily be approached and analysed [12]. They are successfully applied in system analysis, optimal control, signal analysis and many more areas see [13]. However, wavelets are just another basis set which offers considerable advantages over alternative basis sets and allows us to tackle problems not accessible with conventional numerical methods, these main advantages are discussed in [14]. Legendre wavelets as a specific kind of wavelets has been widely applied for solving FDEs, for example see [10, 13, 15, 16, 17], at this juncture, it is worth mentioning that Legendre wavelets has mutual spectral accuracy, orthogonality and other useful properties of wavelets. The main purpose of this paper is to apply Legendre wavelets operational method to obtain the numerical solution of the nonlinear fractional-order Brusselator system given by

$$\begin{aligned} D^{\alpha} y_1(x) &= a - (\mu+1) y_1(x) + y_1^2(x) y_2(x) \\ D^{\beta} y_2(x) &= \mu y_1(x) - y_1^2(x) y_2(x) \end{aligned} \tag{1}$$

subject to $y_i(0) = d_i$ for $i = 1, 2$.
where $a > 0,\ \ \mu > 0,\ \ \ \alpha,\ \ \beta \in (0, 1]$ and d_i are constants.
The equation (1) have been studied by many authors, for example, Gafiychuk and Datsko investigate the stability of fractional-order Brusselator system in [18]. In [19], Wang and Li proved that the solution of fractional-order Brusselator system has a limit cycle using numerical method. Jafari et al. used the variational iteration method to investigate the approximate solutions of this system [20], in [21] Bernstein polynomial operational matrix of fractional order integration was used for the approximate solution of the system (1), polynomial least squares method was also used to obtain the approximate solution of (1) in [22]. The paper is organised as follows. Section 2 introduce some basic definitions and mathematical preliminaries of fractional calculus, in section 3 we first defined shifted Legendre polynomial then, described the basics of wavelets and Legendre wavelets ,in section 4 we introduce the shifted Legendre operational matrix of fractional order derivative and Legendre wavelet operational matrix of fractional order derivative , in section 5 application of the Legendre wavelets operational matrix of fractional order derivative together with collocation method on Brusselator system is shown. In section 6 two numerical examples are considered to demonstrate the accuracy of the scheme.

2. Preliminaries

2.1. Fractional derivative and integral

Here, we recall some basic definitions and properties of fractional calculus that are used in this article. There are various definitions for fractional differentiation [2]. The Riemann-Liouville definition has certain disadvantage when we try to model a real-world phenomenon [2]. However, the Caputo's definition is more reliable in application and so we use this definition for fractional derivatives. But, the most frequently encountered definition of an integral of fractional order is the Riemann-Liouville integral, in which the fractional integral operator I of a function $f(t)$ is defined as:

Definition 2.1 *By refering to [1] (page 69), the Riemann-Liouville integral I of fractional order α of $f(t)$ is given by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad t > 0, \alpha \in \mathbb{R}^+ \tag{2}$$

Where $\Gamma(\cdot)$ is the gamma function, its fractional derivative of order $\alpha > 0$ is given by

$$(D_l^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^m (I^{m-\alpha} f)(x),$$

$$(\alpha > 0, \;\; m-1 < \alpha < m)$$

Some properties of I^{α} are as follows:

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t), \;\; \alpha > 0, \;\; \beta > 0 \tag{3}$$

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha} \tag{4}$$

Definition 2.2 *By refering to [1] Theorem 2.1,The Caputo fractional derivative D^{α} of a function $f(t)$ is defined as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{5}$$

$$n-1 < \alpha \leq n, \;\; n \in \mathbb{N}.$$

The following are some properties of Caputo fractional derivatives

$$D^{\alpha} C = 0, \quad (C \;\; is \;\; constant) \tag{6}$$

$$D^{\alpha} t^{\beta} = \begin{cases} 0, \;\; \beta \in \mathbb{N} \cup \{0\} \quad and \quad \beta < \lceil \alpha \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \;\; \beta \in \mathbb{N} \cup \{0\} \;\; and \;\; \beta \geq \lceil \alpha \rceil \quad or\, \beta \notin \mathbb{N} \;\; and \;\; \beta > \lfloor \alpha \rfloor, \end{cases} \tag{7}$$

Where $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α
Similar to the integer order differentiation, the Caputo fractional differential operator is a linear operator, since,

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x) \tag{8}$$

where λ and μ are constants.

3. Legendre Polynomial and Legendre Wavelets

3.1. Shifted Legendre polynomial

The well known Legendre polynomials of degree m are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae

$$L_{m+1}(t) = \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t),$$

$$m = 1, 2, \cdots,$$

where $L_0(t) = 1$ and $L_1(t) = t$. For one to use these polynomials on the interval $[0, 1]$, we define the so called shifted Legendre polynomials by using the change of variable $t = 2x - 1$. Let the shifted Legendre polynomials $L_m(2x-1)$ be denoted by $P_m(x)$. Then $P_m(x)$ can be obtained as follows:

$$\begin{aligned} P_{m+1}(x) = \;& \frac{(2m+1)(2x-1)}{m+1} P_m(x) \\ & - \frac{m}{m+1} P_{m-1}(x), \;\; m = 1, 2, ..., \end{aligned} \tag{9}$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The analytical form of the shifted Legendre polynomials $P_m(x)$ of degree m is given by:

$$P_m(x) = \sum_{k=0}^{m} (-1)^{m+k} \frac{(m+k)!x^k}{(m-k)!(k!)^2}. \tag{10}$$

Note that $P_m(0) = (-1)^m$ and $P_m(1) = 1$. The orthogonality condition is

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} \frac{1}{2m+1}, & for\ \ m = n, \\ 0, & for\ \ m \neq n. \end{cases} \tag{11}$$

3.2. Wavelets and Legendre Wavelets
Wavelets are family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [?]

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right) \qquad a, b \in \mathbb{R} \quad a \neq 0$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, \quad b = nb_0 a_0^{-k}, \quad a_0 > 1, \quad b_0 > 1$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{n,k}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0)$$

where $\psi_{n,k}$ form a wavelet basis for $L^2(\mathbb{R})$.
Legendre wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments: k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval $[0, 1]$ by

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m + \frac{1}{2}} L_m(2^{k+1}t - (2n+1)), & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0, & otherwise. \end{cases} \tag{12}$$

where $m = 0, 1, \cdots, M$ and $n = 0, 1, \cdots, 2^k - 1$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for the orthonormality and the $L_m(t)$ is the well known Legendre polynomial defined in section 3.1
Using shifted Legendre polynomial $P_m(t)$ as defined in section 3.1 one can write the Legendre Wavelets as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m + \frac{1}{2}} P_m(2^k t - n), & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0, & otherwise. \end{cases} \tag{13}$$

with the same range of m and n as in (11).

3.3. Function approximations
Any function $f(t)$ which is square integrable in the interval $[0, 1)$ can be expanded into Legendre wavelet series as [16]

$$f(t) = \sum_{n=o}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \psi_{n,m} \tag{14}$$

where the coefficient $C_{n,m}$ is given by

$$C_{n,m} = \langle f(t), \psi_{n,m}(t) \rangle$$

The convergence of the Legendre wavelet series (13) is established in [17]. If the infinite series in (13) is truncated, then it can be written as

$$f(t) \approx \sum_{n=o}^{2^k-1} \sum_{m=0}^{M} C_{n,m}\psi_{n,m} = \mathbf{C}^T\Psi(t) \tag{15}$$

where $\mathbf{C}$ and Ψ are $2^k(M+1) \times 1$ matrices given by

$$\mathbf{C} = [c_{0,0}, c_{0,1}, ..., c_{0,M}, c_{1,0}, c_{1,1}, \cdots, c_{1,M}, \cdots, c_{(2^k-1),0}, c_{(2^k-1),1}, \cdots, c_{(2^k-1),M}]^T \tag{16}$$

$$\Psi(t) = [\psi_{0,0}, \psi_{0,1}, ..., \psi_{0,M}, \psi_{1,0}, \psi_{1,1}, \cdots, \psi_{1,M}, \cdots, \psi_{(2^k-1),0}, \psi_{(2^k-1),1}, \cdots, \psi_{(2^k-1),M}]^T \tag{17}$$

4. Legendre Wavelet Operational Matrix of Fractional Order Derivative

In this section, we derive the Legendre wavelet operational matrix of the fractional derivative by first transforming the wavelets to shifted Legendre polynomials, we then make use of the shifted Legendre operational matrix of the fractional derivative derived in [23], and finally we derive the Legendre wavelet operational matrix of the fractional derivative.

4.1. Transformation matrix of the Legendre wavelets to Legendre polynomials

An arbitrary function $y(t) \in L^2[0,1]$ can be expanded into shifted Legendre polynomials as

$$y(x) = \sum_{m=0}^{M} r_m P_m(x) = R\Psi'(x)$$

where the shifted Legendre coefficient vector R and the shifted Legendre vector $\Psi'(x)$ are given by

$$R = [r_0, r_1, \cdots, r_M] \tag{18}$$

$$\Psi'(x) = [P_0(x), P_1(x), \cdots, P_M(x)]^T \tag{19}$$

The Legendre wavelet may be expanded in to $(M+1)$-terms shifted Legendre polynomials as

$$\Psi_{2^k(M+1)\times 1}(t) = \Phi_{2^k(M+1)\times(M+1)}\Psi'_{(M+1)\times 1} \tag{20}$$

where Φ is the transformation matrix of the Legendre wavelet to Legendre polynomial. E.g when $M = 2$ and $k = 1$ we have

$\Psi(t) = [\psi_{0,0}(t), \psi_{0,1}(t), \psi_{0,2}(t), \psi_{1,0}(t), \psi_{1,1}(t), \psi_{1,2}(t)]^T$
$\Psi'(x) = [P_0(x), P_1(x), P_2(x)]^T$

where

$$\left.\begin{aligned} \psi_{0,0}(t) &= \sqrt{2} = \sqrt{2}P_0(t) \\ \psi_{0,1}(t) &= \sqrt{6}(4t-1) = \sqrt{6}P_0(t) + 2\sqrt{6}P_1(t) \\ \psi_{0,2}(t) &= \sqrt{10}(24t^2 - 12t + 1) = 2\sqrt{10}P_0(t) + 6\sqrt{10}P_1(t) + 4\sqrt{10}P_2(t) \end{aligned}\right\} 0 \le t < \tfrac{1}{2}$$

$$\left.\begin{aligned} \psi_{1,0}(t) &= \sqrt{2} = \sqrt{2}P_0(t) \\ \psi_{1,1}(t) &= \sqrt{6}(4t-3) = -\sqrt{6}P_0(t) + 2\sqrt{6}P_1(t) \\ \psi_{1,2}(t) &= \sqrt{10}(24t^2 - 36t + 13) = 3\sqrt{10}P_0(t) - 6\sqrt{10}P_1(t) + 4\sqrt{10}P_2(t) \end{aligned}\right\} \tfrac{1}{2} \le t < 1$$

Thus, in this case
$\Phi = \begin{cases} \Phi_1 = [a_{i,j}]_{6\times 3}, & 0 \le t < \frac{1}{2} \\ \Phi_2 = [b_{i,j}]_{6\times 3}, & \frac{1}{2} \le t < 1. \end{cases}$
where,

$$\Phi_1 = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{6} & 2\sqrt{6} & 0 \\ 3\sqrt{10} & 6\sqrt{10} & 4\sqrt{10} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Phi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ -\sqrt{6} & 2\sqrt{6} & 0 \\ 3\sqrt{10} & -6\sqrt{10} & 4\sqrt{10} \end{bmatrix}$$

4.2. Shifted Legendre operational matrix of fractional order derivative
The fractional derivative of order α of the vector $\Psi'(t)$ as shown in [23] can be expressed by

$$D^{\alpha}\Psi'(t) = \mathbf{F}^{(\alpha)}\Psi'(t), \tag{21}$$

where $\mathbf{F}^{(\alpha)}$ is the $(m+1)\times(m+1)$ operational matrix of fractional derivative of order α defined as:

$$\mathbf{F}^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \Sigma_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,0,k} & \sum\limits_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum\limits_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{k=\lceil\alpha\rceil}^{i}\theta_{i,0,k} & \sum\limits_{k=\lceil\alpha\rceil}^{i}\theta_{i,1,k} & \cdots & \sum\limits_{k=\lceil\alpha\rceil}^{i}\theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{k=\lceil\alpha\rceil}^{m}\theta_{m,0,k} & \sum\limits_{k=\lceil\alpha\rceil}^{m}\theta_{m,1,k} & \cdots & \sum\limits_{k=\lceil\alpha\rceil}^{m}\theta_{m,m,k} \end{pmatrix}$$

where $\theta_{i,j,k}$ is given by:

$$\theta_{i,j,k} = (2j+1)\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l-\alpha+1)} \tag{22}$$

Check [23] for more details on $\mathbf{F}^{\alpha}$

4.3. Legendre wavelet operational matrix of fractional order derivative
Now, we derive Legendre wavelet operational matrix of fractional order derivative.
Let

$$D^{\alpha}\Psi(x) = \mathbf{H}^{(\alpha)}\Psi(x) \tag{23}$$

where $\mathbf{H}^{(\alpha)}$ is the Legendre wavelet operational matrix of fractional derivative. Using Eq.(19) and (20) we get

$$D^{\alpha}\Psi(x) = D^{\alpha}\Phi\Psi'(x) = \Phi D^{\alpha}\Psi'(x) = \Phi\mathbf{F}^{(\alpha)}\Psi'(x) \tag{24}$$

from Eq. (22) and (23) we have

$$\mathbf{H}^{(\alpha)}\Psi(x) = \mathbf{H}^{(\alpha)}\Phi\Psi'(x) = \Phi\mathbf{F}^{(\alpha)}\Psi'(x) \tag{25}$$

Thus, the Legendre wavelet operational matrix of fractional derivative $\mathbf{H}^{(\alpha)}$ is given by

$$\mathbf{H}^{(\alpha)} = \Phi\mathbf{F}^{(\alpha)}\Phi^{-1} \tag{26}$$

5. Applications of Legendre Wavelet Operational Matrix of Fractional Order Derivative on Fractional Order Brusselator System

To solve problem (1) we first approximate $y_1(x)$, $y_2(x)$, $D^{\alpha}y_1(x)$, $D^{\beta}y_2(x)$ as

$$y_1(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{n,m}\psi_{n,m} = \mathbf{C}^T\Psi(x) \tag{27}$$

$$y_2(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} s_{n,m}\psi_{n,m} = \mathbf{S}^T\Psi(x) \tag{28}$$

where $\mathbf{S} = [s_{0,0}, s_{0,1}, ..., s_{0,M}, s_{1,0}, s_{1,1}, \cdots, s_{1,M}, \cdots, s_{(2^k-1),0}, s_{(2^k-1),1}, \cdots, s_{(2^k-1),M}]^T$ and $\mathbf{C}$ as defined in (16) are the unknown vector.

Now, using (23), (27) and (28) we get

$$D^{\alpha}y_1(x) \approx \mathbf{C}^T D^{\alpha}\Psi(x) \approx \mathbf{C}^T\mathbf{H}^{(\alpha)}\Psi(x) \tag{29}$$

$$D^{\beta}y_2(x) \approx \mathbf{S}^T D^{\beta}\Psi(x) \approx \mathbf{S}^T\mathbf{H}^{(\beta)}\Psi(x) \tag{30}$$

Substituting these equations in (1) we get

$$\begin{aligned} \mathbf{C}^T\mathbf{H}^{(\alpha)}\Psi(x) &= a - (\mu+1)\mathbf{C}^T\Psi(x) + (\mathbf{C}^T\Psi(x))^2\mathbf{S}^T\Psi(x) \\ \mathbf{S}^T\mathbf{H}^{(\beta)}\Psi(x) &= \mu\mathbf{C}^T\Psi(x) - (\mathbf{C}^T\Psi(x))^2\mathbf{S}^T\Psi(x) \end{aligned} \tag{31}$$

Also, by substituting initial conditions of (1) in to (27) and (28) we have

$$\begin{aligned} y_1(0) &\approx \mathbf{C}^T\Psi(0) = d_1 \\ y_2(0) &\approx \mathbf{S}^T\Psi(0) = d_2 \end{aligned} \tag{32}$$

Now to find the solution $y_1(x)$ and $y_2(x)$, we collocate (31) at $2^k(M+1)-2$ points. For suitable collocation points we use the first $2^k(M+1)-2$ shifted Legendre polynomial roots $P_{2^k(M+1)}(x)$, these equations together with (32) generates $2^k(M+1)$ non-linear equations which can be solved using Newton's iterative method. Consequently $y_1(x)$ and $y_2(x)$ given in (27) and (28) can be calculated

6. Illustrative Examples

In this section, we demonstrate the effectiveness of the proposed method by solving two numerical examples.

Example 6.1 *We consider fractional-order Brusselator system given in [21] by*

$$D^{\alpha}y_1(x) = -2y_1(x) + y_1^2(x)y_2(x)$$
$$D^{\beta}y_2(x) = y_1(x) - y_1^2(x)y_2(x)$$
$$y_1(0) = 1, \quad y_2(0) = 1$$

We use the technique described in section 5 to solve this problem with $M = 3$ and $k = 0$, first, we approximate the equation with Legendre wavelet as follows

$$\begin{aligned} \mathbf{C}^T\mathbf{H}^{(\alpha)}\Psi(x) &= -2\mathbf{C}^T\Psi(x) + (\mathbf{C}^T\Psi(x))^2\mathbf{S}^T\Psi(x) \\ \mathbf{S}^T\mathbf{H}^{(\beta)}\Psi(x) &= \mathbf{C}^T\Psi(x) - (\mathbf{C}^T\Psi(x))^2\mathbf{S}^T\Psi(x) \end{aligned} \tag{33}$$

For the case $\alpha = \beta = 0.98$.
By collocating (33) at the first three roots of $P_4(x)$ we obtain six nonlinear algebraic equations. we also approximate the initial condition as in (32) as,

$$\begin{aligned} c_{00} - \sqrt{3}c_{01} + \sqrt{5}c_{02} - \sqrt{7}c_{03} &= 1 \\ s_{00} - \sqrt{3}s_{01} + \sqrt{5}s_{02} - \sqrt{7}s_{03} &= 1 \end{aligned} \tag{34}$$

solving these nonlinear equations together with (34) we obtain the unknown values of **C** and **S** Hence the solutions

$$y_1(x) = 1 - 1.0791x + 0.2711x^2 - 0.0638x^3$$
$$y_2(x) = 1 + 0.0151x + 0.4185x^2 - 0.2624x^3$$

Fig.1 and Fig.2 shows the comparison of this solution with the solution obtain as equation (18) in [22].

For the case $\alpha = \beta = 1$.
The six nonlinear algebraic euations obtained after collocating (33) are solved together with (34) and we have the solution as

$$y_1(x) = 1 - 1.0120x + 0.1211x^2 + 0.1517x^3$$
$$y_2(x) = 1 + 0.0096x + 0.4069x^2 - 0.2461x^3$$

Fig.4 and Fig.3 shows the comparison of this solution with the solution obtain as equation (19) in [22]

Example 6.2 *We consider the following Brusselator system solved in [20]*

$$D^{\alpha}y_1(x) = 0.5 - 1.1y_1(x) + y_1^2(x)y_2(x)$$
$$D^{\beta}y_2(x) = 0.1y_1(x) - y_1^2(x)y_2(x)$$
$$y_1(0) = 0.4, \quad y_2(0) = 1.5$$

We use the same technique described in section 5 as utilized in the first example. Fig.5 and Fig.6 shows the comparison of the solutions $y_1(x)$ and $y_2(x)$ with the solution obtain as equation (19) in [20]

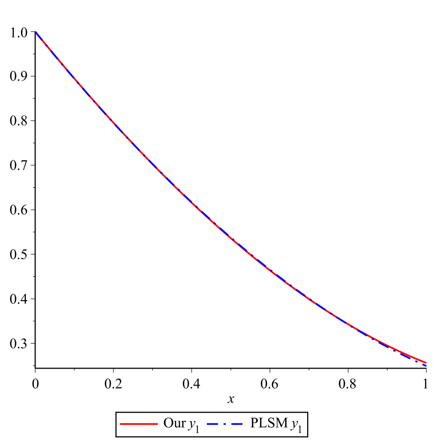

Figure 1. Comparison of y_1 and PLSM y_1 when $\alpha = \beta = 0.98$ for example 6.1

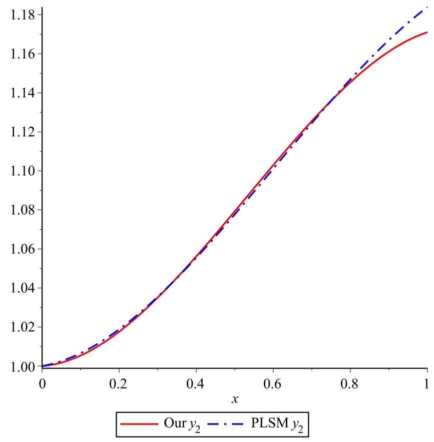

Figure 2. Comparison of y_2 and PLSM y_2 when $\alpha = \beta = 0.98$ for example 6.1

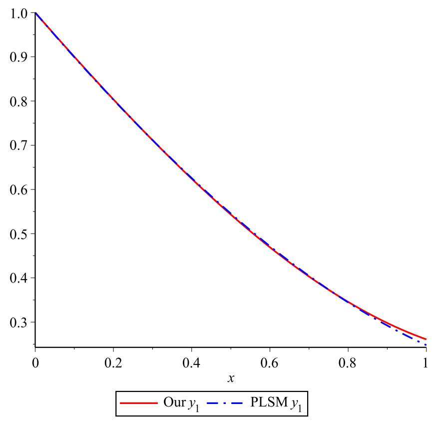

Figure 3. Comparison of y_1 and PLSM y_1 when $\alpha = \beta = 1$ for example 6.1

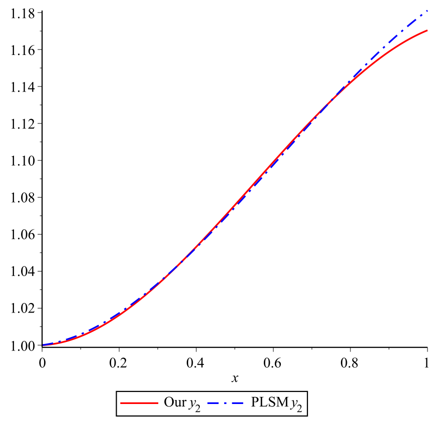

Figure 4. Comparison of y_2 and PLSM y_2 when $\alpha = \beta = 1$ for example 6.1

7. Conclusion

A general formulation for the Legendre wavelet operational matrix of fractional order derivative has been derived through wavelet-polynomial transformation, the procedure is easy to use and yet obtain a very good result. This matrix is used to approximate numerical solution of fractional order Brusselator system. Our numerical findings are compared with previous results and it illustrates the accuracy of the method.

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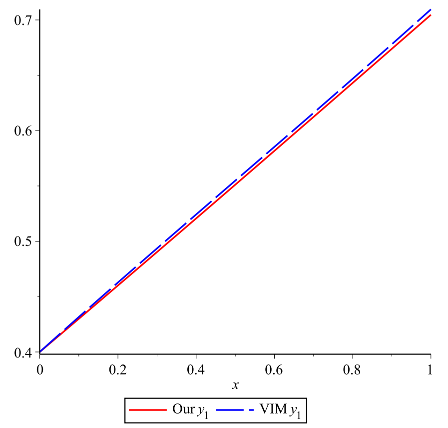

Figure 5. Comparison of y_1 and VIM y_1 when $\alpha = \beta = 0.98$ for example 6.2

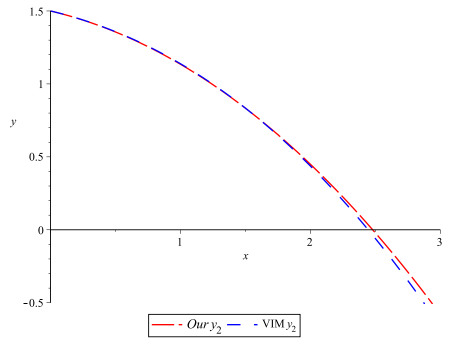

Figure 6. Comparison of y_2 and VIM y_2 when $\alpha = \beta = 0.98$ for example 6.2

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