Are numerical studies of long term dynamics conclusive: the case of the Hénon map

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Abstract. Numerical studies of the Hénon map show that there is an abundance of periodic windows close to classical parameter values. In this work properties of these periodic windows are studied. It is shown that in order to detect periodic windows in simulations one has to compute trajectories in a sufficient precision, which in many cases is much higher than the standard double precision. Moreover, it is shown that even if computations are carried out in a sufficient precision, one may need an extremely large number of iterations to observe convergence to a sink starting from random initial conditions, and hence the underlying steady state behavior may be practically undetectable using the standard trajectory monitoring based approach.

1. Introduction

Numerical simulations play an important role in studying dynamics of nonlinear systems. Simulations are used to compute long trajectories to study steady-state behaviour, to provide data for bifurcation diagrams, to compute Lyapunov exponents to classify attractors, etc. However, due to inherent properties of digital computers, results found by numerical simulations are almost never exact [1, 2]. Nevertheless, computer generated solutions are often accepted as true solutions.

In this work, we will show examples that simulation studies of nonlinear systems as simple as the Hénon map cannot, in general, provide conclusive answer to the question what is the true nature of the attractor existing for fixed parameter values. It follows that what we observe in many examples reported in the literature, and what is claimed to be a chaotic trajectory, might in fact be a transient to a periodic steady-state or an artifact caused by rounding errors.

2. Detection of periodic windows for the Hénon map

The *Hénon map* [3] is a two-parameter, invertible map of a plane into itself

$$h(x, y) = (1 + y - ax^2, bx), \tag{1}$$

displaying a wide array of dynamical behaviors as its parameters are varied. In [3], the map h is numerically studied with parameter values $(a, b) = (1.4, 0.3)$, referred to in the following as the *classical parameter values*, and it is claimed that "depending on the initial point, the sequence of points obtained by iteration of the mapping either diverges to infinity or tends to a strange attractor". For the classical parameter values the so-called *Hénon attractor* is observed in simulations.

Fig. 1(a) shows a bifurcation diagram for the Hénon map. The value of b is fixed ($b = 0.3$) and 10000 values of a in the range $a \in [0, 1.42]$ are selected. For each parameter value 10000 iterates are skipped

and the x variable for the next 50000 iterates is plotted. Period-doubling route to chaos and low-period windows in the chaotic range are clearly visible.
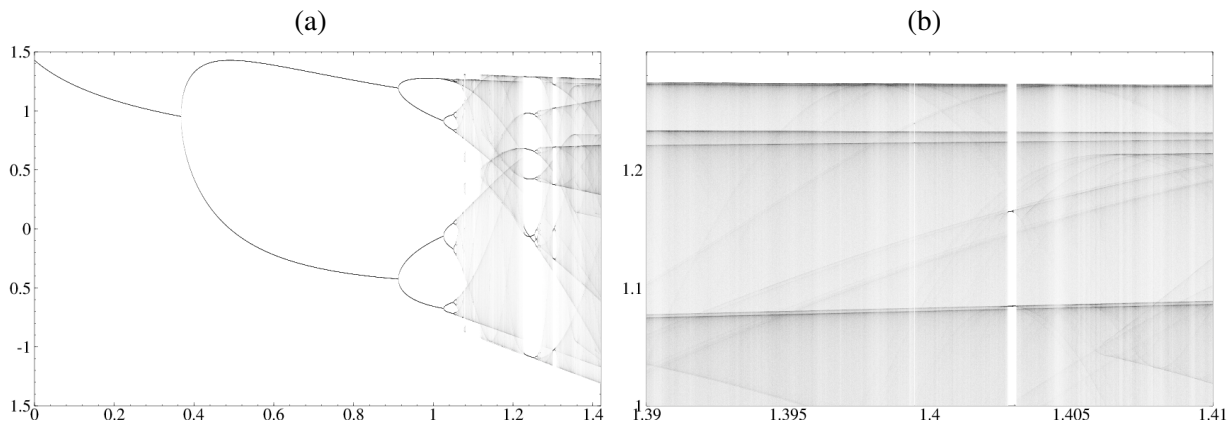

Figure 1. Bifurcation diagram for the Hénon map, $x \in [-1.5, 1.5]$ versus a; $b = 0.3$, (a) 10000 parameter values $a \in [0, 1.42]$, 10000 iterates skipped, 50000 iterates plotted, (b) 10000 parameter values $a \in [1.39, 1.41]$, 100000 iterates skipped, 50000 iterates plotted, $x \in [1.0, 1.3]$.

A bifurcation diagram for a smaller interval $a \in [1.39, 1.41]$ is shown in Fig. 1(b). The number of skipped iteration has been increased to 100000. One can see a single wide periodic window and a few narrower ones (not clearly visible). Important questions are how many periodic windows can be detected using selected grid of parameter values, and whether the number of skipped iterations in a given periodic window is sufficient to converge to the corresponding sink. In the remaining part of the paper we will discuss these problems in detail.

The most natural way to find periodic windows is to using the monitoring trajectory based approach. In this method, a number of iterates are computed in the hope that a trajectory reaches a steady-state. Next, we take the current iterate as the new initial point and check if the trajectory periodically returns very close to this point. This method has been used in [4] to find periodic windows in a small neighborhood of the classical parameter values. In an example search test 10^8 points in the parameter space filling uniformly the interval $[1.3999, 1.4001] \times 0.3$ have been considered. For each point 1000 random initial conditions have been selected and for each initial condition 10^7 iterations have been computed to detect stable periodic orbits. Note that the total number of iterations needed to carry out such a test is 10^{17}. In spite of very long computation time, only 461 periodic windows intersecting the region $Q_0 = [1.3999, 1.4001] \times [0.2999, 0.3001]$ have been found. This failure is caused by a number of factors. First, detection of periodic windows may require arithmetic precision higher than standard double-precision. Second, periodic windows may be too narrow to be detected with a chosen sampling of the parameter space. Third, the number of iterations to observe the convergence may be very large.

A more systematic approach to detect periodic windows has been proposed in [5]. It is based on finding a carefully selected set of periodic orbits existing for fixed parameter values, and then using continuation in the parameter space to find parameter values sustaining a sink. In this method, two close points in the parameter space are selected and for each of them the existence of periodic orbits corresponding to specific symbol sequences is verified using the Biham-Wenzel method [6]. Symbol sequences admissible for only one of the two considered points correspond to periodic windows intersecting the interval defined by these points. Positions of periodic windows are found using the continuation technique. We start at the endpoint where the orbit exist, modify parameter values moving towards the second endpoint where the orbit is not admissible. When moving along the interval in the parameter space, the position of the orbit is updated, eigenvalues of the corresponding Jacobian matrix are calculated, and stability conditions are verified. When a point in parameter space with a stable periodic behavior is found, the calculations are stopped with a candidate of a point belonging to the

periodic window. The existence of a stable periodic orbit is verified using the interval Newton method. For more details see [5]. Selection of symbol sequences to be considered for a given sequence length is based on results obtained for the lower-period windows. This step is necessary since considering all symbol sequences for larger periods is not feasible.

Using this method, many (tens of thousands) low-period windows very close to $(1.4, 0.3)$ have been found. These include several periodic windows within the distance 10^{-20} from the classical parameter values. The closest one is the period-115 window at the distance less than 7×10^{-22} from the point $(1.4, 0.3)$.

3. Convergence properties

In this section, we study properties of periodic windows intersecting a small neighborhood of the classical parameter values $(a, b) = (1.4, 0.3)$. Computation are carried out in the double-precision arithmetic or in multiple-precision arithmetic using the MPFR library [7]. Interface to multiple-precision interval arithmetic calculations is provided by the CAPD library [8].

Fig. 2 shows a trajectory of the Hénon map with parameters $a = 1.400000009849371$ and $b = 0.300000019143266$ for the initial point $(x_0, y_0) = (0, 0)$ computed in the standard double precision arithmetic. 4×10^{11} iterations are skipped in the hope that the steady-state has been reached, and the next 10000 points are plotted using red dots. One can see that the plot is visually indistinguishable from the well-known Hénon attractor. However, if we wait longer the trajectory converges to the period-31 stable periodic orbit plotted as blue circles in Fig. 2.
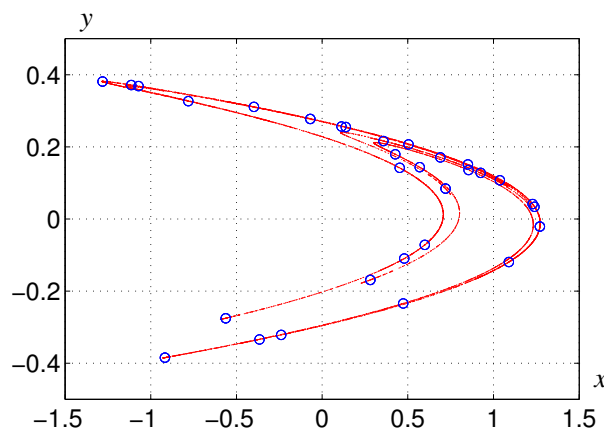

Figure 2. Trajectory of the Hénon map computed in double-precision, $a = 1.400000009849371$, $b = 0.300000019143266$, $(x_0, y_0) = (0, 0)$, 10000 points after skipping 4×10^{11} iterations are plotted using red dots, 10000 points after skipping 5×10^{11} iterations are plotted using blue circles.

The parameter values considered belong to one of the periodic windows detected using the symbolic dynamics based approach [5]. The existence of the stable periodic orbit has been rigorously confirmed using the interval Newton method [9, 10]. Therefore, we can be sure that the observed periodic behaviour is not a rounding error artifact. We have tested various initial conditions and confirmed in simulations that all trajectories converge to the same period-31 sink, which indicates that this sink is perhaps the only attractor existing for the considered parameter values. This example shows that finite pieces of trajectories, even if they are very long, are inconclusive as to the true nature of the steady-state of the system. It follows that we need to be very cautious drawing any conclusions from study of such finite trajectories. For example from the fact that the largest Lyapunov exponent computed along a transient trajectory is positive it does not follow that the steady state is chaotic. In the example considered, the maximum Lyapunov exponent becomes negative only after the steady-state is reached.

Let us assume that the considered parameter values (a, b) belong to a periodic window, i.e. there exist a stable periodic orbit for these parameter values. One can show that sufficiently close to the classical parameter values the Hénon map has infinitely many periodic orbits. This follows from the existence of nontrivial symbolic dynamics with positive topological entropy [11]. Most of these orbits are unstable.

It follows that even if parameter values belong to a periodic window and the sink is the only attractor for the system, it may take an arbitrarily large number of iterations to observe convergence to this sink. Given the iteration number n and any of the unstable periodic orbits one may choose an initial point sufficiently close to this unstable orbit so that only after more than n iterations a trajectory leaves a small neighborhood of the unstable periodic orbit and then it may converge to the sink. An interesting property of stable periodic orbits is the average convergence time starting from random initial conditions. If the average convergence time is large then we have little chance of finding the sink observing short trajectories. Let $n_{\mathrm{conv}}(\varrho)$ denote the number of iterations which are required to converge to the sink with probability $\varrho \in (0, 1)$ starting from random initial conditions. To approximate $n_{\mathrm{conv}}(\varrho)$, we first compute trajectories for N random initial points and record the corresponding convergence times $n_1, n_2, \ldots, n_N$. Next, we sort the convergence times $n_1 \leq n_2 \leq \cdots \leq n_N$ and find k such that $k \approx \varrho N$. n_k serves as an approximation of $n_{\mathrm{conv}}(\varrho)$. In the following, we will use $\varrho = 0.5$.

Later, it will be shown that it is difficult to get a reliable approximation of n_{conv} for parameter values close to the classical case. To investigate convergence properties we will use the notion of the immediate basin of attraction and its radius [4]. Let $A = (z_0, z_1, \ldots, z_{p-1})$ be a periodic attractor. We say that a point z belongs to the *immediate basin of attraction* $\mathrm{B}_\varepsilon(A)$ of the attractor A if its trajectory converges to the attractor and does not escape further than ε from it, i.e.

$$\mathrm{B}_\varepsilon(A) = \left\{ z \colon \|(h^n(z) - h^n(z_k))\| \leq \varepsilon \; \forall n \geq 0 \text{ and } \lim_{n \to \infty} \|(h^n(z) - h^n(z_k))\| = 0 \text{ for some } 0 \leq k < p \right\}. \quad (2)$$

We will use ε to be 1% of the diameter of the smallest ball enclosing the attractor. The condition $z \in \mathrm{B}_\varepsilon(A)$ can be verified numerically by computing the trajectory of z in arithmetic of sufficient precision and verifying whether conditions in (2) hold. Computing accurately the measure $\mu(\mathrm{B}_\varepsilon(A))$ of the immediate basin of attraction requires very fine sampling of the state space and is computationally expensive. It is easier to find connected components of $\mathrm{B}_\varepsilon(A)$, each containing one of the points z_k. In this way, the search is limited to grid points in the state space which are connected to points already identified as belonging to $\mathrm{B}_\varepsilon(A)$. To further reduce the computation time needed to find a lower bound of $\mu(\mathrm{B}_\varepsilon(A))$ we will use the notion of the *immediate basin radius of the attractor A* at the point z_k which is defined as

$$\mathrm{r}_\varepsilon(z_k) = \sup \{ r \colon \|z - z_k\| \leq r \Rightarrow z \in \mathrm{B}_\varepsilon(A) \}. \quad (3)$$

Computation of $\mathrm{r}_\varepsilon(z_k)$ is straightforward. For example, the bisection method can be used to obtain an accurate approximation of the largest r such that the condition $z \in \mathrm{B}_\varepsilon(A)$ is satisfied for test points in a ball of radius r centered at z_k.

The *minimum immediate basin radius of the attractor A* is defined as

$$\mathrm{r}_\varepsilon(A) = \min_{0 \leq k < p} \mathrm{r}_\varepsilon(z_k). \quad (4)$$

If the minimum immediate basin radius of a sink is smaller than the arithmetic precision used then trajectories will most likely escape from the sink even if computations are started at a stable periodic point; in general we will not be able to observe the sink in simulations (compare [1]). Provided that the computations are carried out in a sufficient precision, a trajectory initiated at a point closer to the sink than its immediate basin radius converges to the sink without leaving its small neighborhood; in simulations we will see a stable periodic behavior.

To illustrate the above definitions let us consider the point $(a, b) = (1.4028, 0.3)$ in the parameter space. This point belongs to a period-9 window. Position of one of the stable periodic points existing for $(a, b) = (1.4028, 0.3)$ is shown in Fig. 3 using the star symbol. The border of the connected component containing this point of the immediate basin of attraction of the periodic orbit is plotted in green. The immediate basin radius at this point is plotted as a red circle. Chaotic transient trajectories are plotted as blue dots.
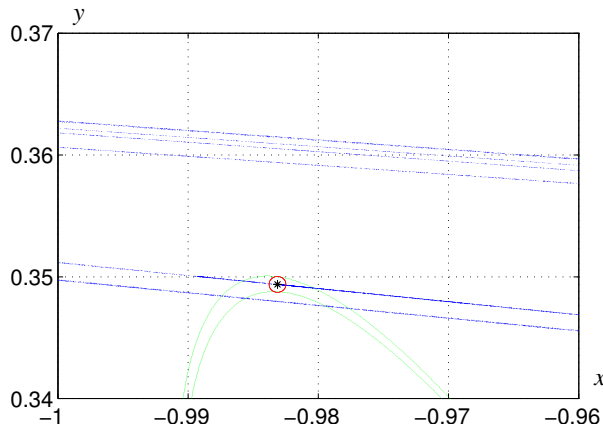
Figure 3. A stable period–9 point existing for $a = 1.4028$, $b = 0.3$ (the star symbol); chaotic transient trajectories (blue dots); border of the connected component of the immediate basin of attraction (green lines).

Convergence properties depend on the measure of the intersection of a chaotic set with the immediate basin. We will show that the measure of the immediate basin and the minimum immediate basin radius can be used to estimate the average time needed for a trajectory to converge to the sink. Let us first consider the case $b = 0$. When $b = 0$, the Hénon map reduces to $h(x) = 1 - ax^2$, which after a linear change of variable is equivalent to the logistic map $f(x) = ax(1 - x)$ (with different a). Let us first study convergence properties for a selected periodic window. For the logistic map each periodic window corresponds to a symbol sequence $s = (s_0, s_1, \ldots, s_{p-1})$, with $s_k \in \{-1, +1\}$, where the number of positive symbols is odd. In [12], it has been shown how to efficiently find the position of a periodic window for a given symbol sequence using the Newton method applied to find bifurcation points. Applying this procedure to the symbol sequence $s = (---+--+--+)$ gives the window's endpoints $(\underline{a}, \overline{a}) \approx (3.96776003545, 3.96776027772)$. Stability of the periodic orbit $(x_0, x_1, \ldots, x_{p-1})$ depends on the value of $\lambda = (f^p)'(x_0) = f'(x_{p-1}) \cdots f'(x_1)f'(x_0)$. If $|\lambda| < 1$ then the orbit is asymptotically stable. Fig. 4(a) shows λ versus a for the periodic window considered. The left and the right endpoints correspond to saddle-node and period-doubling bifurcation points with $\lambda = 1$ and $\lambda = -1$, respectively.
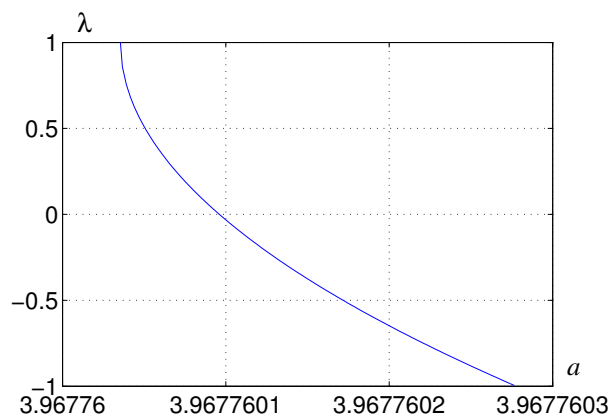

Figure 4. The derivative $\lambda = (f^p)'(x_0)$ across the periodic window of the logistic map corresponding to the symbol sequence $s = (---+--+--+)$.

For the logistic map the state space is one-dimensional and for a given stable periodic orbit $(x_0, x_1, \ldots, x_{p-1})$ it is relatively easy to find p connected components of the immediate basin of attraction of the sink containing points x_k. Let us denote by L the total length of these connected components. The total length L versus the parameter a across the selected periodic window is shown in Fig. 5(a). One can see that when the parameter approaches the saddle-node bifurcation point the total length L goes to zero. This is a consequence of the fact that at the saddle-node point the immediate basin of attraction disappears. Somewhat unexpected is the monotonicity of L across the window. The maximum is observed at the right window's endpoint where the period-doubling bifurcation occurs. One could expect
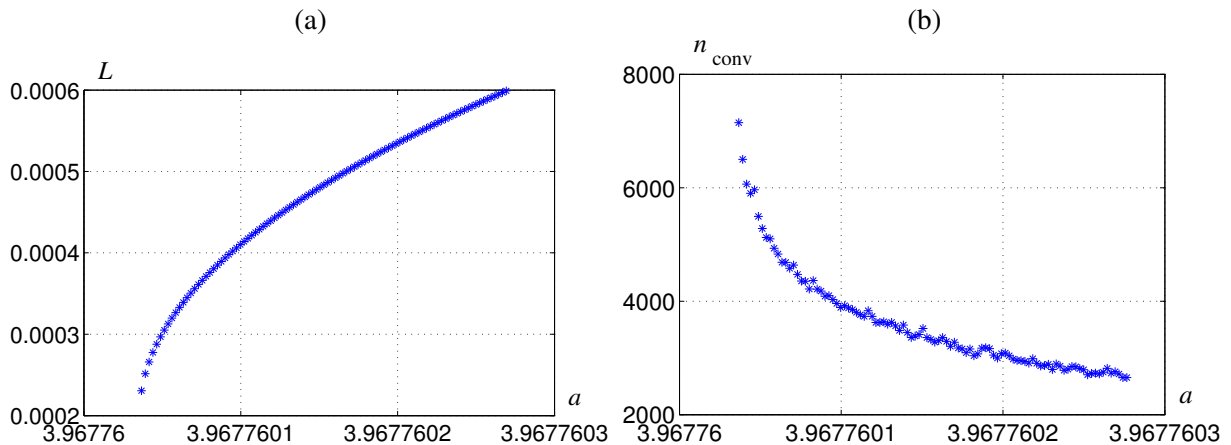
Figure 5. Convergence properties in the periodic window of the logistic map corresponding to the symbol sequence $s = (---+--+--+)$, (a) measure of subset of $B_\varepsilon(A)$ composed of connected components containing points x_k, (b) average number of iterations $n_{conv}(0.5)$ to converge to the sink.

that when λ is small in magnitude, then the immediate basin of attraction should be large. However, this is not true. The total length L for superstable orbit (with $\lambda = 0$) is smaller than for larger a when λ is nonzero. Average convergence times across the periodic window are shown in Fig. 5(b). These results have been obtained by selecting 100 equidistant parameter values belonging to the periodic window. For each parameter value 10000 trajectories starting from random initial conditions have been computed, convergence time was recorded, and the average convergence time has been found. Generally, the average convergence time decreases when a increases. It escapes to infinity when parameter approaches the saddle-node bifurcation point (left window's endpoint).

Note that there is a strong correlation between L and n_{conv} (compare Fig. 5(a) and (b)). Below, we derive an approximate formula for the relation between L and n_{conv}. We will assume that for parameter values considered there exist a single periodic attractor (for the logistic map it is known that there exists at most one attractor), and that transient trajectories visit the invariant set $\mathcal{A}$ containing all periodic orbits according to the uniform distribution. The true distribution is not flat, but we will see that anyway the resulting formula produces acceptable results. Let us assume that the measure of $\mathcal{A}$ is D. The probability that at least one out of n points selected randomly from the set of measure D belongs to a subset of measure L is $\varrho = 1 - (1 - L/D)^n$. Solving for n with the assumption $L \ll D$ yields

$$n = \frac{\log(1 - \varrho)}{\log(1 - \frac{L}{D})} \approx \frac{D \log(1 - \varrho)^{-1}}{L}. \tag{5}$$

When a is close to 4 the measure of the set $\mathcal{A}$ is close to one. Assuming that $\varrho = 0.5$ yields

$$n \approx 0.693 \cdot L^{-1}. \tag{6}$$

Thus, we can expect that the average number n_{conv} of iterations to converge to a sink whose immediate basin has measure L is inversely proportional to L. To verify this relationship, in Fig. 6 we plot n_{conv} versus L for eleven windows of the logistic map with periods $p \in [9, 26]$. The corresponding window width are in the range $[10^{-15}, 10^{-6}]$. One can see that the plot is composed of short intervals. Each interval corresponds to parameter values belonging to one of the periodic windows considered. For smaller L the plot is more noisy. This is related to the fact that for small window width (close to 10^{-15}) the minimum immediate basin radius of the periodic attractor is also very small. In consequence, the effects caused by finite precision of the double precision arithmetic become visible. Data points in Fig. 6

lie close to a straight line

$$\log n_{\mathrm{conv}} = c_1 \log L + c_0, \tag{7}$$

with $c_1 = -0.982$, $c_0 = 0.7642$, which is plotted in Fig. 6 in green. One can see that c_1 is close to -1, which confirms the approximate formula (6). We may use (7) to extrapolate what is the average convergence time n_{conv} for very small L when direct computation of n_{conv} is not feasible. For example when $L = 10^{-30}$ we expect that the average convergence time is around 6×10^{29}. Using one of the fastest computers available nowadays with the performance close to 50 petaflops (5×10^{16} floating point operations per second) with the assumption that three floating point operations are needed to compute one iteration of the logistic map, one would need on average more than one million years of computations to see the convergence.
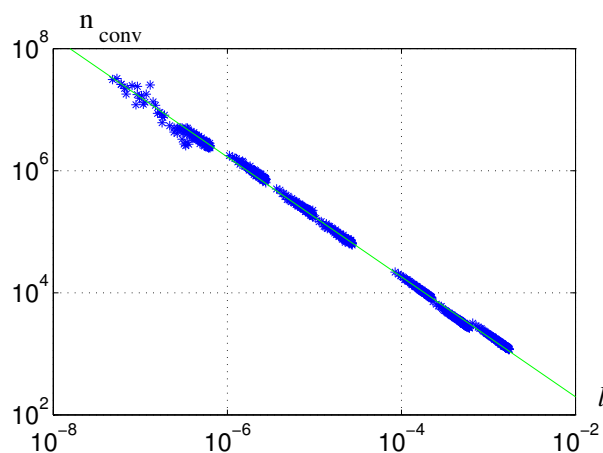

Figure 6. The average number n_{conv} of iterations needed to observe convergence versus the measure L of the immediate basin for several periodic windows of the logistic map.

Let us now consider periodic windows of the Hénon map close to classical parameter values. Fig. 7(a) shows the relationship between the minimum immediate basin radius r_ε and the precision (in bits) needed to see the sink in simulations. This number is computed as the minimum number of bits of multi-precision arithmetic used such that the computer generated trajectory initiated exactly (with the given precision) at the sink position does not escape from a small neighborhood of the sink. One can see that the plot is close to the line: $-\log_2 \mathrm{r}_\varepsilon$ plotted in Fig. 7(a) as a red solid line. This phenomenon can be easily explained in the following way. If the arithmetic precision does not allow us to represent points belonging to a neighborhood of a periodic point with the radius r_ε then most likely a trajectory of a periodic point represented in this precision will escape from this little neighborhood.

To study convergence properties, in Fig. 7(b) we plot the relationship between the minimum immediate basin radius r_ε and average number n_{conv} of iterations to converge to a sink with probability $\varrho = 0.5$. This data was based on computing trajectories starting at 10000 randomly chosen initial conditions for several periodic windows intersecting the interval $(a, b) \in [1.39, 1.41] \times \{0.3\}$. For small r_ε the number of initial points was decreased due to very long computation times. Fitting data points shown in Fig. 7(b) to the model

$$\log n_{\mathrm{conv}} = c_1 \log \mathrm{r}_\varepsilon + c_0. \tag{8}$$

gives $c_1 = -0.7589$, $c_0 = 0.0566$. This model is plotted in Fig. 7(b) as a green line. Note that c_1 is significantly smaller than 1 in absolute value in opposite to the logistic map case (compare (7)). This is related to the fact that the chaotic set for the Hénon map is not one-dimensional, but has a complex fractal structure. The results obtained indicate that we may use formula (8) to extrapolate the average return time for periodic orbits with very small minimum immediate basin radius.
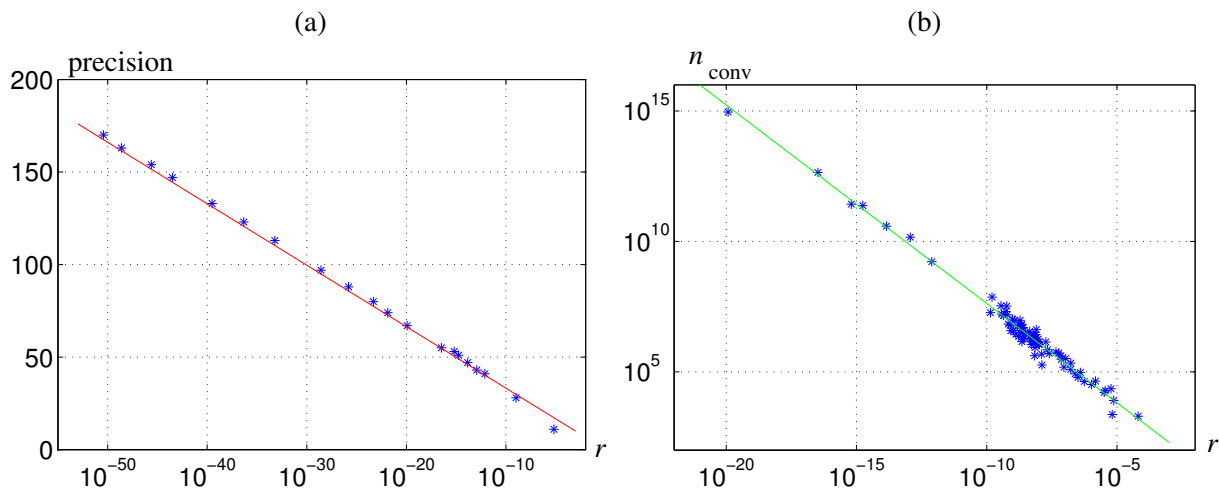
Figure 7. Properties of periodic windows of the Hénon map close to classical parameter values; (a) precision needed to observe the sink versus the minimum immediate basin radius r_ε, (b) The average number n_{conv} of iterations needed to observe convergence versus r_ε.

Let us now apply the results presented above to an example periodic window. Let us consider the closest periodic window reported in [5] with the distance from the point $(1.4, 0.3)$ smaller than 7×10^{-22} and the period equal to 115. Computer arithmetic precision necessary to detect this sink is 170 bits. We verified that if the computations are carried out in lower precision then the trajectory escapes from the sink even if it is started exactly (with the precision used) at the sink position. The minimum immediate basin radius for this sink is $r_\varepsilon \approx 3.72 \times 10^{-51}$. Using formula (8) we obtain $n_{conv} \approx 2 \times 10^{38}$. We expect that so many iterations are needed to observe with probability $\varrho = 0.5$ the convergence to the corresponding periodic orbit starting from random initial conditions. Assuming that 10^{15} iterations can be computed in one second (which is far beyond capabilities of even fastest supercomputers) one would need more than 10^{16} years to reach $n_{conv} = 2 \times 10^{38}$. Since it is impractical to compute so many iterations, we conclude that this and similar periodic windows are practically undetectable using simulation based approach even if computations are carried out with the required precision.

4. Conclusions

We have shown that in order to detect in simulation periodic windows lying very close to classical parameter values of the Hénon map one has to use arithmetic precision with a sufficient number of bits, which is usually much larger than the standard 53 bits precision. It follows that the true nature of the corresponding attractors cannot be verified using standard double-precision computations. We have estimated the average number of iterations needed to converge to corresponding sinks starting from random initial conditions. We have shown that this number can be so large (above 10^{30}) even for sinks of moderate length (with period 100 or so) that these sinks are practically undetectable in simulations even if computations are carried out with the required precision. Although in our study we restricted ourselves to the case of a simple two-dimensional map we believe that the results carry over to many other discrete and continuous dynamical systems.

We conclude with the statement that most of the simulation based results concerning chaotic behaviour of nonlinear systems published in the literature are irrelevant to the question what is the true character of the underlying attractor — no conclusions in this matter can be obtained by studying finite pieces of trajectories.

Acknowledgments

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