

Nonequilibrium phase transition in directed small-world-Voronoi-Delaunay random lattices

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Abstract. On directed small-world-Voronoi-Delaunay random lattices in two dimensions with quenched connectivity disorder we study the critical properties of the dynamics evolution of public opinion in social influence networks using a simple spin-like model. The system is treated by applying Monte Carlo simulations. We show that directed links on these random lattices may lead to phase diagram with first- and second-order social phase transitions out of equilibrium. Keywords: Opinion dynamics, Sociophysics, Majority vote, Nonequilibrium.

1. Introduction

Some collective or social behavior how workers decide whether or not to go on strike [1] or how a school of fish aligns into one direction for swimming [2] can be explained using the Ising model [3, 4]. The Ising model has been used to explain other behaviors as consensus formation [5], a fragmentation into many different opinions, or a leadership effect when a few people change the opinion of lots of others as also in the social impact model [6]. Other behaviors where opinions follow the majority of the neighbourhood and similar to it have been successfully explained using Ising-type model as the voter model of Liggett [7] and Schelling [5]. Most of these cited models and others can be found out in reference [8]. Beyond the Ising model many other models have been employed to study the social behavior of a set of individual or agents located on the nodes or sites of complex networks [9]. Some of these social systems also have been studied on complex networks of interacting agents. In most of these systems the interactions between the agents are directed, i.e., the links between agents act only in one direction (outwards or inwards). We investigate Sánchez-López-Rodríguez (SLR) [10] social model in two dimensions evolving on *directed* small-world-Voronoi-Delaunay (SWVD) random lattices in two dimensions with quenched connectivity disorder. The SRL model describes very well the dynamics of public opinion in social influence networks. In the present work we show that directed networks may lead to a highly nontrivial phase diagram including a social first- and second-order phase transitions out of equilibrium.

2. SLR Model on *directed* SWVD random lattice sizes

We consider a simple spin-like model (SLR), on *directed* SWVD random lattice by a set of spin variables S_i taking the values ± 1 situated on every site i of a *directed* SWVD random lattice with $N = L \times L$ sites, where L is the side of the square cluster. In this random lattice, similar to Sánchez



et al. [10], we start from a two-dimensional SWVD random lattice consisting of sites linked to their k (where $3 < k < 20$ and different for each site of network) nearest neighbors by both outgoing and incoming links. Then, with probability p , we replace nearest neighbor outgoing links by new outgoing links to different sites chosen at random. After repeating this process for every link, we are left with a network with a density p of SWVD *directed* links. Therefore, with this procedure every site will have k outgoing links and varying (random) number of incoming links. Then, the spins or actors are then placed at the network sites. Then, any actor is connected by k outgoing links to other actors or mates and can be in one of two possible S_i states taking the values ± 1 . Depending on the state of their mates and neighborhood, an actor may change its state according to a majority rule (ferromagnetic). In order to implement this, we introduce the payoff function:

$$G_i = 2S_i \sum_{j=1}^k S_j, \quad (1)$$

where the sum is carried out over the k mates of actor i . The external noise or social temperature T is included to allow some degree of randomness in the time evolution. Then, for a given value of the external temperature, the update of the model is then performed as follows: At each time step, an actor (network site) is randomly chosen and its corresponding G_i is computed according to Eq. (1).

- (1): If $G_i < 0$, $S_i \rightarrow -S_i$. The actor i is opposing the mates neighborhood majority and the change of its actual state is accepted.
- (2): If $G_i > 0$, the change of its current state is accepted with the probability

$$P_i = e^{-G_i/T}, \quad (2)$$

wich depends on temperature, i.e, an unfavorable change. Therefore, this model is a nonequilibrium one, since detailed balance is not satisfied.

The simulations have been performed on different *directed* SWVD random lattice sizes comprising a number $N = 250, 500, 1000, 2000, 4000, 8000$ and 16000 of sites. For each system size quenched averages over the connectivity disorder are approximated by averaging over $R = 100$ independent realizations. For each simulation we have started with a uniform configuration of spins. We ran 4×10^5 Monte Carlo steps (MCS) per spin with 2×10^5 configurations discarded to reach steady state. We do not see any significant change by increasing the numbers R and MCS. So, for the sake of saving computer time, the present values seem to give reasonable results for our simulation.

From the magnetization per spin, $m = \sum_i S_i/N$, we can derive the average the magnetization, the susceptibility, and the fourth-order magnetic cumulant,

$$M(T) = [\langle |m| \rangle]_{\text{av}}, \quad (3)$$

$$\chi(T) = \frac{N[\langle m^2 \rangle_{\text{av}} - \langle |m| \rangle_{\text{av}}^2]}{T}, \quad (4)$$

$$U_4(T) = 1 - \left[\frac{\langle m^4 \rangle_{\text{av}}}{3 \langle |m| \rangle_{\text{av}}^2} \right]. \quad (5)$$

In the above equations $\langle \dots \rangle$ stands for thermodynamic averages and $[\dots]_{\text{av}}$ for averages over different realizations.

In order to verify the order of the transition of this model, we apply finite-size scaling (FSS). In the case of a first-order phase transition, we then expect, for large system sizes, an asymptotic FSS behavior of the form [11, 12],

$$\chi_{\text{max}} = a_\chi + b_\chi N + c_\chi/N + \dots \quad (6)$$

for the maximum of the susceptibility, $\chi_{\max}$, where a_χ , b_χ and c_χ are constants.

Otherwise, in the case of a second-order phase transition, we then expect, for large system sizes, an asymptotic FSS behavior of the form

$$[\langle |m| \rangle]_{\text{av}} = L^{-\beta/\nu} f_m(x) [1 + \dots], \quad (7)$$

$$\chi = L^{\gamma/\nu} f_\chi(x) [1 + \dots], \quad (8)$$

The $1/\nu$, β/ν , and γ/ν are the usual critical exponents ratio, and $f_i(x)$ are FSS functions with

$$x = (T - T_c) L^{1/\nu} \quad (9)$$

being the scaling variable. The dots in the brackets $[1 + \dots]$ indicate corrections-to-scaling terms. We calculated the error bars from the fluctuations among the different realizations. Note that these errors contain both, the average thermodynamic error for a given realization and the theoretical variance for infinitely accurate thermodynamic averages which are caused by the variation of the quenched, random geometry of the lattices.

The correlation length exponent (ν) can be estimated from $T_c(L) = T_c + bL^{-1/\nu}$, where $T_c(L)$ is the pseudo-critical temperature for the lattice size L , T_c is the critical temperature in the thermodynamic limit, and b is a non-universal constant.

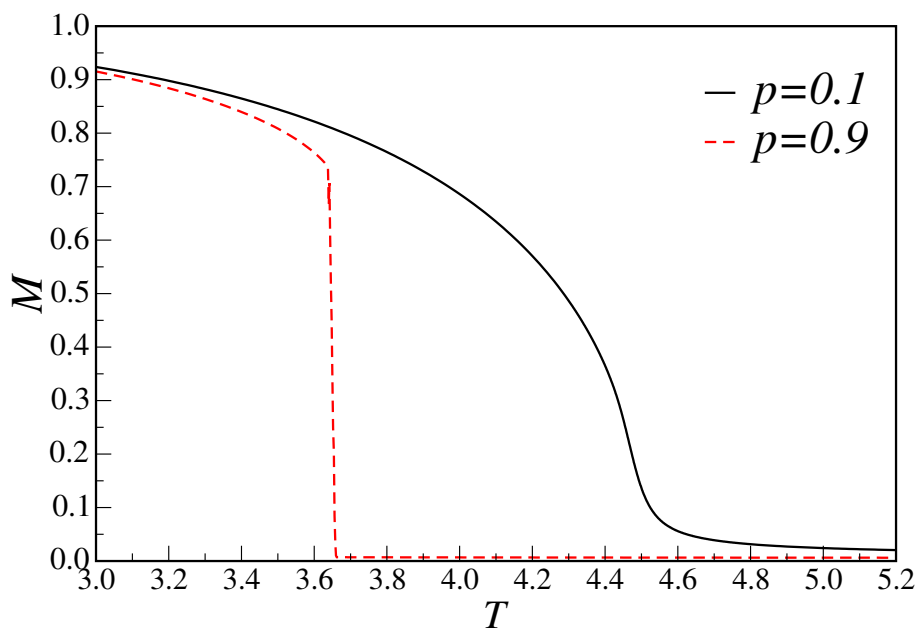


Figure 1. (color online) Magnetization as a function of T for $N = 16000$ and rewiring probabilities $p = 0.1$ and 0.9 .

3. Results and Discussion

In Figure 1 we show the behavior of the magnetization versus temperature for $N = 16000$ and rewiring probabilities $p = 0.1$ and 0.9 . One can see a typical behavior of a second and first-order phase transition for $p = 0.1$ and 0.9 , respectively. In order to estimate the critical temperature we calculate the fourth-order Binder cumulant given by eq. (5). It is well known that these quantities are asymptotically independent of the system size and should intercept at the critical

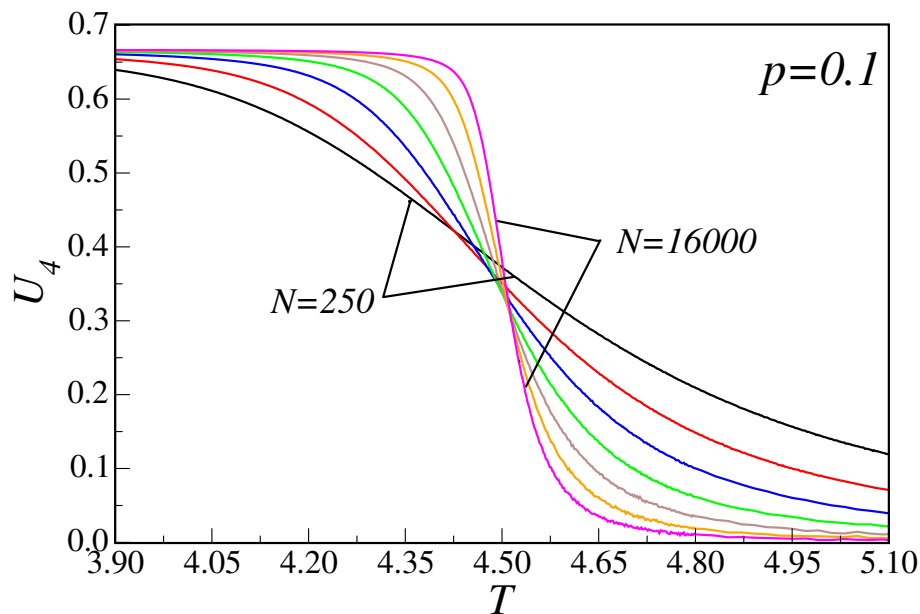


Figure 2. (color online) Plot of the fourth-order Binder cumulant as a function of T for various lattice sizes with $N = 250, 500, 1000, 2000, 4000, 8000$, and 16000 and rewiring probability $p = 0.1$.

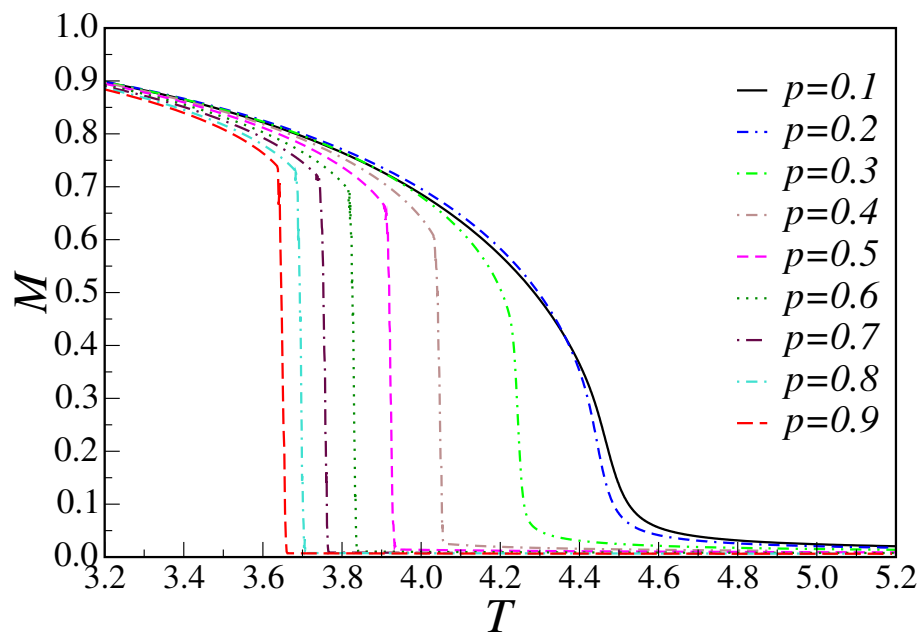


Figure 3. (color online) The same as Figure 1 for $N = 16000$ and various rewiring probabilities with $p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, and 0.9 .

temperature [13]. In Figure 2 the fourth-order Binder cumulant is shown as a function of the T for several lattice sizes for the rewiring probability $p = 0.1$. Taking the largest lattices we have $T_c = 4.508(3)$ and $U_4^* = 0.342(3)$. One can see that U_4^* is different from the universal

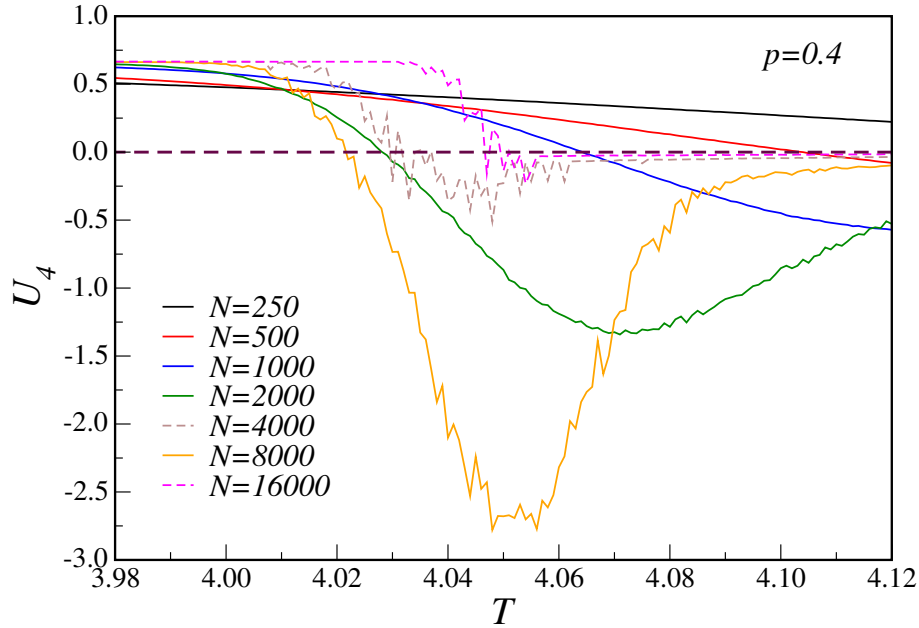


Figure 4. (color online) Same plot as Figure 2 for the fourth-order Binder cumulant as a function of T , but now for $p = 0.4$.

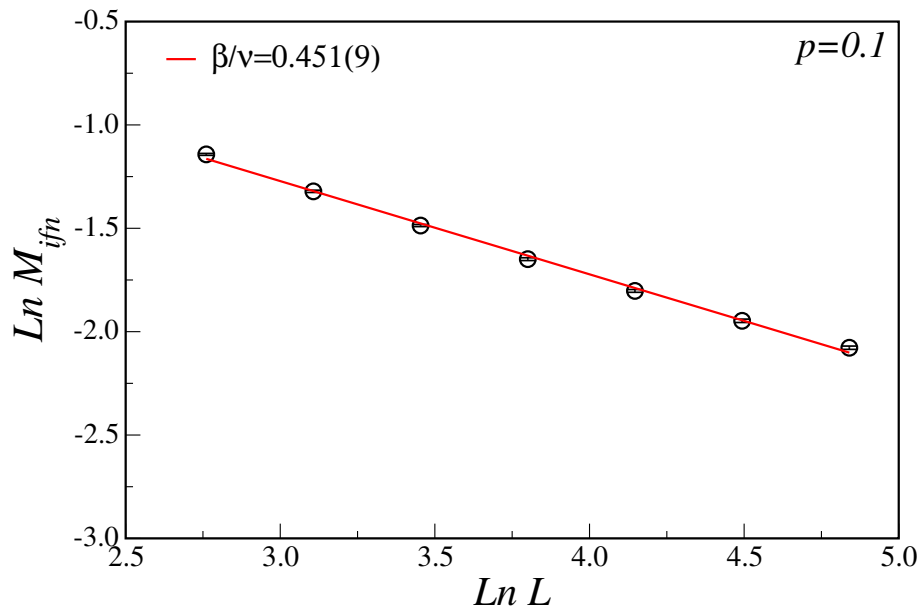


Figure 5. (color online) Plot of the logarithm of the modulus of the magnetization at the inflection point as a function of the logarithm of L . The solid line is the best linear fit.

value $U_4^* \sim 0.61$ valid for Ising models both on regular $d = 2$ lattices and on Voronoi-Delaunay random lattices [14, 15, 16].

In Figure 3 the magnetization as a function of T for $N = 16000$ and various rewiring probabilities with $p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, and 0.9 . From the above results

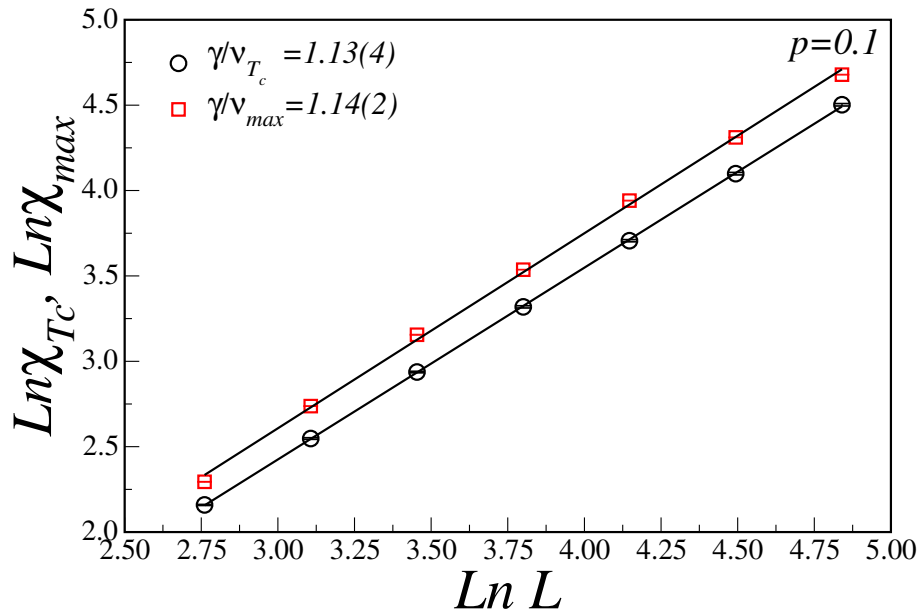


Figure 6. (color online) Log-log plot of the susceptibility χ at T_c and the susceptibility maxima $\chi_{\max}$ as a function of the logarithm of L . The solid lines are the best linear fits.

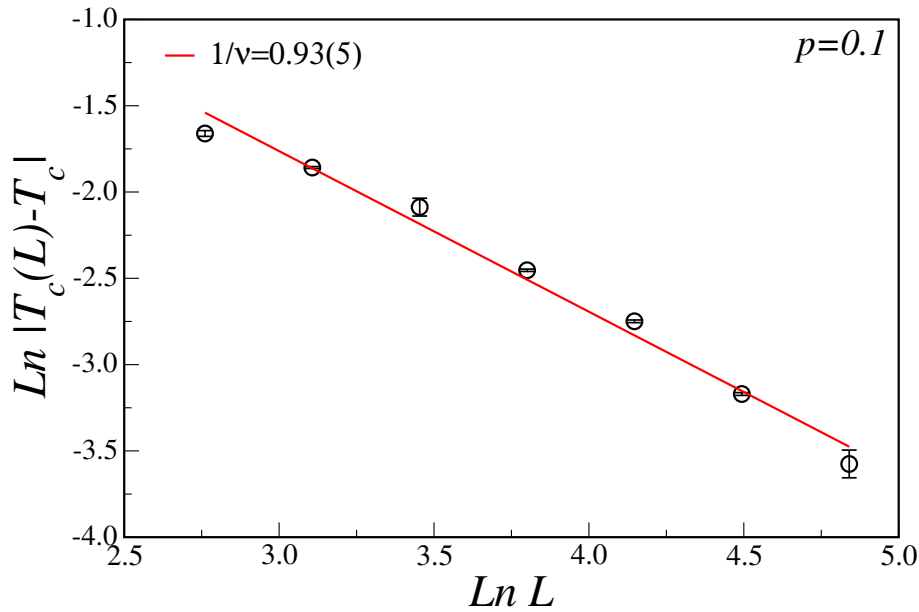


Figure 7. (color online) Plot the $\ln [T_c(L) - T_c]$ versus $\ln L$ for $p = 0.1$. The solid line is the best linear fit.

we can estimate the location of the tricritical point separating the first-order from the second-order transition. As for $p = 0.3$ one still has a second-order behavior and for $p = 0.4$ one has already a first-order transition, we could say that the tricritical point is located at $p = 0.35(5)$. Accordingly, the corresponding tricritical temperature is given by $T_t \approx 4.18$.

Figure 4 shows the fourth-order Binder cumulant as a function of T for $p = 0.4$ where one

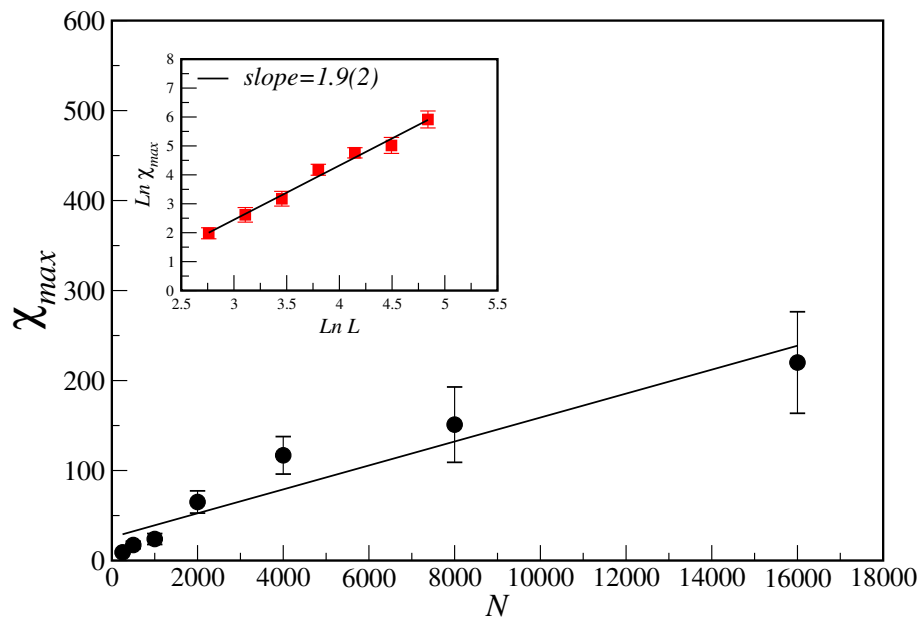


Figure 8. (color online) Plot of the susceptibility maxima $\chi_{\max}$ as a function of N for $p = 0.9$. The solid lines are the best linear fit. Note that in the inset we plot $\text{Ln}(\chi_{\max})$ versus $\text{Ln}(L)$, where $L = \sqrt{N}$.

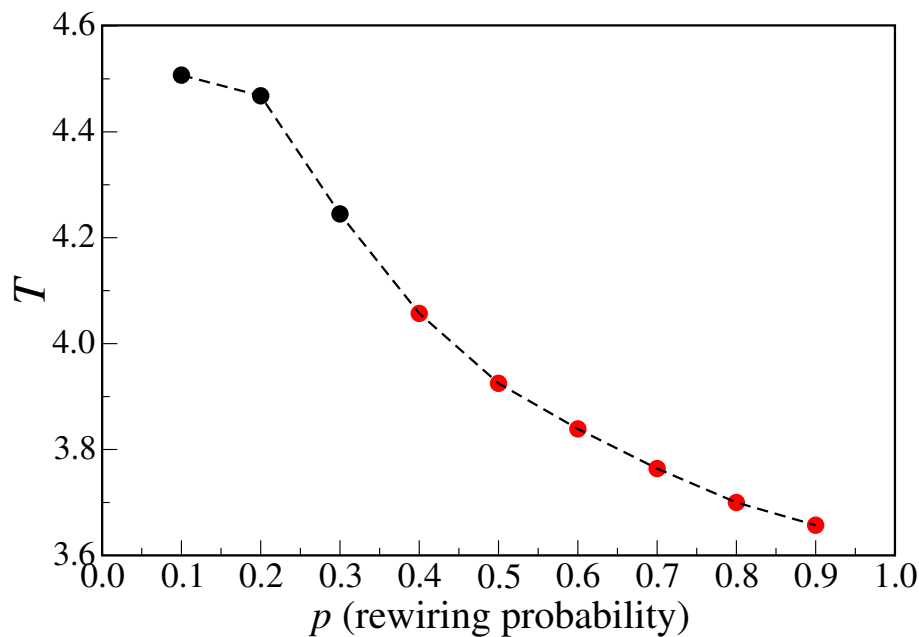


Figure 9. (color online) Plot of the transition temperature versus p for $p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 . Note that for $p > 0.35$ the transition is of the first-order type.

has a first-order transition.

We have also computed the modulus of the magnetization at the inflection point and the magnetic susceptibility at T_c . The logarithm of these quantities as a function of the logarithm

of L are presented in Figures 5 and 6, respectively. A linear fit of these data gives β/ν from the magnetization and γ/ν from the susceptibility. In addition, we plotted in Figure 6 the logarithm of the maximum value of the susceptibility $\chi_{\max}$ as a function of $\ln L$ for $p = 0.1$. One can also see that the exponents ratios $\beta/\nu = 0.451(9)$ and $\gamma/\nu = 1.13(4)$ are different from $\beta/\nu = 0.53(2)$ and $1/\nu = 1.06(5)$ of the SRL model and also of $\beta/\nu = 0.125$ and $\gamma/\nu = 1.75$ obtained for a regular $d = 2$ lattice, but they obey the hyper-scaling relation (within the error bars)

$$2\frac{\beta}{\nu} + \frac{\gamma}{\nu} = d, \quad (10)$$

where $d = 2$ for equilibrium models.

In Figure 7 it is shown a plot of $\ln [T_c(L) - T_c]$ as a function of $\ln L$ for $p = 0.1$. A linear fit of these data gives $1/\nu = 0.93(5)$. Figure 8 displays the maximum value of the susceptibility $\chi_{\max}$ as a function of N for $p = 0.9$. For a first-order phase transition we expect, for large system sizes, an asymptotic FSS behavior of the form given by eq. 6 [11, 12]. One can see the first-order nature of the transition. This fact is also illustrated in the inset of the figure where we have a log-log plot of the physical quantities as function of the lattice size.

Regarding the inset of Fig. 8, we have additionally done a linear fit using relations (6) with $c_\chi = 0$ and allowing the lattice dimension exponent vary as $N = L^d$ in those equations. We obtain, in this case, an exponent $slope = 1.9(2)$ for $p = 0.9$, which is indeed close to $d = 2$, as expected.

The phase diagram so obtained is depicted in Fig. 9 in the temperature T versus rewiring probability p . For $p \leq 0.35(5)$ we have a second-order phase transition, otherwise it is a first-order phase transition, indicating that there exists a nonequilibrium tricritical point at $p_c = 0.35(5)$.

4. Conclusion

In summary, the SRL social model on *directed* SWVD random lattice presents a second-order phase transition for $p \leq 0.35(5)$, otherwise it is a first-order phase transition with a nonequilibrium tricritical point at $p_c = 0.35(5)$. For $p = 0.1$ we find out the exponents ratios $\beta/\nu = 0.451(9)$, $\gamma/\nu = 1.13(4)$ and $1/\nu = 0.93(5)$ that are different for SRL social model on square lattice, where $\beta/\nu = 0.53(2)$ and $1/\nu = 1.06(5)$. Our results are also different of Ising model 2d, where $\beta/\nu = 0.125$, $\gamma/\nu = 1.75$ and $\nu = 1$. Therefore, the SRL social model is not robust and the results depend on the topology of the system.

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