

## Resonance regions of extended Mathieu equation

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**Abstract.** One of the mechanisms of energy transfer between degrees of freedom of dusty plasma system is based on parametric resonance. Initial stage of this process can be described by equation similar to Mathieu equation. Such equation is studied by analytical and numerical approach. The numerical solution of the extended Mathieu equation is obtained for a wide range of parameter values. Boundaries of resonance regions, growth rates of amplitudes and times of onset are obtained. The energy transfer between the degrees of freedom of dusty plasma system can occur over a wide range of frequencies.

### 1. Introduction

Dusty plasma is an ionized gas containing micrometer-sized charged dust particles. In a gas-discharge plasma such particles obtain high negative charge and can have abnormally high temperatures, even higher than electron and ion temperatures [1,2]. Mean kinetic energy of dust particles vertical and horizontal motion may be significantly different [3,4]. Phenomenon of energy transfer between different degrees of freedom of a dusty plasma system, which causes such heating, are of great interest. One of such mechanisms is based on parametric resonance and can be described by an extended Mathieu equation [3,4]:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2(1 + h \cos \omega_p t)x = \eta(t), \quad (1)$$

where  $\lambda$  is a friction coefficient,  $h$  is an amplitude of modulation,  $\omega_0$  is an eigenfrequency of system,  $\omega_p$  is a frequency of parameter and  $\eta(t)$  is a stochastic force with zero mean value.

Mathieu equation [5] is used to describe dust particles motion in various articles [6,7]. In [6,7] the equation is derived by assuming, that the dust particle charge fluctuates harmonically. In this article extended Mathieu equation is derived by considering features of the near-electrode sheath in gas-discharge plasma and random fluctuations of the dust particle charge [3,4]. The parameters of equation (1) in such case have values significantly different from the ones considered in [6,7].

Classical Mathieu equation is studied for  $h \ll 1$ ,  $\lambda \ll 1$  and  $|\omega_p - 2\omega_0/n| \ll \omega_0$ , approximations of third and lower orders of accuracy is obtained in [8]. This range of parameters is too small for dusty plasma, since there can be significant friction and values of  $h$  may be above 3.

Connection of energy transfer between different degrees of freedom in dusty plasma and extended Mathieu equation is discussed in the second section. Analytical solution of the extended Mathieu



equation is derived in the third section. Method of numerical solution is presented in the fourth section and the results of numerical simulation are presented and discussed in the fifth section.

## 2. Energy transfer between different degrees of freedom

The motion of dust particles is described by model presented in [3]. Since  $x$  and  $y$  axes are symmetrical in such system, only fluctuations along  $x$  and  $z$  axes are considered:

$$\begin{cases} m\ddot{x} = F^{inter} + F^{trap} + F^{fr}, \\ m\ddot{z} = F^{inter} + F^{fr} + F^{grav} + F^{el}, \end{cases} \quad (2)$$

where  $F^{inter}$  is the force of interaction between dust particles defined by Yukawa potential,  $F^{fr}$  is the friction force,  $F^{grav}$  is the gravity force,  $F^{el}$  is electrostatic force, which balances gravity force, and  $F^{trap}$  is the force of potential trap, which prevents negatively charged particles from scattering horizontally.

The amplitude of oscillations is small, comparing to inter-particle distance before the resonance starts, so the Taylor series expansion can be used:

$$\begin{cases} \ddot{x} = -a_1x + a_2xz + a_3x^3 + a_4xz^2 + \dots - 2\lambda\dot{x}, \\ \ddot{z} = -b_1z + b_2z^2 + b_3z^3 + b_4z^2x + \dots - 2\lambda\dot{z} + g\delta q(t). \end{cases} \quad (3)$$

First and fourth terms is much larger than the others in the conditions of standard laboratory dusty plasma experiment, so:

$$\ddot{x} \approx -(a_1x - a_4xz^2) - 2\lambda\dot{x}. \quad (4)$$

Substitution of one harmonic of vertical motion  $z \approx A_z \cos(\omega_z t)$  in equation (4) leads to extended Mathieu equation (1), where  $\omega_0^2 = a_1 - a_4A_z^2/2 + \dots$ ,  $\omega_p = 2\omega_z$  and  $h = -a_4A_z^2/\omega_0^2$ .

Since resonance leads to growth of the amplitude, impact of other terms in (3) becomes greater. So equation (1) can't be used to describe late stages of the energy transfer between degrees of freedom but it can be used to describe the first stages of energy transfer.

## 3. Analytical solution

Approximations of boundaries of the resonance regions can be derived analytically acting by analogy with [8] and using averaging over an ensemble of distributions of  $\eta(t)$ . This approach works only for  $h \ll 1$ ,  $|\varepsilon| = |\omega_p - 2\omega_0/n| \ll \omega_0$ , where  $n$  is number of the resonance region.

The first resonance region boundaries for  $\lambda = 0$  are obtained with fourth-order accuracy by a method similar to one in [8].

$$-\omega_0\varepsilon \pm \frac{\omega_0^2 h}{2} = 0, \quad (9)$$

$$-\omega_0\varepsilon - \frac{\varepsilon^2}{4} \pm \frac{\omega_0^2 h}{2} + \frac{\omega_0^2 h^2}{32} = 0, \quad (10)$$

$$-\omega_0\varepsilon - \frac{\varepsilon^2}{4} \pm \frac{\omega_0^2 h}{2} + \frac{\omega_0^2 h^2}{32} - \frac{9\omega_0 h^2 \varepsilon}{2^8} = 0, \quad (11)$$

$$-\omega_0\varepsilon - \frac{\varepsilon^2}{4} \pm \frac{\omega_0^2 h}{2} + \frac{\omega_0^2 h^2}{32} - \frac{9\omega_0 h^2 \varepsilon}{2^8} + \frac{h^4}{3 \cdot 2^{15}} + \frac{9\varepsilon^2 h^2}{2^{13}} = 0. \quad (12)$$

And next three equations are approximations of second, fourth and sixth orders of accuracy for second resonance region derived for the same conditions.

$$\begin{cases} -2\omega_0\varepsilon + \frac{\omega_0^2 h^2}{12} = 0, \\ -2\omega_0\varepsilon - \frac{5\omega_0^2 h^2}{12} = 0; \end{cases} \quad (13)$$

$$\begin{cases} -2\omega_0\varepsilon + \frac{\omega_0^2 h^2}{12} - \varepsilon^2 - \frac{2\omega_0\varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2^7} = 0, \\ -2\omega_0\varepsilon - \frac{5\omega_0^2 h^2}{12} - \varepsilon^2 - \frac{2\omega_0\varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2^7} = 0; \end{cases} \quad (14)$$

$$\begin{cases} -2\omega_0\varepsilon + \frac{\omega_0^2 h^2}{12} - \varepsilon^2 - \frac{2\omega_0\varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2^7} - \frac{\varepsilon^2 h^2}{9} - \frac{\omega_0\varepsilon h^4}{2^{10}} + \frac{h^6}{3 \cdot 5 \cdot 2^{12}} = 0, \\ -2\omega_0\varepsilon - \frac{5\omega_0^2 h^2}{12} - \varepsilon^2 - \frac{2\omega_0\varepsilon h^2}{9} + \frac{\omega_0^2 h^4}{9 \cdot 2^7} - \frac{\varepsilon^2 h^2}{9} - \frac{\omega_0\varepsilon h^4}{2^{10}} + \frac{h^6}{3 \cdot 5 \cdot 2^{12}} = 0. \end{cases} \quad (15)$$

Approximations of third and fourth orders of accuracy of boundaries of the first resonance region are almost the same even for  $h > 1$ . Equations (14) and (15) are almost the same as well. These approximations are closer to data obtained numerically than approximations of lower orders of accuracy. However none of this approximations are close to data obtained numerically for  $h > 1$ .

It is usually assumed that the friction causes the amplitude to decay as  $e^{-\lambda t}$  [8]. Such assumption leads to

$$\left( \frac{h\omega_0}{2} \right)^2 - \varepsilon^2 - 4\lambda^2 = 0. \quad (16)$$

This approximation is in serious disagreement with numerical solution. This is due to the fact that this approach takes into account only terms of zero-order of accuracy with  $\lambda$ . So this approach is applicable only for  $\lambda \ll 1$  and for solution of the first order of accuracy.

The other way to describe resonance region in the presence of friction is solving equation (1) by an analogy with the case of zero friction without any additional assumptions. Such approach gives results closer to data obtained numerically. It also explains the shift of the bottom point of resonance region in terms of amplitude of modulation  $h$ . For first resonance region in approximation of the first order of accuracy this approach gives

$$\frac{\omega_0^4 h^2}{4} - \omega_0^2 \varepsilon^2 - 4\lambda^2 \left( \omega_0 + \frac{\varepsilon}{2} \right)^2 = 0 \quad (17)$$

instead of (16).

In equation (16)  $h^{\min} = 4\lambda/\omega_0$  when  $\varepsilon = 0$ , in equation (17)  $h$  is minimum for

$$\varepsilon_{fr} = -\frac{2\omega_0\lambda^2}{\omega_0^2 + \lambda^2} \quad (18)$$

and equals

$$h_{fr}^{\min} = \frac{2(\varepsilon^2 \lambda^2 + 4\varepsilon \lambda^2 \omega_0 + \varepsilon^2 \omega_0^2 + 4\lambda^2 \omega_0^2)^{1/2}}{\omega_0^2}. \quad (19)$$

Comparison of the boundaries specified by equations (16) and (17) and data obtained numerically is presented on figure 1.

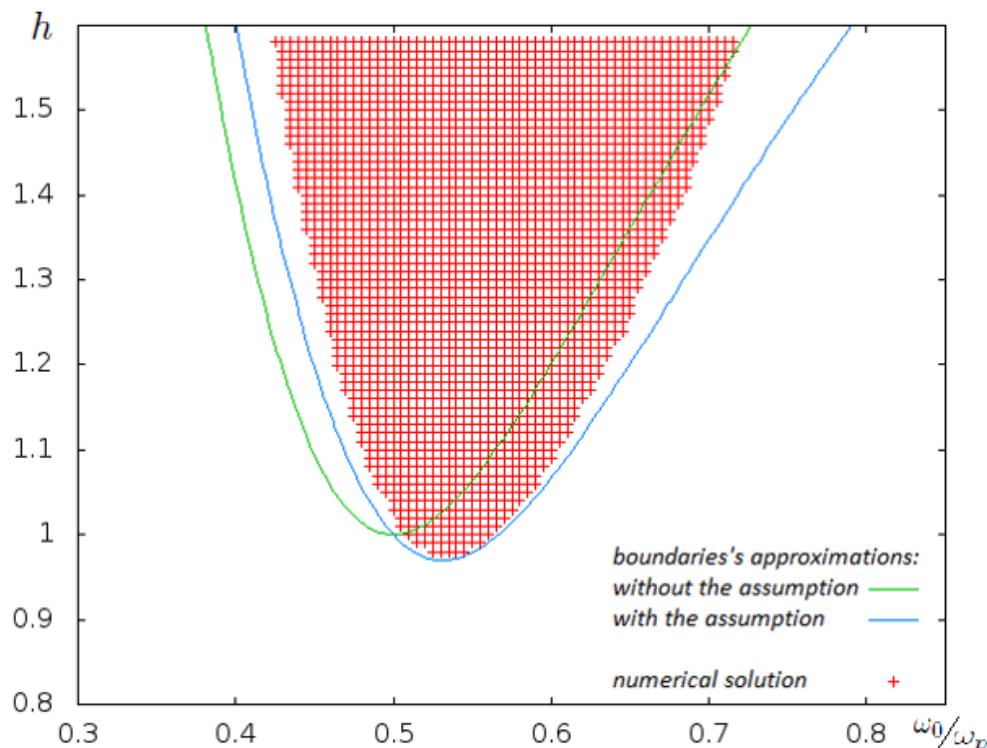


Figure 1. Boundaries of the resonance region obtained analytically with the assumption of exponential decay (blue line) and without this assumption (green line) and the resonance region obtained numerically (red crosses).

#### 4. Numerical method

In case of numerical detection of resonance regions, growth rates of the amplitude and the time of onset (to be explained below) equation (1) is solved by numerical integration for each fixed set of parameters. The velocity Verlet algorithm is used. Initial values are  $x(0) = 0$  and  $\dot{x}(0) = 1$  and aren't affecting the results. Time step is  $\Delta t = 10^{-5}$  sec.

Criterion of the resonance occurrence is exponential growth of the amplitude. Moment of the amplitude value from which the growth starts is considered to be the time of onset. Thus the time of onset is obtained with an accuracy of quarter period.

#### 5. Results of the numerical simulation

Resonance regions for equations (1) is found for  $h < 4$  and  $\omega_0/\omega_p < 4$ . Resonance regions for fixed  $\omega_0 = 50$  and  $\lambda = 10$  are shown on figure 2 and ones for fixed  $\omega_p = 70$  and  $\lambda = 10$  is shown on figure 3.

Figure 2 shows that resonance regions are very close to each other for  $h > 2$ . So resonance occurs at almost any value of  $\omega_0/\omega_p > 0.35$ . Since this result is obtained for fixed  $\omega_0$  (corresponding to horizontal oscillations), it shows that fixed harmonic of horizontal oscillations is heated by a wide spectrum of vertical oscillations, not only those which is close to  $2\omega_0/n$ . Figure 3 shows the same result for fixed  $\omega_p$ . So each harmonic of vertical oscillations warms up a wide spectrum of horizontal oscillations with frequencies above  $0.35\omega_p$ .

A wide spectrum of dust particles oscillations is also observed in a standard laboratory dusty plasma experiment. The possibility of participation of a wide range of oscillation frequencies in the resonant

energy transfer between the degrees of freedom, allows to explain a series of experiments [9] and makes this result very important for the description of the energy transfer in dusty plasmas.

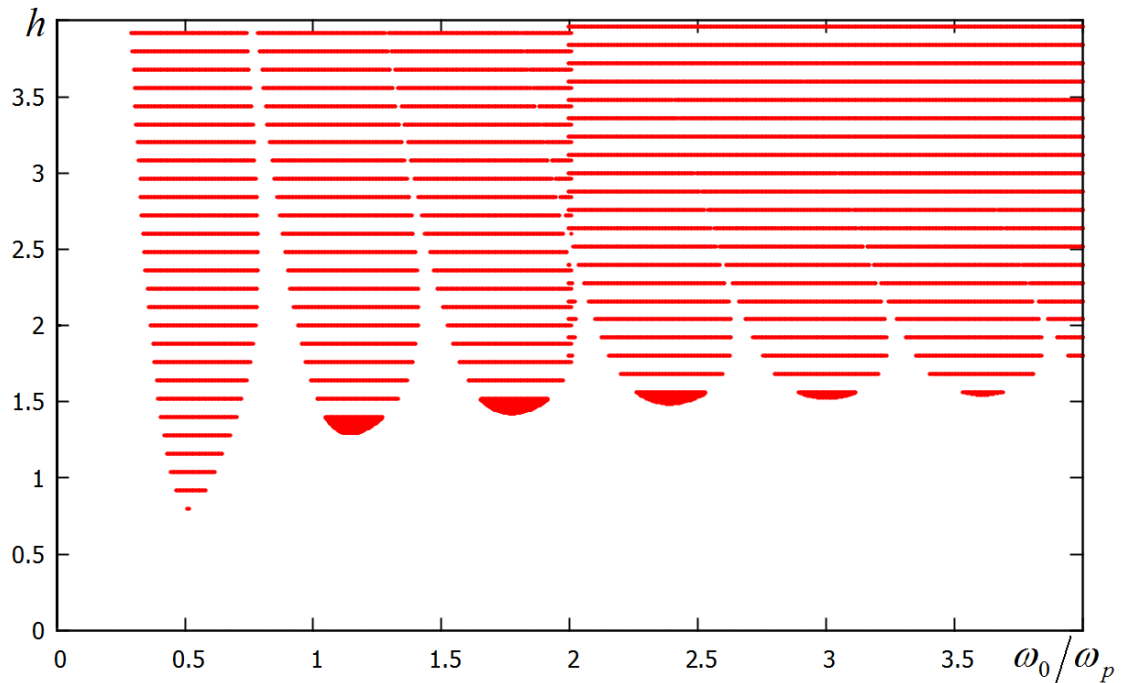


Figure 2. Resonance regions obtained numerically for  $\omega_0 = 50$  and  $\lambda = 10$ .

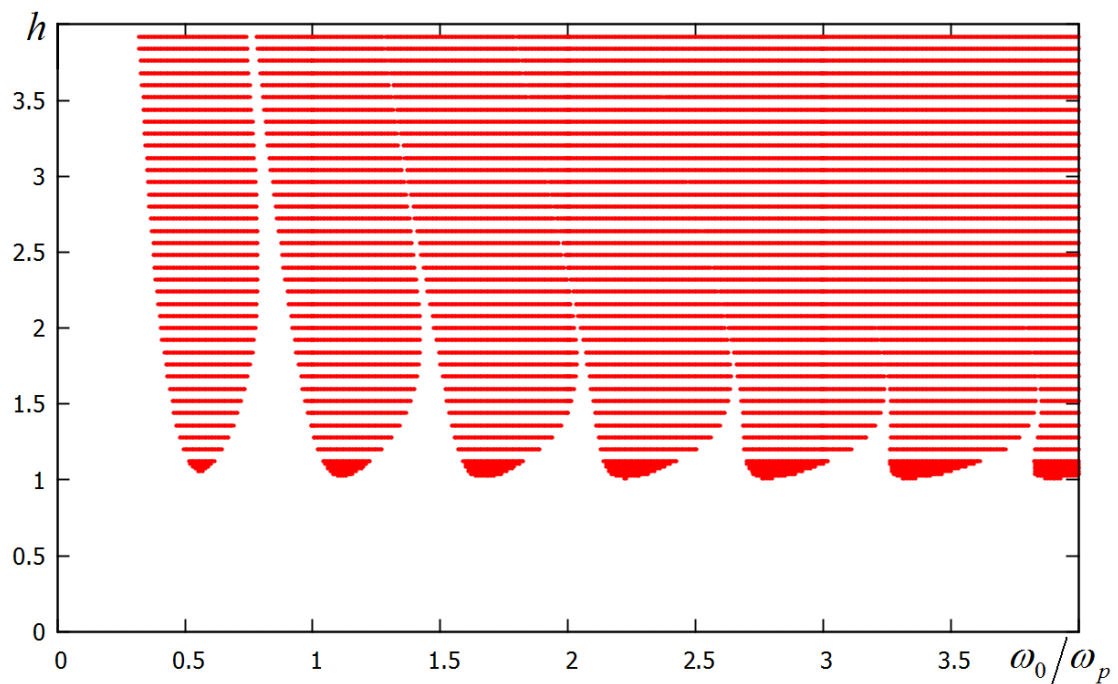


Figure 3. Resonance regions obtained numerically for  $\omega_p = 70$  и  $\lambda = 10$ .

Stochastic force  $\eta(t)$  isn't affecting previous results, however it is affecting the time of onset. The time of onset is stochastic and has normal distribution (figure 4) because of the stochastic force.

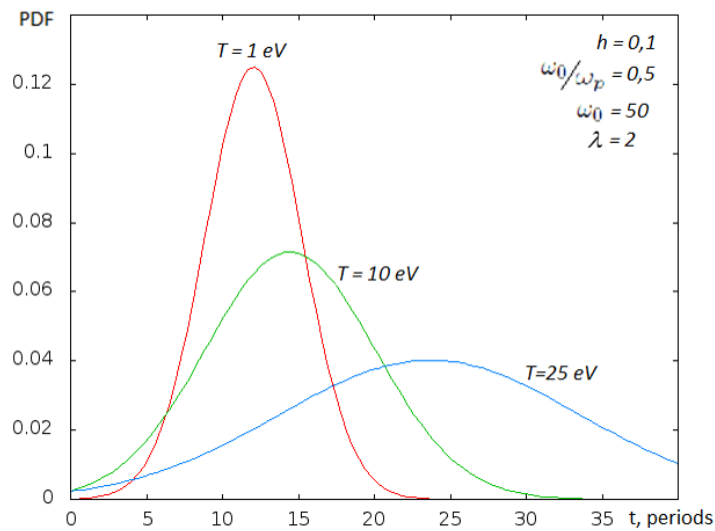


Figure 4. Probability density function of the time of onset obtained numerically for various mean kinetic energies  $T$  of the dust particle.

The more amplitude of stochastic force (corresponding to mean value of kinetic energy of dust particle) the more mean value and dispersion of the time of onset. However even for large values of the kinetic energy (25eV) the time of onset hardly can be over 40 periods of oscillations.

## 6. Conclusions

Connection between energy transfer between degrees of freedom in dusty plasma and extended Mathieu equation is shown. Approximations of boundaries of the resonance regions of various orders of accuracy are derived analytically. The shift of the point in which resonance occurs with a minimum value of  $h$  is shown by simulation and described analytically. Resonance regions, growth rates of the amplitude and the times of onset are obtained using numerical simulation. It is shown that the resonance regions obtained this way are close to each other for  $h > 2$ . This shows that resonance can occur at almost any values of  $\omega_0/\omega_p > 0.35$ . And since spectral peaks of vertical motion in dusty plasma are wide, each harmonic of vertical oscillations are exciting a wide spectrum of horizontal oscillations and single harmonic of horizontal oscillations are excited by a wide spectrum of vertical oscillations. Stochastic force shifts the timing of the resonance onset upward and increases the width of the onset time distribution.

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