International Conference on Computer Simulation in Physics and Beyond 2015 IOP Publishing
Journal of Physics: Conference Series **681** (2016) 012024 doi:10.1088/1742-6596/681/1/012024

On Behavior of Stochastic Synchronization Models

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Abstract. We consider N-component synchronization models defined in terms of stochastic particle systems with special interaction. For general (nonsymmetric) Markov models we discuss phenomenon of the long time stochastic synchronization. We study behavior of the system in different limit situations related to appropriate changes of variables and scalings. For $N = 2$ limit distributions are found explicitly.

1. Model

Synchronization models studied in the present paper are motivated by some computer science applications (parallel computations [1], wireless sensor networks [2, 3] etc.). These models surprisingly have a lot in common with interacting particle systems in physics. Recall the definition of N-component stochastic synchronization system. Let $x_j \in \mathbb{R}^d$ represent the state of a component j, $j = \overline{1, N}$. The dynamics of the system is a continuous time stochastic process $x(t) = (x_1(t), \ldots, x_N(t))$, $t \geq 0$, which evolution is composed of two parts called respectively a free dynamics and a synchronizing interaction. The interaction between components is happened only at some random epochs $0 = T_0 < T_1 < T_2 < \cdots$ and has a form of instantaneous synchronizing jumps $x(T_n) \mapsto x(T_n + 0)$ to be precised later on. The free dynamics means that between successive epochs of interaction the components $x_j(t)$ evolve independently. Namely there is a family of mutually independent stochastic processes $x_j^\circ(t)$, $j = \overline{1, N}$, such that

$$x_j(t) - x_j(T_{n-1} + 0) = x_j^\circ(t) - x_j^\circ(T_{n-1} + 0), \qquad T_{n-1} < t \leq T_n, \qquad n \geq 1, \quad j = \overline{1, N}.$$

Before going into further details, let us reformulate the model in terms of particle systems. Instead of components we will speak of N particles with coordinates x_j.

To introduce a pair-wise synchronizing interaction between particles it is convenient to assume that particles can share information about each other by sending and receiving messages. Imagine, for example, that at time T_q the particle $j_1^{(q)}$ sends a message to some another particle $j_2^{(q)}$. For breavity notation let us write $j_1 = j_1^{(q)}$ and $j_2 = j_2^{(q)}$. The message contains information on the current value of x_{j_1}. Assume that messages reach their destinations instantly. After receiving the message from j_1 the particle j_2 ajusts its coordinate to the value x_{j_1}: $x_{j_2}(T_q + 0) = x_{j_1}(T_q)$. This is the only jump in the system at the time T_q: $x_j(T_q + 0) = x_j(T_q)$

for all $j \neq j_2$. Define a (linear) map $M_{j_1 j_2} : \left(\mathbb{R}^d\right)^N \to \left(\mathbb{R}^d\right)^N$ such that x and $y = M_{j_1 j_2} x$ differ only in the j_2-component:

$$y_{j_2} = x_{j_1}, \quad y_k = x_k, \quad k \neq j_2, \qquad x = (x_1, \ldots, x_N), \quad y = (y_1, \ldots, y_N).$$

Thus the synchronizing jump can be expressed as $x(T_q + 0) = M_{j_1 j_2}\, x(T_q)$.

The key idea of this synchronization model is that the sequence $\mathcal{H} = \left\{ \left(T_n;\, j_1^{(n)}, j_2^{(n)} \right) \right\}_{n \geq 1}$ should be random. Consider point processes

$$\boldsymbol{T}^{j \to k} := \{T_n : (T_n;\, j, k) \in \mathcal{H}\}, \quad (j \neq k), \qquad \boldsymbol{T}^{j \to} := \bigcup\nolimits_k \boldsymbol{T}^{j \to k}, \qquad \boldsymbol{T} := \bigcup\nolimits_{j,k} \boldsymbol{T}^{j \to k}$$

where, for brevity, we have put $\boldsymbol{T}^{j \to j} = \varnothing$. Evidently, $\boldsymbol{T}$ is the sequence of synchronization epochs $\{T_n\}_{n \geq 1}$. For any interval $[t_1, t_2] \subset \mathbb{R}_+$ and any route (j_1, j_2) denote

$$\nu_{j_1 j_2}([t_1, t_2]) := \text{card}\,(\{(T_n;\, j_1, j_2) : T_n \in [a, b]\}) = \text{card}\,\left(\boldsymbol{T}^{j_1 \to j_2} \cap [a, b]\right),$$

the random number of messages sent from j_1 to j_2 during the time interval $[a, b]$. Reasonable assumptions covering a wide field of applications are the following.

M1) Message flows $\boldsymbol{T}^{j \to}$ generated by different particles j are independent.

M2) The particle j sends messages to another particle k with the rate $\beta_{jk} \geq 0$, i.e., for any j and $t, h \geq 0$

$$\begin{aligned} \mathsf{P}\,(\nu_{jk}([t, t+h]) = 1) &= \beta_{jk} h + o(h), \qquad k \neq j, \\ \mathsf{P}\left(\sum_{k:\, k \neq j} \nu_{jk}([t, t+h]) = 0 \right) &= 1 - \beta_j h + o(h) \quad \text{as } h \to 0 \qquad \text{where } \beta_j := \sum_{k:\, k \neq j} \beta_{jk}. \end{aligned}$$

(If $\beta_{jk} = 0$ then $\boldsymbol{T}^{j \to k} = \varnothing$, that is, there is no message flows on the route (j, k).)

M3) For any j the message flow $\boldsymbol{T}^{j \to}$ generated by the particle j is nonzero, i.e., $\sum\limits_{k:\, k \neq j} \beta_{jk} > 0$.

With the set of parameters $\{\beta_{jk},\, k \neq j\}$ we associate a directed graph $\mathcal{G}$ with vertices $1, \ldots, N$ and directed arcs (j, k) such that $\beta_{jk} > 0$.

M4) The graph $\mathcal{G}$ is strongly connected, i.e., any pair of vertices can be connected by a directed path.

Assumptions M1–M2 implies that

$$\begin{aligned} \mathsf{P}\left(\text{card}\,(\boldsymbol{T} \cap [a, b]) \geq 2\right) &= \mathsf{P}\left(\sum_{j,k:\, k \neq j} \nu_{jk}([t, t+h]) \geq 2 \right) = o(h), \\ \mathsf{P}\left(\text{card}\,(\boldsymbol{T} \cap [a, b]) = 0\right) &= 1 - \beta h + o(h) \qquad \text{where } \beta := \sum_j \beta_j = \sum_{j \neq k} \beta_{jk}. \end{aligned}$$

Put $X_0 = 0$. A point process $\boldsymbol{X} = \{X_1, X_2, \ldots, X_n, \ldots\} \subset (0, +\infty)$ is said to be a Poisson process of rate $\gamma > 0$ if $\{X_n - X_{n-1}\}_{n \geq 1}$ is a sequence of independent and identically distributed exponential random variables with the mean γ^{-1} that is $\mathsf{P}\left(X_n - X_{n-1} > s\right) = \exp(-\gamma s)$ for $s \geq 0$. Differences $X_n - X_{n-1}$ are called inter-event intervals. For further details the interested reader is referred to [4].

From Assumptions M1–M3 and the classic point processes theory it follows that

- any message flow $\boldsymbol{T}^{j_1 \to j_2}$ is a Poisson process of rate $\beta_{j_1 j_2}$;
- the flows $\boldsymbol{T}^{j \to k}$ corresponding to different routes (j, k) are mutually independent;

- the point processes $\boldsymbol{T}^{j\to}$ and $\boldsymbol{T} = \{T_n\}_{n\geq 1}$ are Poisson with rates β_j and β respectively.

So the mechanism of *synchronizing interaction* is very transparent: at epochs of the message flow $\boldsymbol{T}^{j_1\to j_2}$ we apply the map $M_{j_1 j_2}$.

Before specifying the free dynamics we fix some notation. Given a random vector $Z \in \mathbb{R}^d$, let P_Z and ψ_Z denote its probability distribution and characteristic function respectively, i.e.,

$$P_Z(A) := \mathsf{P}\,(Z \in A)\,, \quad A \subset \mathbb{R}^d, \qquad \psi_Z(a) := \int_{\mathbb{R}^d} e^{\mathrm{i}\,a\cdot q}\, P_Z(dq) = \mathsf{E}\,\exp\left(\mathrm{i}\,a\cdot Z\,\right), \quad a \in \mathbb{R}^d,$$

where E is the expectation with respect to P and $a \cdot q$ denotes the Euclidian scalar product of two vectors a and q in $\mathbb{R}^d$. It is readily seen that the function ψ_Z is the Fourrier transform of the probability measure P_Z.

Recall [5, 6] that an $\mathbb{R}^d$-valued stochastic process $z(t)$, $t \geq 0$, is called a Lévy process if 1) it has independent and stationary increments; 2) it is stochastically continuous; 3) $z(0) = 0$. A remarkable property of Lévy processes is a representation for $\psi_{z(t)}$,

$$\psi_{z(t)}(a) = \psi_{z(s+t)-z(s)}(a) = \exp\left(t\phi_z(a)\right), \qquad t \geq 0, \quad s > 0,$$

where ϕ_z has the Lévy-Khintchine form [5]. The function $\phi_z : \mathbb{R}^d \to \mathbb{C}$ is called a characteristic exponent of the Lévy process $z = z(t)$. It is easy to see that $\phi_z(a)$ is continuous, $\phi_z(0) = 0$ and $\mathrm{Re}\,\phi_z(a) \leq 0$. Our main assumption about the *free dynamics* is the following.

F) The free dynamics $x_j^\circ(t)$, $j = \overline{1, N}$, are independent Lévy processes. For further convenience, the characteristic exponent for the particle j will be denoted as $\phi_{x_j^\circ}(a) = -\eta_j(a)$, i.e.,

$$\mathsf{E}\,\exp\left(\mathrm{i}\,\lambda_j \cdot x_j^\circ(t)\,\right) = \exp\left(-t\,\eta_j(\lambda_j)\right), \qquad \lambda_j \in \mathbb{R}^d.$$

Many interesting examples of one-particle dynamics satisfy Assumption F. In particular, if $x_j^\circ(t)$ is a Brownian motion with diffusion matrix Σ and drift vector $v_j \in \mathbb{R}^d$ then $\eta_j(\lambda_j) = -\mathrm{i}\,v_j \cdot \lambda_j + \frac{1}{2}\,(\Sigma\Sigma^\top \lambda_j) \cdot \lambda_j$. Sometimes we will need stronger assumptions.

F0) Assumption F holds and the functions η_j are such that $\mathrm{Re}\,\eta_j(\lambda_j) > 0$ for all $\lambda_j \neq 0$.

FS$\boldsymbol{\alpha}$) Assumption F0 holds and each $x_j^\circ(t)$ is an α-stable Lévy process. In other words, the characteristic exponents $-\eta_j(\lambda_j)$, $j = \overline{1, N}$, satisfy the following condition: there exists $\alpha \in (0, 2]$ such that

$$\eta_j(s\lambda_j) = s^\alpha \eta_j(\lambda_j) \qquad \forall s > 0, \quad \lambda_j \in \mathbb{R}^d. \tag{1}$$

The number α is called the stability index. The classical examples of stable Lévy dynamics are Brownian motions with zero drift ($\eta_j(\lambda_j) = \frac{1}{2}\,(\Sigma\Sigma^\top \lambda_j) \cdot \lambda_j$, $\alpha = 2$). More information about stable laws can be found in [7, 8].

The only assumption about *initial configuration* $x(0)$ is that it is independent of the free dynamics and all message flows $\boldsymbol{T}^{j\to k}$.

Under above assumptions $x(t)$ is a Markov process. Synchronization model is called symmetric if $\eta_j = \eta$ for all j and $\beta_{jk} = c/(N-1)$ for all $j \neq k$.

2. Stochastic synchronization

It is clear from definition of $x(t)$ that collective behavior of particle systems with synchronization is a superposition of two tendencies. Indeed, the processes $x_j^\circ(t)$, $j = \overline{1, N}$, are mutually independent and have independent increments. Thus, due to the free dynamics, the spread of the particle system has the tendency to increase with the course of time. An opposite tendency is produced by synchronizing jumps which try to decrease the spread of the particle configuration.

In order to state precise results on a long-time behavior of $x(t)$ it is convenient to use the concept of convergence in distribution. We don't touch here all nuances of this notion and refer interested readers to [9]. We need only the following useful statement which can be found in most probability textbooks: an $\mathbb{R}^d$-valued random sequence $\{Z_n\}_{n=1}^{\infty}$ converges in distribution to a random vectors Z iff $\psi_{Z_n}(a) \to \psi_Z(a)$ for all $a \in \mathbb{R}^d$.

The long time behavior of different *symmetric* synchronization models was studied in a series of papers [3, 10–14]. In particular, it is known that

a) components $x_j(t)$ have no limits in distribution as $t \to \infty$,

b) any difference $x_j(t) - x_k(t)$ converge in distribution to a nontrivial probability law,

c) an *improved process* $y(t) = (y_1(t), \dots, y_N(t)) \in (\mathbb{R}^d)^N$ such that $y_i(t) = x_i(t) - M(t)$ and $M(t) := N^{-1}\sum_{m=1}^{N} x_m(t)$ *is ergodic.*

In particular, there exists a limit in distribution of $y(t)$ as $t \to \infty$. Evidently, the process $y(t)$ describes the particle system $x(t)$ viewed by an observer placed in the center of mass $M(t)$.

In the presence of b) we say that for the system $x(t)$ a *long time stochastic synchronization* takes place. Note that statement c) is stronger than statement b).

The goal of this and subsequent papers is to obtains results similar to a)–c) for *non-symmetric* synchronization models. We are interested in the joint distribution of the total system $x(t) = (x_1(t), \dots x_N(t)) \in (\mathbb{R}^d)^N$ and, hence, in the joint characteristic function (c.f.)

$$\chi_{1,2,\dots,N}(t;\,\lambda) = \mathsf{E}\,\exp\Big(\mathrm{i}\sum_j \lambda_j \cdot x_j(t)\Big), \qquad \lambda_j \in \mathbb{R}^d, \quad \lambda = (\lambda_1, \dots, \lambda_N).$$

It contains all information about distribution of the system and, in particular, about its marginal distributions. For instance, $\psi_{x_1(t)}(\lambda_1) = \chi_{1,2,\dots,N}(t;\,\lambda_1, 0, \dots, 0)$ etc. But the function $\chi_{1,2,\dots,N}$ appears to be very complicated. Due to the synchronizing interaction the particles $x_1(t), \dots x_N(t)$ are dependent. Hence $\chi_{1,2,\dots,N}(t;\,\lambda_1, \dots, \lambda_N) \neq \psi_{x_1(t)}(\lambda_1)\cdots\psi_{x_N(t)}(\lambda_N)$ for any finite $t > 0$ and, as we will see later, this feature survives also in the asymptotic limit $t \to \infty$. Nevertheless, we will get a nice description of the long time behavior of $x(t)$ by making an appropriate change of variables. Consider $y(t)$ and $M(t)$ as defined in item c). Let $\psi_{y(t),M(t)}$ be their joint c.f.,

$$\psi_{y(t),M(t)}(u, \rho) = \mathsf{E}\,\exp\Big(\mathrm{i}\sum_j u_j \cdot y_j(t) \, + \mathrm{i}\,\rho \cdot M(t)\Big), \quad u_j, \rho \in \mathbb{R}^d, \quad u \in (\mathbb{R}^d)^N,$$

$\psi_{y(t)}(u) = \psi_{y(t),M(t)}(u, 0)$ and $\psi_{M(t)}(\rho) = \psi_{y(t),M(t)}(0, \rho)$ be c.f.s of $y(t)$ and $M(t)$. The results on their long time behavior are summarized in next two theorems.

Theorem 1. *Let Assumptions M1-M4 and F0 hold.*

i) None of the vectors $M(t)$, $x_j(t)$, $j = \overline{1, N}$, has a limit in distribution as $t \to \infty$.

ii) For any $u \in (\mathbb{R}^d)^N$ $\psi_{y(t)}(u) \to \psi_{y(\infty)}(u)$ as $t \to \infty$ where $\psi_{y(\infty)}(u)$ is a c.f. of some probability law on $(\mathbb{R}^d)^N$.

Item *ii* implies that $y(t)$ has a limit in distribution. Remark that $\sum_j y_j(t) \equiv 0$, hence all distributions $P_{y(t)}$ and $P_{y(\infty)}$ are supported on the linear subspace $\Big\{y : \sum_j y_j = 0\Big\}$.

Theorem 2. *Let Assumptions M1-M4 and FSα hold. Denote $m(t) := M(t)/t^{1/\alpha}$. Then*

$$\forall u, \rho \qquad \psi_{m(t)}(\rho) \to \psi_{m(\infty)}(\rho), \qquad \psi_{y(t),m(t)}(u, \rho) \to \psi_{y(\infty)}(u)\psi_{m(\infty)}(\rho) \qquad (t \to \infty)$$

where $\psi_{m(\infty)}(\rho)$ is a c.f. of some probability law on $\mathbb{R}^d$.

This theorem implies that the joint distribution of $(y(t), m(t)) = (y(t), M(t)/t^{1/\alpha})$ has a limit as $t \to \infty$ and, after passage to the limit, the components y and m become independent.

Below we give a detailed study of the case $N = 2$. For $N > 2$ complete proofs of Theorems 1 and 2 demand some additional constructions which might exceed the size of the present publication. So the case $N \geq 3$ will be considered in another paper.

3. Complete treatment for $N = 2$

Fix $\lambda_1, \lambda_2 \in \mathbb{R}^d$ and denote $\varepsilon_{12} = \eta_1(\lambda_1) + \eta_2(\lambda_2)$, $\varepsilon_1 = \eta_1(\lambda_1 + \lambda_2)$, $\varepsilon_2 = \eta_2(\lambda_1 + \lambda_2)$,

$$f_{12}(t) = \chi_{1,2}(t;\, \lambda_1, \lambda_2), \qquad f_1(t) = \chi_{1,2}(t,;\, \lambda_1 + \lambda_2, 0), \qquad f_2(t) = \chi_{1,2}(t;\, 0, \lambda_1 + \lambda_2).$$

By using Markov property of $x(t)$ it is straightforward to check that the functions f_{12}, f_1, f_2 satisfies the following system of linear differential equations

$$\frac{d}{dt}\begin{pmatrix} f_{12}(t) \\ f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} -\varepsilon_{12} - \beta_{12} - \beta_{21} & \beta_{12} & \beta_{21} \\ 0 & -\varepsilon_1 - \beta_{21} & \beta_{21} \\ 0 & \beta_{12} & -\varepsilon_2 - \beta_{12} \end{pmatrix} \begin{pmatrix} f_{12}(t) \\ f_1(t) \\ f_2(t) \end{pmatrix} \tag{2}$$

or in short notation $\frac{d}{dt}\vec{f} = \mathcal{A}(\boldsymbol{\varepsilon})\,\vec{f}$, $\boldsymbol{\varepsilon} := (\varepsilon_{12}, \varepsilon_1, \varepsilon_2)$. Hence $\vec{f}(t) = \exp(t\mathcal{A}(\boldsymbol{\varepsilon}))\,\vec{f}(0)$ is the solution. To study behavior of $\vec{f}(t)$ as $t \to \infty$ let us find an eigensystem of $\mathcal{A}(\boldsymbol{\varepsilon})$: $\mu\vec{v} = \mathcal{A}(\boldsymbol{\varepsilon})\,\vec{v}$. The simplest pair is $\mu_1(\boldsymbol{\varepsilon}) = -\varepsilon_{12} - \beta_{12} - \beta_{21}$ and $\vec{v}_1(\boldsymbol{\varepsilon}) = (1, 0, 0)^\top$. The next two eigenvalues can be found in an explicit form containing radical fuctions. After some algebra we get their expansions as $\boldsymbol{\varepsilon} \to 0$, namely,

$$\mu_2(\boldsymbol{\varepsilon}) = -\frac{\beta_{12}\varepsilon_1 + \beta_{21}\varepsilon_2}{\beta_{12} + \beta_{21}} + o(\|\boldsymbol{\varepsilon}\|), \qquad \mu_3(\boldsymbol{\varepsilon}) = -\beta_{12} - \beta_{21} + o(1). \tag{3}$$

The corresponding eigenvectors $\vec{v}_2(\boldsymbol{\varepsilon})$ and $\vec{v}_3(\boldsymbol{\varepsilon})$ approaches as $\boldsymbol{\varepsilon} \to 0$ to the column vectors $(1, 1, 1)^\top$ and $(0, -\beta_{21}, \beta_{12})^\top$ respectively. It is important to remark that $\mu_2(\boldsymbol{\varepsilon})$ and $\mu_3(\boldsymbol{\varepsilon})$ do not depend on ε_{12}. Note that $\boldsymbol{\varepsilon} \to 0$ as $(\lambda_1, \lambda_2) \to 0$. Denote $S = \big\{(\lambda_1, \lambda_2) \in \mathbb{R}^{2d} \,:\, \lambda_1 + \lambda_2 \neq 0\big\}$. If the pair $(\lambda_1, \lambda_2) \in S$ is sufficiently small then all eigenvalues have strictly negative real parts. This follows from Assumption F0. Thus $\vec{f}(t) \to 0$ as $t \to +\infty$ for small $(\lambda_1, \lambda_2) \in S$. Since $\chi_{1,2}(t;\, 0, 0) \equiv 1$ the limit of $\chi_{1,2}(t;\, \lambda_1, \lambda_2)$ is discontinuous at $(\lambda_1, \lambda_2) = 0$. By the Lévy continuity theorem the stochastic process $(x_1(t), x_2(t))$ has no limit in distribution as $t \to \infty$. Now item i of Theorem 1 follows.

Note that if $N = 2$ then $y = (y_1, y_2) = \frac{1}{2}(x_1 - x_2, x_2 - x_1)$. Therefore it is enough to consider c.f. of $r(t) := x_1(t) - x_2(t)$, i.e., $\psi_{r(t)}(\lambda) = \mathsf{E}\, e^{\mathrm{i}\lambda\cdot r(t)} = \chi_{1,2}(t;\, \lambda, -\lambda)$. We can use system of equations (2) with substitution $(\lambda_1, \lambda_2) = (\lambda, -\lambda)$. Since now $f_1(t) = f_2(t) = 1$ for all t we have to solve only the equation for $\chi_{1,2}$:

$$\frac{d}{dt}\,\psi_{r(t)}(\lambda) = (-\eta_1(\lambda) - \eta_2(-\lambda) - \beta_{12} - \beta_{21})\,\psi_{r(t)}(\lambda) + \beta_{21} + \beta_{12}.$$

Solving it and letting $t \to +\infty$ we get the proof of item ii of Theorem 1

$$\psi_{r(t)}(\lambda) \to \psi_{r(\infty)}(\lambda) = \frac{\beta_{12} + \beta_{21}}{\beta_{12} + \beta_{21} + \eta_1(\lambda) + \eta_2(-\lambda)}\,. \tag{4}$$

Consider now a joint distribution of $(r(t), m(t)) = \left(x_1(t) - x_2(t)\,,\, \dfrac{x_1(t) + x_2(t)}{2t^{1/\alpha}}\right)$ where α is the stability index from Assumption $\mathbf{FS}\boldsymbol{\alpha}$**)** . The joint c.f. is

$$\psi_{r(t),m(t)}(\lambda, \rho) = \mathsf{E}\, e^{\mathrm{i}\lambda\cdot r(t) + \mathrm{i}\rho\cdot m(t)} = \chi_{1,2}\Big(t;\, \lambda + \frac{\rho}{2t^{1/\alpha}}, -\lambda + \frac{\rho}{2t^{1/\alpha}}\Big), \qquad \lambda, \rho \in \mathbb{R}^d.$$

Using eigenvectors of $\mathcal{A}(\boldsymbol{\varepsilon})$ we decompose $\vec{f}(0) = \sum\limits_{k=1}^{3} c_{f,k}(\boldsymbol{\varepsilon})\vec{v}_k(\boldsymbol{\varepsilon})$ and rewrite solution of (2) as

$$\vec{f}(t) \;=\; \sum_{k=1}^{3} e^{t\mu_k(\boldsymbol{\varepsilon})} c_{f,k}(\boldsymbol{\varepsilon})\vec{v}_k(\boldsymbol{\varepsilon})\,. \tag{5}$$

Journal of Physics: Conference Series **681** (2016) 012024 doi:10.1088/1742-6596/681/1/012024

Since this representation holds for any fixed $\boldsymbol{\varepsilon}$ and t to obtain $\psi_{r(t),m(t)}(\lambda,\rho)$ we substitute

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(t)=\Big(\eta_1\Big(\lambda+\frac{\rho}{2t^{1/\alpha}}\Big)+\eta_2\Big(-\lambda+\frac{\rho}{2t^{1/\alpha}}\Big),\eta_1\Big(\frac{\rho}{t^{1/\alpha}}\Big),\eta_2\Big(\frac{\rho}{t^{1/\alpha}}\Big)\Big)$$

into (5). Note that $\boldsymbol{\varepsilon}(t)\to\boldsymbol{\varepsilon}(\infty):=(\eta_1(\lambda)+\eta_2(-\lambda),0,0)$ as $t\to\infty$. It follows from the above analysis that the first and the third terms in (5) vanish as $t\to\infty$. The second term is a product of two multipliers. The first multiplier, $e^{t\mu_2(\boldsymbol{\varepsilon})}|_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(t)}$ depends on ρ and does not depend on λ. The second one, $c_{f,2}(\boldsymbol{\varepsilon})\vec{v}_2(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(t)}$, depends on the both variables ρ and λ. By continuity of $c_{f,2}(\boldsymbol{\varepsilon})$ and $\vec{v}_2(\boldsymbol{\varepsilon})$ we obtain that $c_{f,2}(\boldsymbol{\varepsilon})\vec{v}_2(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(t)}\to c_{f,2}(\boldsymbol{\varepsilon})\vec{v}_2(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\infty)}$ as $t\to\infty$. Recalling definition of $\boldsymbol{\varepsilon}(\infty)$ we conclude that $c_{f,2}(\boldsymbol{\varepsilon})\vec{v}_2(\boldsymbol{\varepsilon})|_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\infty)}$ depends only on λ and its value is the same as the value of $\psi_{r(\infty)}(\lambda)$ calculated already in (4). Let us evaluate the limit of $t\mu_2(\boldsymbol{\varepsilon}(t))$ as $t\to\infty$. By the stability property (1) we have $\eta_j\Big(\rho t^{-1/\alpha}\Big)=t^{-1}\eta_j(\rho)$. Using (3) we get

$$t\mu_2(\boldsymbol{\varepsilon}(t))=-\frac{\beta_{12}\eta_1(\rho)+\beta_{21}\eta_2(\rho)}{\beta_{12}+\beta_{21}}+o(1),\qquad t\to\infty.$$

So we have proved that

$$\psi_{r(t),m(t)}(\lambda,\rho)\longrightarrow\frac{\beta_{12}+\beta_{21}}{\beta_{12}+\beta_{21}+\eta_1(\lambda)+\eta_2(-\lambda)}\exp\Big(-\frac{\beta_{12}\eta_1(\rho)+\beta_{21}\eta_2(\rho)}{\beta_{12}+\beta_{21}}\Big).$$

This is even more than it was stated in Theorem 2. Indeed, putting $(\lambda,\rho)=(0,\rho)$ we obtain not only convergence of $\psi_{m(t)}(\rho)$ to some limit $\psi_{m(\infty)}(\rho)$ but also an explicit formula for the limiting c.f., namely,

$$\psi_{m(\infty)}(\rho)=\exp\Big(-\frac{\beta_{12}\eta_1(\rho)+\beta_{21}\eta_2(\rho)}{\beta_{12}+\beta_{21}}\Big).$$

It is readily seen that distribution of $m(\infty)$ is α-stable.

Distributions similar to (4) were previously obtained in context of synchronization models in [13, 14]. They are related to the class of Linnik distributions.

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