

The Bayesian Reliability Assessment and Prediction for Radar System Based on New Dirichlet Prior Distribution

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Abstract. This article studies on Bayesian reliability growth models of complex system based on new Dirichlet prior distribution when the sample of system is small. The model briefly describes expert experience as uniform distribution, then equivalent general Beta distribution of uniform distribution can be solved by optimization method when prior parameters are variables, mean is constraint condition, and variance is regarding as the optimization objective. The optimization method solves the problem of how to determine values of hyper-parameters of new Dirichlet distribution when these parameters have no specific physical meaning. Because the multidimensional numerical integration of posterior distribution is very difficult to calculate, WinBUGS software is employed to establish Bayesian reliability growth model based on a new Dirichlet prior distribution, and two practical cases are studied under this model in order to prove validity of model. The analysis results show that the model can improve the precision of calculation, and it is easy to use in engineering.

1. Introduction

With lots of new developing products more and more complex, the procedure of new product developing usually have following characteristics, such as smaller sample, non-homogeneous, heritage and requiring higher reliability. In general, development of a new product is the process of design-test-redesign-retest, the last test is continuing and evolution of early one, there is therefore lot of history information and early test data. At the same time, many experts have prior knowledge of a new product development. Therefore, Bayesian reliability growth model is usually used to the reliability assessment of new product when having prior information, history experience and data of similar product.

In recently, the main Bayesian reliability growth models are Smith model [1], Barlow Scheuer model [2], Logistic model [3], discount factor method [4], and so on. These models usually base on non-prior information hypothesis as well as testing information discount principle, so they can only give the reliability assessment of current testing product, rather than reliability estimation of final product. In 1991, Mazzuchi T. A. and Soyer R. [5], [6], [7] presented the reliability growth model of product based on Dirichlet prior distribution. The marginal distribution and joint distribution of Dirichlet distribution were used to combining opinion and experience of experts as well as test information of similar products. Meanwhile these distributions can predict product reliability and achieve management of reliability growth, such as how to foresee cost and periods of product before test and/or during test, how to trace the process of reliability growth of product, how to determine the



stop time, etc. Erkanli, Mazzuchi and Soyer [8] discussed the calculation issue of reliability growth model based on Sequential Dirichlet prior distribution, and introduced the application of MCMC in computation for Bayesian posterior distribution. The Sequential Dirichlet family has $(m+1)$ independent parameters, namely $\beta, \alpha_1, \dots, \alpha_m$, by choosing $\{\alpha_i\}$ the Sequential Dirichlet prior describes well knowledge of engineer, relevant decision methods were discussed in van Dorp et al [9]. Moreover, Somerville, Dietrich and Mazzuchi [10] gave the analysis method base on Sequential Dirichlet prior distribution as well as its application on the accelerated life test (ALT), considering step stress ALT the inverted sequence Dirichlet prior distribution had used because of product reliability decreasing during whole test period. Based on Dirichlet prior distribution Zhang and Li [11] [12] discussed the issue of Bayesian reliability of multinomial distribution aiming at robustness of likelihood function. Yu [13] and Liu [14] studied on the issue of reliability growth based on Sequential Dirichlet distribution. Furthermore, Li and Wu [15] pointed out some disadvantages of the Sequential Dirichlet distribution, such as reliability prior information of difference stages is described by only one parameter β , which is not consistent with various present practices. In fact the prior density of various phases generally is single peak, therefore in order to accord with this characteristic, the value of β must be larger value, i.e. the variance of various phases is all small. So sequential prior Dirichlet distribution has become a dominant prior distribution in this situation, thus the prior information will has a main effect on calculation of posterior information based on a small sample test, i.e. test results will lightly influence on posterior inference. Furthermore, all of prior parameters of Dirichlet distribution must be determined before reliability growth test, but in engineering experts usually give accurate reliability assessment of next phase only under knowing test results and innovative approach of last one.

Therefore, Li and Wu [16] studied further on a new prior distribution class in their paper, and established Bayesian reliability growth model based on the new Dirichlet prior distribution. Moreover, the value of reliability assessment and its confidence are described by two parameters in each reliability growth phase. However, parameters of the new prior distribution class $\alpha_1, \alpha_2, \dots, \alpha_m$ have not definitely physical meaning, it is difficult to determine their values by experts experience, and will be “respectable” when it will be able to solve the problem of practice engineering.

In order to solve the above problems, this paper focuses on Bayesian reliability growth model of new product developing based on a new Dirichlet prior distribution. The physical meaning of the hyper-parameters of the new Dirichlet prior distribution has been defined and determined by optimization method. Moreover, discussing the problem of how to calculate multidimensional numerical integration of posterior distribution, WinBUG software package is employed to describe Bayesian reliability growth inference model of a new Dirichlet prior distribution. Finally, the paper presents an application of the model to two cases, the results show that the model is brief and effective and can be used easily to the reliability growth assessment.

2. Bayesian reliability growth model and a new prior distribution

2.1. Model hypothesis

I. Binomial distribution model is usually used to reliability analysis of complex large system. Assuming products have been undergone m test period. In each test stage, there are n_k products undergoing a time-terminated test, s_k products survived. Therefore, we can collect data of success and failure of product.

II. After each test stage, correcting products and removing failure. Therefore, reliability of products is an evolution process during the whole test phases. Assuming the reliability of product is R_k in the k th period, thus,

$$0 \leq R_1 \leq R_2 \leq \dots \leq R_{m-1} \leq R_m \leq 1 \quad (1)$$

Based on above hypothesizes and combining the information of similar products as well as expert experience, this paper studies on how to assess reliability growth of product by Bayesian theory and technology.

2.2. Likelihood function

According to the model assumption, success data of the k^{th} time-terminated test stage s_k follows a Binomial distribution, and the parameters are n_k and R_k , i.e.

$$s_k \sim \text{Binomial}(n_k, R_k) \quad (2)$$

Likelihood function as following.

$$L(R_k; n_k, s_k) = \binom{n_k}{s_k} R_k^{s_k} (1 - R_k)^{n_k - s_k} \quad (3)$$

2.3. Prior distribution family

I. Order Dirichlet distribution

The expression of order Dirichlet distribution is following,

$$\pi(\tilde{R} | \tilde{\alpha}, \beta) = \frac{\Gamma(\beta)}{\prod_{i=1}^{m+1} \Gamma(\beta \alpha_i)} \prod_{i=1}^{m+1} (R_{i-1} - R_i)^{\beta \alpha_i - 1} \quad (4)$$

Where, $\tilde{R} = (R_1, R_2, \dots, R_{m-1}, R_m)$, $\beta > 0$, $\alpha_i > 0$, $\sum_{i=1}^{m+1} \alpha_i = 1$, the β parameter is shape parameter, the α parameter is location parameter and $R_0 = 0$, $R_{m+1} = 1$. Moreover, the α parameter can be determined by prior information and well describes engineering experience of reliability assessment of various phases.

According to characteristics of Dirichlet distribution marginal distributions of above several distributions are a Beta distribution, namely,

$$R_i \sim \text{Beta}(\beta \alpha_i^*, \beta(1 - \alpha_i^*)) \quad (5)$$

Where, $\alpha_i^* = \sum_{k=1}^i \alpha_k$. Furthermore, according to characteristics of Beta distribution the following expressions are given.

$$E[R_i] = \sum_{j=1}^i \alpha_k = \alpha_i^* \quad (6)$$

$$\alpha_i = E[R_i] - E[R_{i-1}] \quad (7)$$

Thus, the α parameter expresses reliability growth between each stage. In general product reliability increases quickly in the early phase of product development, corresponding value of α is bigger. Contrary in the late of product development α is usually small because it is difficult to have bigger improvement in such stage. The β parameter describes the degree of confidence of estimate value, assuming α is determined, the larger β the smaller standard deviation of prior, therefore confidence of reliability is higher, the reverse is true.

Inspection of above analysis of order Dirichlet distribution shows that in whole period of product development there is only a shape parameter β describing the variance of product reliability in

various test phases, however experts usually need to know test results and correct approach of last phase so that they can give accurate reliability assessment of next one.

II. A new prior distribution family based on Beta distribution

A disadvantage of the order Dirichlet distribution is that it only has one shape parameter β to describe the variance of product reliability in various test periods, and is not consistent with present practice. Based on general Bate distribution of various test phases Reference [17] established a new prior distribution family by conditional distribution modality, which is fit for reliability growth model of new product. The distribution eliminates some of disadvantage of order Dirichlet distribution, and can express well the prior information.

Regarding the k^{th} test interval, constructing general Beta distribution of $(R_{k-1}, 1)$ as reliability distribution of product in this interval, then,

$$g_k(R_k | R_{k-1}) = g_k(R_k | R_{k-1}; a_k, b_k) = \frac{(1-R_{k-1})^{1-a_k-b_k}}{B(a_k, b_k)} (R_k - R_{k-1})^{a_k-1} (1-R_k)^{b_k-1} I_{(0,1)}(R_k) \quad (8)$$

Where $a_k > 0, b_k > 0, R_0 = 0$, $B(\cdot, \cdot)$ is Beta function, And $B(a_k, b_k) = \frac{\Gamma(a_k)\Gamma(b_k)}{\Gamma(a_k + b_k)}$. Then, the

density function of joint prior distribution of $\tilde{R} = (R_1, R_2, \dots, R_{m-1}, R_m)$ described as following,

$$\pi(\tilde{R} | \tilde{\alpha}, \tilde{\beta}) = \prod_{k=1}^{m+1} g_k(R_k | R_{k-1}) \quad (9)$$

Equations (8) and (9) show that there are two parameters in each test period, which can accurately express experts' experience. Moreover, for each reliability growth stage the reliability estimation and its confidence can be described by both parameters. Therefore, the new prior distribution is better than the order dirichlet prior distribution.

2.4. Mathematics characteristic of prior distribution family

The condition distribution, expressed by equation (8), is general Beta distribution in $(R_{k-1}, 1)$, which characteristic is similar with standard Beta distribution in $(0, 1)$. Several typical Beta density curves show in figure 1.

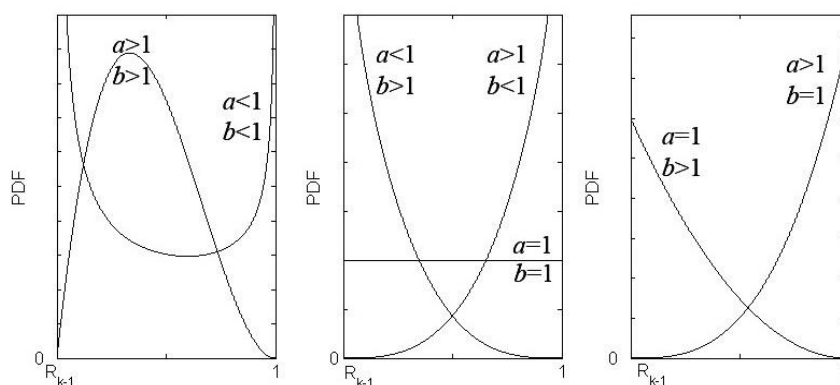


Figure 1. Several typical Beta density curves.

Considering general Beta distribution in $(R_{k-1}, 1)$, when R_1, R_2, \dots, R_{k-1} is known, we can solve the condition mean and condition variance of R_k by equation (8) as following,

$$\mu_k^* = \frac{a_k + b_k R_{k-1}}{a_k + b_k}$$

$$v_k^* = \frac{a_k b_k (1 - R_{k-1})^2}{(a_k + b_k)^2 (a_k + b_k + 1)} \quad (10)$$

Based on above analysis of characteristic, it stands to reason that the physical meaning of parameters of a new prior distribution family is not clear. Therefore, it is the first step that how to determine parameters of prior based on prior information, if we wish to use the new prior distribution for reliability assessment of product.

3. The method of determining prior parameters

The prior information of reliability is used to presenting by continuous interval modality. For instance, product reliability of k^{th} test phase, $R_k \in (R_{k,L}, R_{k,H})$, may be given according to information of similar product and expert experience. In this paper, firstly prior information of product is described by uniform distribution, then the general Beta distribution is conducted by optimization method, which mostly approximate to the uniform distribution, when prior parameters are variables as well as mean is constraint condition, and variance is regarding as the optimization objective. Then the special Beta distribution is used to describing prior distribution as depicted in figure 2.

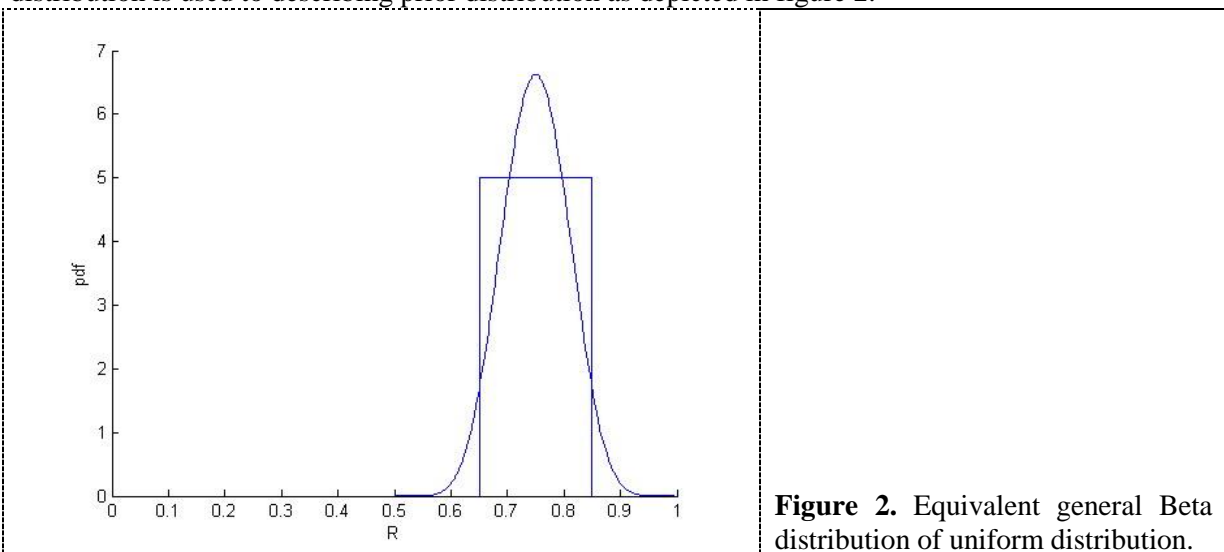


Figure 2. Equivalent general Beta distribution of uniform distribution.

If there are more prior information of product reliability in k^{th} test phase, for instance h continuous phases $(R_{k,L}^j, R_{k,H}^j)$ $j=1, 2, \dots, h$, are given, then the mean and variance are shown as following.

$$ER_k = \frac{R_{k,L} + R_{k,H}}{2} \quad (11)$$

$$VR_k = \frac{(R_{k,H} - R_{k,L})^2}{12} \quad (12)$$

How to determine the prior parameters, one of natural ideas is that firstly making the mean and variance of general Beta distribution expressed by equation (10) to equal the mean and variance of uniform distribution $U(R_{k,L}, R_{k,H})$ expressed by equation (11) and (12), then we can determine parameters of prior distribution by solving equation (11). Unfortunately, the solutions of equation (11)

are sometime negative, which is not according with non-negative characteristic of prior parameters. Therefore, this paper solves prior parameters of k^{th} test stage by the optimization model describing as following,

$$\begin{aligned} & \text{Min } (v_k^* - VR_k)^2 \\ & \text{s.t.} \\ & \begin{cases} \mu_k^* = ER_k \\ a_k > 0, b_k > 0 \end{cases} \end{aligned} \quad (13)$$

After considering equation (10) (11) and (12), the optimization model, equation (13), can solve prior parameters which exactly describe prior information. Where R_{k-1} of equation (10) can be substituted by condition mean of $k-1^{\text{th}}$ test stage.

If there are more prior information depicting product reliability R_k of k^{th} test phase, such as the number of continuous phases $(R_{k,L}^j, R_{k,H}^j)_{j=1, 2, \dots, h}$ is h , then the mean and variance as following,

$$ER_k = \frac{1}{h} \sum_{j=1}^h \frac{R_{k,L}^j + R_{k,H}^j}{2} \quad (14)$$

$$VR_k = \frac{1}{h} \sum_{j=1}^h \left[\frac{(R_{k,H}^j - R_{k,L}^j)^2}{12} + \frac{(R_{k,H}^j + R_{k,L}^j)^2}{4} \right] - \frac{1}{h^2} \left(\sum_{j=1}^h \frac{R_{k,H}^j + R_{k,L}^j}{2} \right)^2 \quad (15)$$

Of course, the prior parameters of product reliability of k^{th} test phase can also be solved by optimization model expressed by equation (13).

4. The method of calculating value of reliability by WinBUGS

The Bayesian statistical theory is one main branch of statistical theories, since 1763 the first paper of Bayesian statistical theory had been published, until 80th in twenty century Bayesian statistical theory has been in theory research phase, integral calculation is a big barrier in his development and application. However, Markov Chain Monte Carlo (MCMC) has been used to Bayesian statistical inference in recently, a main characteristic of this method is Metropolis-Hastings updating and Gibbs sampling, it can solve well the problem of numerical integration and sampling in multidimensional distribution, which is convenient for posterior inference of parameters and accelerate the application of Bayesian theory.

Regarding the theorem and application of MCMC, many research works have been done and make great progress, for instance the software WinBUGS developed in 1991, which is special software, is mainly used for Bayesian analysis of complex statistical model by MCMC approach. Therefore, in order to calculate multidimensional numerical integration of multivariate distribution, such as marginal posterior distribution, mean of posterior and quantile of posterior depicted in section 3 of the paper, this paper employed the software WinBUGS to solve posterior inference of parameters and have enough calculation accuracy.

4.1. Introductions of WinBUGS

MRC (Medical Research Council, Cambridge UK) and Imperial College of Science, Technology and Medicine joined efforts in developing WinBUGS (Bayesian Inference Using Gibbs Sampling), the exclusive software solves Bayesian inference by MCMC method. Gibbs sampling of many common models and distribution can easily finished by WinBUGS, programmer need not to know accurate equation of likelihood function or prior density of parameters, as long as setting prior distribution of

variants and describing the interested model generally, then can easily achieve Bayesian analysis of model rather than complex compilation.

Considering WinBUGS software, Directed Graphical Model can be used in describing the interested model directly, alternatively programming the model codes, and presenting the Dynamic Graph of Gibbs sampling of parameters, based on Smoothing method getting core density estimation graph of posterior distribution, auto correlation graph of sampling value, transfer graph of mean and confidence interval, and so on. Having finished all above works, the result of sampling would be direct and reliable. After Gibbs sampling converging, we can get mean of posterior distribution of parameters, as well as standard variance, 95% confidence interval, quantile etc.

4.2. Description of Bayesian reliability growth model by WinBUGS codes

According to equation (5) which is general Beta distribution in $(R_{k-1}, 1)$, by substitution of variable,

$$X_k = \frac{R_k - R_{k-1}}{1 - R_{k-1}} \quad (15)$$

Then

$$X_k \sim \text{Beta}(a_k, b_k) \quad (16)$$

Where $\text{Beta}(\bullet, \bullet)$ is a standard Beta distribution in $(0, 1)$. For analysis convenient, equation (2), (15) and (16) are rewrite as following,

$$s_k \sim \text{Bino}(n_k, R_k) \quad (17)$$

$$X_k \sim \text{Beta}(a_k, b_k) \quad (18)$$

$$X_k = \frac{R_k - R_{k-1}}{1 - R_{k-1}} \quad (19)$$

Where equation (17) and (18) express the relationship of statistic, equation (19) expresses determine relationship.

According to above equations, Bayesian inference model of reliability growth can be established by WinBUGS codes. Depicting as following,
model {

```

    for (i in 1: N) {
        s[i] ~ dbin(R[i], n[i])
        x[i] ~ dbeta(a[i], b[i])
    }
    for (i in 2: N){
        R[i]<-R[i-1]+(1-R[i-1])*x[i]
    }
    R[1]<-0+(1-0)*x[1]
}
```

5. The cases study

5.1. Case one

Assuming the development of an aero engine is consisted of three experimental stages, in the first stage two success and one failure, in the second stage four success and one failure, the last stage five success and no failure. According to the experts' assessment and history information, the estimation

value of reliability is separately $R = (0.70, 0.90, 0.95)$ in each test stage, and 0.975 is prior value of product reliability after reliability growth test of three stages.

The reference [18] analyzed the reliability growth of product by order Dirichlet distribution, and assuming value of parameter β is 44. Number of Reliability growth test n_i , success number s_i as well as point estimation value of R_i based on prior information have been shown in the second column, the third column and the fourth column of table 1.

Table 1. Reliability growth test data and prior estimate.

t_i	n_i	s_i	R_i	$(R_{i,L}, R_{i,H})$
1	3	2	0.70	(0.4, 1)
2	5	4	0.90	(0.8, 1)
3	5	5	0.975	(0.95, 1)

In fact interval estimation of reliability of various test stages can be presented according to experts' experience and history data, moreover interval estimation describes prior information more accurate than point estimation. Assuming based on experience of experts interval estimation of each test stage $(R_{i,L}, R_{i,H})$ has shown in the fifth column of table 1. Inspection of the table 1 shows that in the first test stage no test data and correcting information of product can be available, therefore estimation value of reliability is not accuracy, and interval estimation for instance given by experts is much wider. After reliability growth test and improvement, the interval estimation is more accuracy and wide decreasing.

The prior parameters of general Beta distribution of various test phases α_k, b_k have been solved by optimization model expressed equation (13), where general Beta distribution is equivalent to uniform distribution $U(R_{k,L}, R_{k,H})$. Mean ER_k and variance V_k of the two distributions are shown in the table 2.

Table 2. Statistic characteristic parameters of equivalent general Beta distribution and uniform distribution.

t_i	Uniform Distribution		Equivalent General Beta Distribution			
	ER_k	V_k	ER_k	V_k	a_k	b_k
1	0.7	0.03000	0.5	0.03000	4.200	1.800
2	0.9	0.00333	0.75	0.00333	3.333	1.667
3	0.975	0.00021	0.95	0.00021	6.001	2.000

The figure 3 is the curve of uniform distribution based on experience of experts as well as the equivalence general Beta distribution.

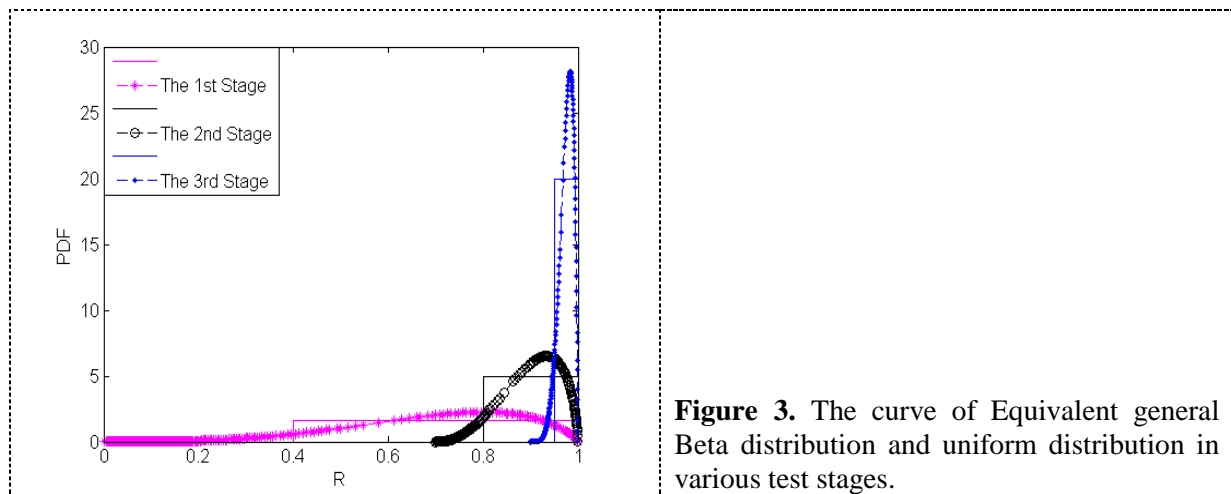


Figure 3. The curve of Equivalent general Beta distribution and uniform distribution in various test stages.

Inspection of this plot shows that prior distribution, which solved by optimization equation (13), has a good fitting with prior information.

After getting prior parameters based on prior information, posterior estimation values of reliability of various test phases can be solved by WinBUGS model expressed in section 3.2, and table 3 shows the results.

Table 3. Reliability posterior estimate values of various test stages

t_i	New Distribution(OPT) ^a		Order Dirichlet ^b		New Distribution(Li) ^c	
	$R_{kL}(0.9)$	V_k	$R_{kL}(0.9)$	V_k	$R_{kL}(0.9)$	V_k
1	0.4757	0.01924	0.6108	0.0047	\	0.0420
2	0.776	0.00555	0.8303	0.0020	\	0.0115
3	0.9395	0.00064	0.9497	0.0005	0.934	0.0015

^a a optimization method presented by this paper.

^b order distribution.

^c method of reference [15].

Notice in table 3 that variance of posterior estimation value based on a new prior distribution is bigger than variance of order Dirichlet distribution, especially in the early two phases. Comparing the lower bound $R_{kL}(0.90)$ of 90% confidence limit, the posterior estimation value based on a new prior distribution is smaller. Whereas the posterior estimation value based on order Dirichlet distribution is evidently effected by prior information, i.e. order Dirichlet distribution has a mainly influence on posterior estimation value.

5.2. Case two

Considering another situation in case one, assuming prior estimation value of product based on experience of expert is not various, the achieved reliability of product is lower because of test results of various stages becoming worse, and supposing test result of various stages (n_i, s_i) is (3, 1), (5, 3) and (5, 4) separately. By the same calculating procedure been used in table 3, posterior estimation values of various test stages are shown in table 4.

Inspection of the table 4 shows that if employing order Dirichlet distribution to depict the expert experience when the test sample is smaller, the prior distribution will be whip hand so that posterior estimation is mainly determined by prior information. It is not consistent with present practice.

According to the above cases, the new prior Dirichlet distribution can accurately describe expert experience, the Bayesian estimation value of product reliability during developing phase can be solved by optimization method presented by this paper. Moreover, the method has a good adaptability when

test sample is small. Bayesian model based on a new Dirichlet distribution can take full advantage of history information, expert experience and test date of various phases.

Table 4. Reliability posterior estimate values based on worse test results of various test stages.

t_i	New Distribution(OPT)		Order Dirichlet	
	$R_{kL}(0.90)$	V_k	$R_{k \cdot L}(0.90)$	V_k
1	0.3049	0.01924	0.5886	0.00452
2	0.606	0.01059	0.8047	0.00230
3	0.8569	0.00233	0.9187	0.00081

6. Conclusion

This paper presents a new method for the application of Bayesian theory and technology to product reliability growth during product development phase. The research work mainly focuses on how to determine prior distribution parameters of a new Dirichlet distribution and presents the relevant optimization model and method, finally demonstrate the validity of Bayesian reliability growth model by WinBUGS software. The conclusions as following:

(1) The optimization method describes experience of expert as a general Beta distribution, which is fitted for Bayesian reliability growth model based on a new Dirichlet prior distribution, and solve the problem of how to determining hyper-parameters.

(2) Regarding the issue of multidimensional numerical integration of posterior, the paper employs WinBUGS software to establish the reliability growth model based on a new Dirichlet prior distribution, and solves posterior inference of parameters at the same time keeping enough calculation accuracy.

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