Analysis of half-spin particle motion in static Reissner-Nordström and Schwarzschild fields

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Abstract. The paper presents the analysis of effective potentials of Dirac equations in static Schwarzschild and Reissner-Nordström (RN) fields. It is shown that in all the explored cases the condition of a particle "fall" to appropriate event horizons is fulfilled. The exception is one of the solutions for the Reissner-Nordström extremal field, for which the existence of the stationary bound state of half-spin particles is possible inside the event horizon.

1. The Reissner-Nordström metric

In the early 2000s, it was believed that there is no stable bound state of Dirac particles in the gravitational field of Reissner-Nordström black holes [1], [2]. However, in 2013 came a surprising result that stable bound states are yet possible in the field of extremal black hole both inside and outside the event horizon [3]. Approach of the authors [3] was to find a solution to the eigenvalue problem for the Dirac Hamiltonian in the background of an extremal black hole. There is much more simple method enabling us to clarify this issue, the effective potential method. In this paper, we address the question of whether stable bound states realizable in an extremal black hole by applying the efective potential method to the Dirac equation, and arrive at the conclusion that such states are indeed unfeasible in the region outside the event horizon, but a single state (rather than two) can exist inside the extremal black hole.

The line element RN metric is

$$ds^2 = f_{R-N}dt^2 - \frac{dr^2}{f_{R-N}} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),\qquad(1)$$

where $f_{R-N} = \left(1 - \frac{r_0}{r} + \frac{r_Q^2}{r^2}\right)$, $r_0 = \frac{2GM}{c^2}$ is the gravitational radius of the Schwarzschild field, $r_Q = \frac{\sqrt{G}Q}{c^2}$ is the "charge" radius, G is the gravitational constant, c is the velocity of light.

For obtaining the Schrödinger-type equation with the self-conjugate Hamiltonian the initial Dirac Hamiltonian must be also self-conjugate.

The self-conjugate Hamiltonian of a half-spin particle of mass m and charge e for the Reissner-Nordström metric was derived in [4]

$$H_\eta = \sqrt{f_{R-N}}\beta m - i\alpha^1\left(f_{R-N}\frac{\partial}{\partial r} + \frac{1}{r} - \frac{r_0}{2r^2}\right)$$
$$-i\sqrt{f_{R-N}}\frac{1}{r}\left[\alpha^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\mathrm{ctg}\theta\right) + \alpha^3\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right] + \frac{eQ}{r}. \tag{2}$$

In (2), α^k, β are Dirac matrices.

After separation of variables (see, for instance, [5]), the system of equations for radial functions $F_{R-N}(\rho), G_{R-N}(\rho)$ takes the form

$$f_{R-N}\frac{dF_{R-N}(\rho)}{d\rho} + \left(\frac{1+\kappa\sqrt{f_{R-N}}}{\rho} - \frac{\alpha}{\rho^2}\right)F_{R-N}(\rho) - \left(\varepsilon - \frac{\alpha_{em}}{\rho} + \sqrt{f_{R-N}}\right)G_{R-N}(\rho) = 0,$$
$$f_{R-N}\frac{dG_{R-N}(\rho)}{d\rho} + \left(\frac{1-\kappa\sqrt{f_{R-N}}}{\rho} - \frac{\alpha}{\rho^2}\right)G_{R-N}(\rho) + \left(\varepsilon - \frac{\alpha_{em}}{\rho} - \sqrt{f_{R-N}}\right)F_{R-N}(\rho) = 0. \tag{3}$$

In (3), dimensionless variables have been introduced $\rho = \frac{r}{l_c}$; $\varepsilon = \frac{E}{m}$; $\alpha = \frac{r_0}{2l_c} = \frac{GMm}{\hbar c} = \frac{Mm}{M_P^2}$, $\alpha_Q = \frac{r_Q}{l_c} = \frac{\sqrt{G}Qm}{\hbar c} = \frac{\sqrt{\alpha_{fs}}}{M_P}m\frac{Q}{|e|}$; $\alpha_{em} = \frac{eQ}{\hbar c} = \alpha_{fs}\frac{Q}{e}$, $l_c = \frac{\hbar}{mc}$ is the Compton wavelength of a Dirac particle; E is the energy of a Dirac particle; $M_P = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass; $\alpha_{fs} \approx 1/137$ is the electromagnetic fine structure constant; $\kappa = \pm\left(j + \frac{1}{2}\right)$.

2. The Reissner-Nordström field with two event horizons ($\alpha^2 > \alpha_Q^2$)

From the system of equations (3), we derive the second-order ordinary differential equation for the function $\psi(\rho)$ proportional to $F(\rho)$.

$$\frac{d^2\psi(\rho)}{d\rho^2} + 2\left(E_{schr} - U_{eff}(\rho)\right)\psi(\rho) = 0. \tag{4}$$

The effective potential has a complicate analytical form, but our concern is not with the precise analytical expression for U_{eff}. It is sufficient to know the behavior of the effective potential in the vicinity of the event horizons and in vicinity of the origin.

$$U_{R-N}(\rho)|_{\rho\to\rho_-} = -\frac{1}{(\rho_- - \rho)^2}\left[\frac{1}{8} + \frac{1}{8}\frac{(\varepsilon\rho_- - \alpha_{em})^2\rho_-^2}{(\alpha - \rho_-)^2}\right] + \mathrm{O}\left(\frac{1}{\rho_- - \rho}\right), \tag{5}$$

$$U_{R-N}(\rho)|_{\rho\to\rho_+} = -\frac{1}{(\rho_+ - \rho)^2}\left[\frac{1}{8} + \frac{1}{8}\frac{(\varepsilon\rho_+ - \alpha_{em})^2\rho_+^2}{(\rho_+ - \alpha)^2}\right] + \mathrm{O}\left(\frac{1}{\rho_+ - \rho}\right). \tag{6}$$

It is well known [6] that quantum-mechanical particle "falls" to the center if the singular effective potential behaves as $-C/\rho^2$ with $C > 1/8$, otherwise it does not "fall" to center. The above expressions for the singular effective potential behaves as $-C/(\rho - \rho_\pm)^2$ in the vicinities of the horizons, with the numerators of these expressions being positive, and no less than $1/8$. It follows that Dirac particles "fall" to the horizons (either inner or outer).

For the Schwarzschild field, $\rho_+ \equiv \rho_0 = 2\alpha$, $\rho_- = 0$, $\alpha_{em} = 0$ in the region $\rho > 2\alpha$ the effective potential (6) in the vicinity of event horizon has a form

$$U_S(\rho)|_{\rho\to 2\alpha} = -\frac{1}{(\rho - 2\alpha)^2}\left[1/8 + 2\alpha^2\varepsilon^2\right] + \mathrm{O}\left(\frac{1}{\rho - 2\alpha}\right). \tag{7}$$

It follows that in this case half-spin particles "fall" to the event horizon with $\rho_0 = 2\alpha$.

3. The Reissner-Nordström extremal field $(\alpha^2 = \alpha_Q^2)$

Consider the effective potential for an extremal black hole, $M = |Q|$ or $\alpha = |\alpha_Q|$.

In this case, there is the only event horizon with the radius $\rho_+ = \rho_- = \alpha$. The effective potential in the vicinity of horizon is more singular than $C/(\rho - \alpha)^2$ and therefore that the Dirac particle "falls" to the horizon, but if we take into account Dokuchaev and Eroshenko solution $\varepsilon = \frac{\alpha_{em}}{\alpha}$ [3], then the effective potential in the vicinity of horizon becomes

$$U_{R-N}^{extr}(\rho)\Big|_{\rho \to \alpha} = -\frac{\frac{1}{4} - \kappa^2 - \alpha^2 + \alpha_{em}^2}{2(\rho - \alpha)^2}. \tag{8}$$

The eigenvalue $\varepsilon = \frac{\alpha_{em}}{\alpha}$ with normalized wave can exist if condition

$$\kappa^2 + \alpha^2 - \alpha_{em}^2 > \frac{1}{4} \tag{9}$$

imposed. We claim that, in the exterior region of an extremal black hole, a stable bound states described by the Dirac equation do not exist. This is consistent with the results of Refs. [1] and [2]. Indeed, let us consider two possibilities. First, $\alpha > |\alpha_{em}|$. In this case, the found effective potential has no extremal point. The effective potential $U_{R-N}^{extr}(\rho)$ is schematically shown in figure 1. Second, $\alpha < |\alpha_{em}|$. The shape of the potential $U_{R-N}^{extr}(\rho)$ differs qualitatively from that in the first case (see figure 2).
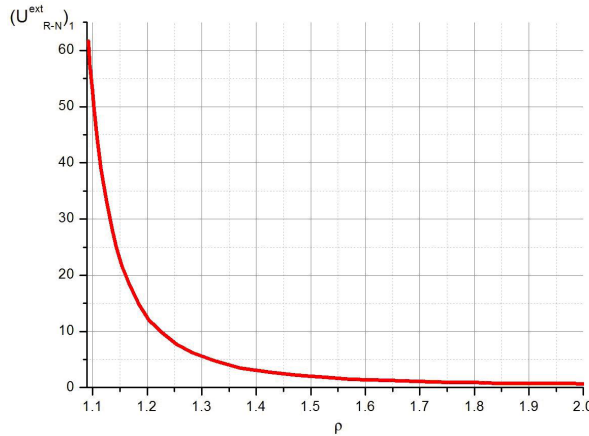

Figure 1. The effective potential in the exterior of the RN extremal black hole for $\alpha > \alpha_{em}$, $\alpha = \alpha_Q = 1, \alpha_{em} = 0.9, \varepsilon = 0.9$.
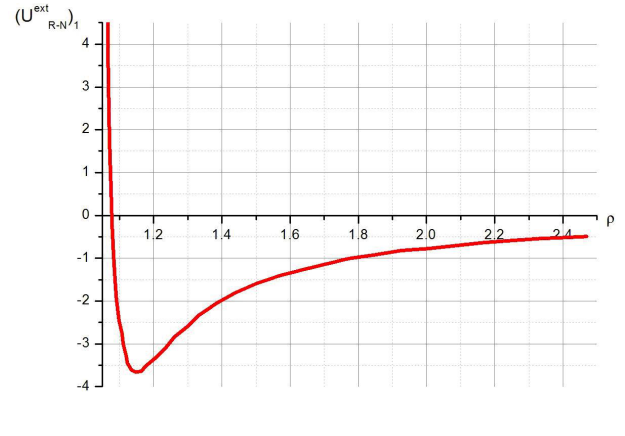
Figure 2. The effective potential in the exterior of the RN extremal black hole for $\alpha < \alpha_{em}$, $\alpha = \alpha_Q = 1, \alpha_{em} = 1.25, \varepsilon = 1.25$.

Let us first assume that the black hole and the particle have like signs of electric charges, $\alpha_{em} = Qe > 0$. Clearly, this energy corresponding to a bound states can not exist because $\varepsilon > 1$ so that this energy level belongs to the continuum part of the spectrum. Let the black hole and the particle have opposite signs of electric charges, $\alpha_{em} = Qe < 0$. It follows that $\varepsilon = -\left|\frac{\alpha_{em}}{\alpha}\right| < -1$, which formally does not prevent to the existence a discrete energy level with $E < -m$. Let us consider the interior region of an extremal Reissner-Nordsröm black hole $(0 < \alpha < \rho)$. The effective potential is positive.
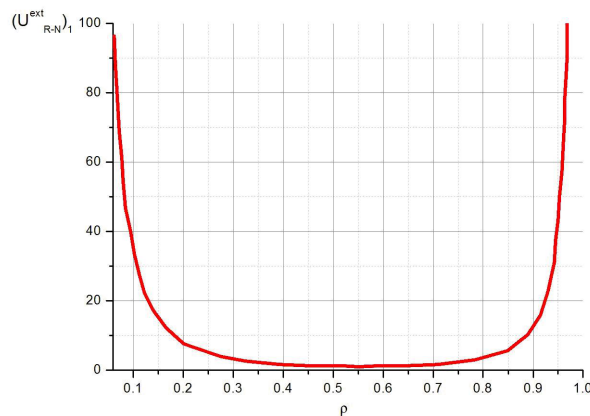
Figure 3. The effective potential in the interior of the RN extremal black hole $0 < \rho < \alpha$, for $\alpha = \alpha_Q = 1, \alpha_{em} = 1.25, \varepsilon = 1.25$.

In the vicinity of the origin, we have $U_{R-N}^{extr}(\rho)\big|_{\rho \to 0} = \frac{3}{8}\frac{1}{\rho^2} + \mathrm{O}\left(\frac{1}{\rho}\right)$ while in the vicinity of the horizon $U_{R-N}^{extr} \sim C/(\rho - \alpha)^2$. Can a discrete state with $\varepsilon = \frac{\alpha_{em}}{\alpha}$ exist there? The answer is positive provided that the particle and the extremal black hole have the same sign of their electric charges. Otherwise the answer is negative.

4. Discussions
The effective potential method allows us to obtain qualitative results consisting in the following: we show that in all the explored cases, but one, the condition of a particle "fall" to appropriate event horizons is fulfilled. The exception is one of the solutions for the RN extreme field with the single event horizon. For this solution, inside the event horizon the possibility of existence of stationary bound states of hslf-spin particles is shown only at like signs of the particle and the black hole. If $\kappa^2 + \alpha^2 - \alpha_{em}^2 > 1/4$ and $\varepsilon = \frac{\alpha_{em}}{\alpha}$, then the effective potential exhibits the second order pole with a positive coefficient, which implies that the black hole becomes quantum-mechanically impenetrable. This should be compared with the usual situation when a particle "falls" to event horizon.

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