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Solar magnetograms editing using discrete Morse theory

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Abstract. Solar magnetic field structure in active regions is complicated for modeling. We construct by magnetogram data a special structure – discrete Morse-Smale complex. We perform simplification of complex using persistence pair elimination. Finally, we obtain a simplified complex that stores main topological features of the magnetogram data and neighbourhood relation between them.

1. Introduction

Matching of the observed magnetic fields in Active Regions (AR) of the Sun with models is a complicated issue. Actually, magnetograms produced by space observatories are too complex [4] to compare with simple models (like a magnetic charge). Thus, simplification of the magnetograms is necessary. However, the greater part of AR formed by the background noise with small intensity values. Influence of the sunspot on the topology (for example, the Euler characteristic) of the entire AR is low [7]. Moreover, AR includes parts of both polarities with distinct field extrema values. Critical points of the scalar field are characterized by the Morse index — the number of negative eigenvalues of Hessian. Thus, in the two-dimensional case, minima have index zero, maxima — index two, and saddle points — index one. We introduce a discrete structure (cell complex), matching minima with vertices, saddle points with edges, and maxima with faces of this structure. After that, we construct a discrete gradient field which consists of ascending and descending integral lines. If intersections of the integral lines are transversal, we obtain cells of the so-called Morse-Smale complex. Every cell contains exactly one minimum, one maximum, and two saddle points. Euler characteristic (χ) defined as an alternating sum of critical point numbers of the different indices. Hence simplification methods must reduce critical point number in a way that do not change χ. Editing, or simplifying of field consists of sequentially eliminations of the persistent pairs “saddle - minimum” or “saddle - maximum”. The global field topology during simplification remains unchanged.

2. Related Work

Discrete Morse theory is a useful tool of the topological data analysis. There are many applications of this in astrophysics [11], geomorphology, particle physics [3] etc.

Discrete Morse theory was stated by Forman [2]. Lewiner [6] was the first who presented technique for computation of a discrete gradient field that agrees with the flow behavior of a scalar function. Sousbie [11], Robins et al. [10] presented faster algorithms using various techniques. Gyulassy et al. [3] proposed a probabilistic approach. Edelsbrunner et al. [1] introduced a topological persistence and persistent pairs cancellation concepts.

In this parer, we show the result of a Morse-Smale complex construction and simplification for a bipolar field (obtained by processing of SDO/HMI data).

3. Preliminaries

Morse theory concerned on the behavior of the smooth scalar functions defined over generic manifolds. At the first, we present some basic definitions, and then extend theory for discrete domains.

3.1. Morse Theory

Consider a smooth function $f : \mathbb{M} \to \mathbb{R}$ defined over two-dimensional manifold $\mathbb{M}$.

Denote as ∇f the *gradient* of f. A point $p \in \mathbb{M}$ is *critical* if $\nabla f(p) = 0$, and it is *degenerate* if a determinant of Hessian matrix $\mathcal{H}_f(p) = d^2 f/dxdy(p)$ takes zero value.

The function f is a *Morse function* if all its critical points are non-degenerate. In addition, require that all values of f in critical points are different. For such a function, *Morse Lemma* states [8] that there exists a chart (x, y) in a neighbourhood U of critical point p such that $x(p) = 0$, $y(p) = 0$ and $f(x, y) = f(p) \pm x^2 \pm y^2$. The number of minus signs in this equation gives an *index (order)* of critical point. One can think about index as the number of independent downward directions. In 2D case, critical points of index 0 are called *minima*, index 1 − *saddles*, index 2 − *maxima*.

Integral line is a curve which is tangent to gradient field in every point. Precisely, it is a curve $l(t) \subset \mathbb{R}^2$ such that $dl(t)/dt = \nabla f$. *Origin* of integral line is $\lim_{t\to-\infty} l(t)$ and *destination* is $\lim_{t\to+\infty} l(t)$. Actually, each integral line has origin and destination in critical points.

Integral lines of f have the following properties:

(i) every point $p \in \mathbb{M}$ belongs to exactly one integral line,

(ii) every two integral lines are fully distinct (but may have the same origin or destination).

From (i) and (ii) follows that the domain of f can be decomposed into regions containing integral lines with common origin or destination. *Ascending (descending) manifold* is a set of points whose integral lines have common origin (destination). Decomposition of $\mathbb{M}$ into ascending (descending) manifolds is called a Morse complex. A critical point with index i has an i-dimensional descending manifold and $2 - i$-dimensional ascending manifold. 1-manifolds of a Morse-Smale complex are called *arcs*.

Morse function f is called *Morse-Smale* if intersection of its ascending and descending manifolds are transversal. We obtain a *Morse-Smale complex* by intersection of ascending and descending manifolds.

3.2. Discrete Morse Theory

Supposing the magnetic field intensity were a Morse function, which critical points correspond to the topological features of the field and arcs define neighbourhood relation between critical points.

Regrettably, a magnetogram data is given over discrete domain and do not comply with criteria of Morse function. Thus, we need to give a generalization of Morse theory for functions defined over discrete space.

For details of such a generalization, namely *discrete Morse theory*, we refer to Forman [2].

How to define correctly a discrete Morse function by given magnetogram values and how to construct a gradient field are highly intricate issues. We use approach stated by Sousbie

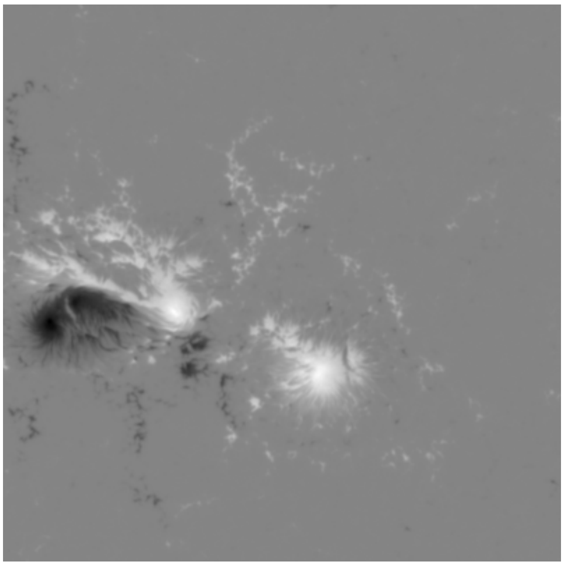

Figure 1. Initial 400 × 400 SDO/HMI magnetogram.

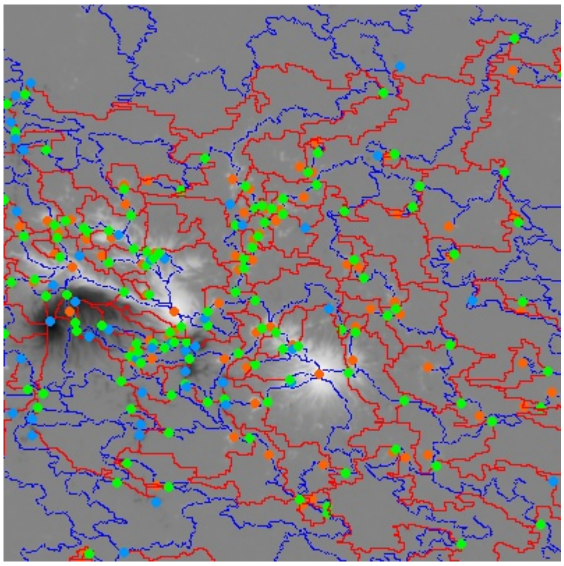

Figure 2. Intermediate result of simplification. Red are maxima, blue are minima, green are saddles. There are 76 maxima, 44 minima, and 117 saddle points (the Euler characteristic $\chi = 3$).

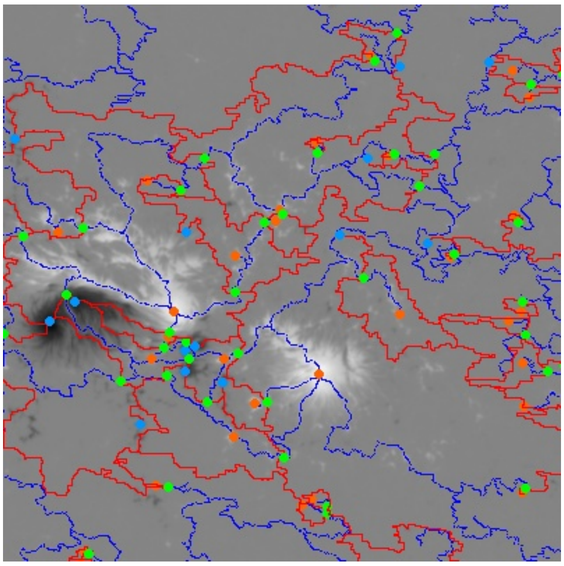

Figure 3. Intermediate result of simplification. There are 29 maxima, 14 minima, and 37 saddle points ($\chi = 6$).

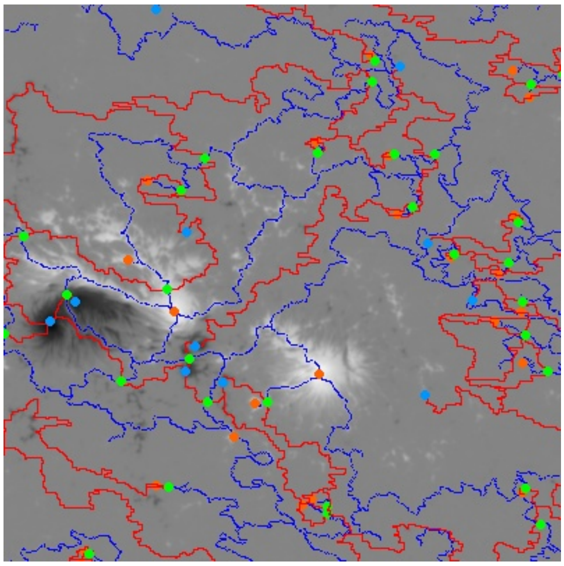

Figure 4. DMC obtained by persistence simplification. Remaining critical points are correspondent to the key topological features. There are 23 maxima, 11 minima, and 29 saddle points ($\chi = 5$).

in [11] implemented for 2D grid (initially, the proposed algorithms were designed for simplicial complex). Some details of implementation we get from Robins et al [10].

3.3. Persistence Simplification
Only a small part of identified topological features (correspondent to critical points of the computed discrete Morse complex) are seemed significant, most of them appear to be noise. the way to eliminate minor topological features is *the persistent pair identification and cancellation.*

For each arc (σ_0, σ_1) of the discrete Morse complex (DMC) we associate a number $|f(\sigma_0) - f(\sigma_1)|$, which is called *persistence* of arc.

Each step of simplification consists of cancellation of pair "maximum – saddle" or "minimum – saddle" connected by arc with the smallest persistence value. After cancellation performed, we recover arcs of DMC incoming the removed vertices.

For more detailed discussion about topological persistence and its applications to discrete Morse theory, we refer to Edelsbrunner et al [1].

4. Results
We have applied a simplification algorithm on the SDO/HMI magnetogram data. We use 2-periodical boundary conditions obtained by reflection and gluing initial rectangle borders in torus $\mathbb{T}^2$. Process of the Morse-Smale simplification presented on Figures 1 – 4. Initial MS-complex contains 5503 maxima, 9330 minima, and 14799 saddle points. MS-complex after editing (Fig. 4) contains 23 maxima, 11 minima, and 29 saddle points.

Our implementation of this algorithm is in C++. Pictures are generated using OpenCV library [9].

5. Future work
Firstly, we are intended to obtain a sequence of edited fields for Active Regions corresponding to flares. After that, we want to look at the dependence of general predictors of field complexity [5] on the simplification level. An other problem is comparison of MS-complexes constructed by the bipolar field with a scalar energy field $B^2/2$.

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