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Yang-Baxter deformations of Minkowski spacetime

Jun-ichi Sakamoto

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

E-mail: `sakajun@gauge.scphys.kyoto-u.ac.jp`

Abstract. We discuss Yang–Baxter sigma deformations of 4D Minkowski spacetime proposed recently. To avoid the degeneracy of the standard bilinear form associated with the familiar coset $ISO(1,3)/SO(1,3)$, we consider a slice of AdS_5 in Poincaré coordinates by embedding the 4D Poincaé group into the 4D conformal group $SO(2,4)$. With this procedure we present the metrics and B-fields as Yang–Baxter deformations which correspond to well-known backgrounds such as T–duals of Melvin backgrounds, Hashimoto–Sethi and Spradlin–Takayanagi–Volovich backgrounds, pp-waves, and T–duals of dS_4 and AdS_4. Finally we consider a deformation with a classical r-matrix of Drinfeld–Jimbo type and explicitly derive the associated metric and B-field.

1. Introduction

Yang–Baxter sigma models are known to describe systematic integrable deformations of two–dimensional non-linear sigma models. They have been proposed in [2] and their classical integrability is shown by constructing the universal expression for the Lax pair [3]. This Klimcik's original work [2] [3] employs deformations of principal chiral models on a Lie group G based on a modified classical Yang–Baxter equation (mCYBE). It has been extended to any symmetric coset spaces [4] and the classical Yang–Baxter equation (CYBE) [5].

An important application of Yang–Baxter sigma models are integrable deformations of the superstring on $\mathrm{AdS}_5 \times S^5$ spacetime, which is abbreviated as $\mathrm{AdS}_5 \times S^5$ superstring. The integrable structure of the $\mathrm{AdS}_5 \times S^5$ superstring has been discovered in [1] in the sense of the existence of a Lax pair. A remarkable deformation that preserves the integrability of the $\mathrm{AdS}_5 \times S^5$ superstring is the q–deformed $\mathrm{AdS}_5 \times S^5$ superstring [6]. The deformation is based on a mCYBE and the associated background including the fermionic sector are given by [42] [43]. Also the Maldacena–Russo background [11] and the Lax pair are obtained from q–deformed $\mathrm{AdS}_5 \times S^5$ superstrings by taking a scaling limit [43] [44].

After that, the above models have been generalized to deformations of the $\mathrm{AdS}_5 \times S^5$ superstring based on a classical Yang–Baxter equation [7]. A CYBE has many (skew-symmetric) classical r–matrices as solutions, in contrast to the mCYBE. Yang–Baxter deformed $\mathrm{AdS}_5 \times S^5$ backgrounds for these classical r–matrices can be identified with the well-known γ–deformation of S^5 [8,9], gravity duals of non-commutative gauge theories [10] [11], Schrödinger spacetimes [12–14] and gravity duals for dipole theories [15–19], as shown in a series of works [20–24]. Remarkably, these deformations can also be applied to non-integrable backgrounds such as $\mathrm{AdS}_5 \times T^{1,1}$ [25]. The deformations of this background [8] [26] can be reproduced as Yang–Baxter deformations [27] . This correspondence between classical r-matrices and deformed backgrounds is called as gravity/CYBE correspondence (for a brief review, see [28]).

Instead of curved spaces, we have generalized this method to flat space in order to study to the applicability of this correspondence [29]. For flat space, there is the obvious problem that the standard bilinear form degenerates if we employ the familiar coset Poincaré group/Lorentz group. A possible resolution is that we consider a conformal embedding of 4D Minkowski spacetime into AdS_5 spacetime in the Poincaré coordinates. This method seems work well because we can reproduce the metric and B–field of well-known backgrounds such as T–duals of Melvin backgrounds [30, 32, 33], Hashimoto–Sethi backgrounds [34], time-dependent backgrounds of Spradlin–Takayanagi–Volovich [35], pp-wave backgrounds, and T–duals of dS_4 and AdS_4. The purpose of this article is to give a short review of this work.

The classical r-matrices considered in this note can be divided into two classes. Yang–Baxter deformations for classical r-matrices of the first class can be interpreted as backgrounds generated by applying a TsT–transformation to 4D Minkowski spacetime in the string theory sense. Since T–duality preserves classical integrability of two dimensional sigma models [36–39], Yang–Baxter sigma models in this class are also classical integrable. In fact, we can construct a general Lax pair in this case [48]. Another class of classical r-matrices containing the dilatation generator $\hat{d}$ correspond to non-twisted backgrounds, such as the T–duals of dS_4 and AdS_4. Although these backgrounds are classically integrable, we have not succeeded in obtaining a formal proof. Hence, it is a conjecture that our Yang–Baxter sigma models for this class have classical integrability.

The structure of this article is as follows. In Section 2, we explain the coset construction of AdS_5 spacetime in Poincaré coordinates. After that we will realize flat space as a slice of the Poincaré AdS_5. In Section 3, we introduce Yang–Baxter deformations of 4D Minkowski spacetime. In Sections 4, we will provide some examples of classical r-matrices and their associated metrics and two-form B–fields. In Section 5, we extend the formulation from the CYBE to the mCYBE and then study a deformation with a classical r-matrix of Drinfeld–Jimbo type. Section 6 is devoted to conclusions and discussion.

2. A coset construction of Minkowski spacetime

Yang–Baxter sigma models need to a non-degenerate bilinear form for their respective group or coset manifolds. The Killing form of the familiar coset space $ISO(1,3)/SO(1,3)$ is however degenerate because the Poincaré algebra is not a semi-simple algebra. A possible resolution is to consider instead an embedding of 4D Minkowski spacetime into the Poincaré AdS_5.

2.1. Coset construction of Poincaré AdS_5 *revisited*

Let us consider AdS_5 spacetime in Poincaré coordinates. For this purpose, it is helpful to use the conformal basis for $\mathfrak{so}(2,4)$:

$$\mathfrak{so}(2,4) = \mathrm{span}_{\mathbb{R}}\{\, p_\mu\,, n_{\mu\nu}\,, \hat{d}\,, k_\mu \mid \ \mu,\nu = 0,1,2,3\,\}\,, \tag{1}$$

where the translation generator p_μ , the Lorentz generators $n_{\mu\nu}$, the dilatation $\hat{d}$ and the special conformal generator k_μ are represented by, respectively,

$$p_\mu \equiv \frac{1}{2}(\gamma_\mu - 2n_{\mu 5})\,, \quad n_{\mu\nu} \equiv \frac{1}{4}[\gamma_\mu, \gamma_\nu]\,, \quad \hat{d} \equiv \frac{1}{2}\gamma_5\,, \quad k_\mu \equiv \frac{1}{2}(\gamma_\mu + 2n_{\mu 5})\,. \tag{2}$$

Here we introduced the gamma matrices γ_μ , $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ and $n_{\mu 5}$ defined by

$$n_{\mu 5} \equiv \tfrac{1}{4}[\gamma_\mu, \gamma_5]\,. \tag{3}$$

The non-vanishing commutation relations are

$$\begin{aligned} [p_\mu, k_\nu] &= 2(n_{\mu\nu} + \eta_{\mu\nu}\,\hat{d}\,)\,, \quad [\hat{d}, p_\mu] = p_\mu\,, \quad [\hat{d}, k_\mu] = -k_\mu\,, \\ [p_\mu, n_{\nu\rho}] &= \eta_{\mu\nu}\,p_\rho - \eta_{\mu\rho}\,p_\nu\,, \quad [k_\mu, n_{\nu\rho}] = \eta_{\mu\nu}\,k_\rho - \eta_{\mu\rho}\,k_\nu\,, \\ [n_{\mu\nu}, n_{\rho\sigma}] &= \eta_{\mu\sigma}\,n_{\nu\rho} + \eta_{\nu\rho}\,n_{\mu\sigma} - \eta_{\mu\rho}\,n_{\nu\sigma} - \eta_{\nu\sigma}\,n_{\mu\rho}\,. \end{aligned} \tag{4}$$

It well known that AdS_5 spacetime is described by a symmetric coset

$$\mathrm{AdS}_5 = \frac{SO(2,4)}{SO(1,4)}\,. \tag{5}$$

Here the Lie algebra $\mathfrak{so}(1,4)$ of the stabilizer group $SO(1,4)$ is given by

$$\mathfrak{so}(1,4) = \mathrm{span}_{\mathbb{R}}\{\ \frac{1}{2}(p_\mu - k_\mu) = -n_{\mu 5}\ |\ \ \mu = 0,1,2,3\ \}\,. \tag{6}$$

So the coset generators are spanned by $\frac{1}{2}(p_\mu + k_\mu) \sim \gamma_\mu$ and $\hat{d} \sim \gamma_5$. A convenient representative of the coset (5) is given by

$$g = \exp\big[\,p_\mu\, x^\mu\,\big]\ \exp\Big[\hat{d}\log z\Big]\ , \tag{7}$$

and makes manifest the $\mathbb{R}^4$ symmetry in the boundary of AdS_5 in Poincaré coordinates. Then, we can easily get the AdS_5 metric in Poincaré coordinates,

$$ds^2 = \mathrm{Tr}(A\overline{P}(A)) = \frac{-(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2 + dz^2}{z^2}\,, \tag{8}$$

where $\overline{P}$ is a coset projector from $\mathfrak{so}(2,4)$ to $\mathfrak{so}(2,4)/\mathfrak{so}(1,4)$ and is defined as

$$\overline{P}(x) \ \equiv\ \gamma_0\,\frac{\mathrm{Tr}(\gamma_0 x)}{\mathrm{Tr}(\gamma_0^2)} + \sum_{i=1}^{3}\gamma_i\,\frac{\mathrm{Tr}(\gamma_i x)}{\mathrm{Tr}(\gamma_i^2)} + \gamma_5\,\frac{\mathrm{Tr}(\gamma_5 x)}{\mathrm{Tr}(\gamma_5^2)} \tag{9}$$

$$=\ \frac{1}{4}\Big[-\gamma_0\ \mathrm{Tr}(\gamma_0 x) + \sum_{i=1}^{3}\gamma_i\ \mathrm{Tr}(\gamma_i x) + \gamma_5\ \mathrm{Tr}(\gamma_5 x)\Big] \qquad \text{for } x \in \mathfrak{so}(2,4)\,. \tag{10}$$

Note that the AdS radius is set to 1 . This coset projector respects the $\mathbb{Z}_2$–grading structure of the coset space $SO(2,4)/SO(1,4)$.

Finally, let us comment on a relation between the projection $\overline{P}$ and the standard method. In the usual way, we expand A as

$$A = g^{-1}dg = e^\mu\frac{1}{2}(p_\mu + k_\mu) + e^5\hat{d} + \omega^\mu\frac{1}{2}(p_\mu - k_\mu)\,, \tag{11}$$

where the vielbein and spin connections are

$$e^\mu = \frac{dx^\mu}{z}\,,\quad e^5 = \frac{dz}{z}\,,\quad \omega^\mu = \frac{dx^\mu}{z}\,. \tag{12}$$

Then it is easy to get the metric on AdS_5 spacetime by computing

$$ds^2 = \eta_{\mu\nu}e^\mu e^\nu + e^5 e^5\,. \tag{13}$$

Now using the relations

$$e^\mu = \frac{1}{2}\,\mathrm{Tr}(\gamma^\mu A)\,,\quad e^5 = \mathrm{Tr}(\hat{d}A)\,, \tag{14}$$

this expression can be rewritten as

$$\begin{aligned} ds^2 &= \frac{1}{4}\eta_{\mu\nu}\mathrm{Tr}(\gamma^\mu A)\mathrm{Tr}(\gamma^\nu A) + \mathrm{Tr}(\hat{d}A)\mathrm{Tr}(\hat{d}A) \\ &= \mathrm{Tr}\left[A\,\frac{1}{4}\eta^{\mu\nu}\gamma_\mu\mathrm{Tr}(\gamma_\nu\,A)\right] + \mathrm{Tr}\Big[A\,\hat{d}\mathrm{Tr}(\hat{d}A)\Big] \\ &= \mathrm{Tr}(A\overline{P}(A))\,. \end{aligned} \tag{15}$$

Thus our method using the coset projector is equivalent to the standard manner.

2.2. A conformal embedding of 4D Minkowski spacetime

Now we can get the coset space for 4D Minkowski spacetime by embedding $ISO(1,3)/SO(1,3)$ into the Poincaré AdS$_5$. A representative of the group element g is

$$g = \exp\left[\, p_\mu\, x^\mu \,\right] . \tag{16}$$

Unlike the coset representative for the Poincaré AdS$_5$ (7), the radial coordinate z does not appear in the above expression. It implies that we realize 4D Minkowski spacetime as a slice of $z = 1$ of the Poincaré AdS$_5$.

Note here that the 4D Poincaré algebra $\mathfrak{iso}(1,3)$ and the 4D Lorentz algebra $\mathfrak{so}(1,3)$ are generated by the following generators, respectively,

$$\begin{aligned} \mathfrak{iso}(1,3) &= \mathrm{span}_{\mathbb{R}}\{\, n_{\mu\nu}\,, p_\mu \mid \mu,\nu = 0,1,2,3 \,\}\,, \\ \mathfrak{so}(1,3) &= \mathrm{span}_{\mathbb{R}}\{\, n_{\mu\nu} \mid \mu,\nu = 0,1,2,3 \,\}\,. \end{aligned} \tag{17}$$

Thus it makes sense to use the generators p_μ to parameterize the coset representative of $ISO(1,3)/SO(1,3)$ as (16). Eventually the left-invariant one-form $A = g^{-1}dg$ is written as a linear combination of p_μ .

To avoid the problem of the degeneracy of the Poincaré group $ISO(1,3)$, we can use the projector for 4D Minkowski spacetime

$$P(x) = \frac{1}{4}\Big[-\gamma_0\ \mathrm{Tr}(\gamma_0 x) + \sum_{i=1}^{3} \gamma_i\ \mathrm{Tr}(\gamma_i x)\Big] \qquad \text{for} \quad x \in \mathfrak{so}(2,4)\,. \tag{18}$$

Here due to the dimension of the coset space, we have dropped $\gamma_5 \sim \hat{d}$ from the projector $\overline{P}$. Because the projected one-form $P(A)$ is expanded in terms of $\gamma_\mu (\mu = 0,1,2,3)$, we can obtain the flat metric

$$ds^2 = \mathrm{Tr}(AP(A)) = -(dx^0)^2 + \sum_{i=1}^{3} (dx^i)^2\,. \tag{19}$$

Note that this computation is equivalent to the standard method using the vierbein like in the Poincaré AdS$_5$ case. This result is the starting point of our argument in the following.

3. Yang–Baxter sigma models for 4D Minkowski spacetime

In this section, we consider Yang–Baxter deformations of two-dimensional sigma models with 4D Minkowski spacetime as target space. Next we present a classification of classical r-matrices with values in $\mathfrak{so}(2,4) \otimes \mathfrak{so}(2,4)$.

3.1. Deformed action

Following the above coset construction of flat space, we introduce Yang–Baxter sigma models for 4D Minkowski spacetime.

The deformed action is given by[1]

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\ \mathrm{Tr}\left[A_\alpha P \circ \frac{1}{1 - 2\eta R_g \circ P}(A_\beta)\right], \tag{20}$$

[1] Here the string tension $T = \frac{1}{2\pi\alpha'}$ is set to 1, and the conformal gauge is taken so as to drop the dilaton coupling to the world-sheet scalar curvature.

where $A_\alpha = g^{-1}\partial_\alpha g$ and g is given in Eq. (16). Here η is a deformation parameter and the action (20) is reduced to the undeformed one for $\eta = 0$. $\gamma_{\alpha\beta} = \mathrm{diag}(-1,1)$ and $\epsilon^{\alpha\beta}$ are the world sheet metric and the anti-symmetric tensor that is normalized as $\epsilon^{\tau\sigma} = 1$, respectively. The operator R_g is defined as

$$R_g \equiv g^{-1}R(gXg^{-1})g\,, \tag{21}$$

where a linear operator $R : \mathfrak{so}(2,4) \to \mathfrak{so}(2,4)$ is a solution of the CYBE,

$$\big[R(M), R(N)\big] - R\left([R(M), N] + [M, R(N)]\right) = 0\,, \qquad M, N \in \mathfrak{so}(2,4)\,. \tag{22}$$

The R-operator is related to the *skew-symmetric* classical r-matrix in the tensorial notation through

$$R(X) = \mathrm{Tr}_2[r(1 \otimes X)] = \sum_i (a_i\,\mathrm{Tr}(b_i X) - b_i\,\mathrm{Tr}(a_i X))\,, \tag{23}$$

where the classical r-matrix is given by

$$r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i) \tag{24}$$

satisfying the CYBE,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\,. \tag{25}$$

The generators a_i, b_i are elements of $\mathfrak{so}(2,4)$.

To obtain deformed backgrounds, it is convenient to rewrite the deformed Lagrangian as

$$L = -\frac{1}{2}\,\mathrm{Tr}\left[A_- P(J_+)\right]\,, \tag{26}$$

where $J_\pm$ is a deformed current defined as

$$J_\pm \equiv \frac{1}{1 \mp 2\eta R_g \circ P}\, A_\pm\,, \qquad A_\pm = A_\tau \pm A_\sigma\,. \tag{27}$$

By solving the equations

$$(1 \mp 2\eta P \circ R_g) P(J_\pm) = P(A_\pm)\,, \tag{28}$$

the projected deformed current $P(J_\pm)$ is determined. Then the metric and NS-NS two-form are given by the symmetric and skew-symmetric parts regarding to worldsheet coordinates in the deformed Lagrangian, respectively.

3.2. Classification of classical r-matrices

One can organize the classical r-matrices as follows:

(a) $r =$ Poincaré $\wedge$ Poincaré
  1. abelian e.g., $r \sim p_3 \wedge n_{12}$, 2. non-abelian e.g., $r \sim p_1 \wedge n_{12}$,

(b) $r =$ Poincaré $\wedge$ non-Poincaré
  1. abelian e.g., $r \sim n_{12} \wedge \hat{d}$, 2. non-abelian e.g., $r \sim p_0 \wedge \hat{d}$,

(c) $r =$ non-Poincaré $\wedge$ non-Poincaré
  1. abelian e.g., $r \sim k_1 \wedge k_2$, 2. non-abelian e.g., $r \sim k_0 \wedge \hat{d}$.

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Here, given a classical r-matrix $r = a \wedge b$, the word (non-)abelian means that a and b (do not) commute each other.

We consider Yang–Baxter deformations of 4D Minkowski spacetime for the classes (a)-1 and (b)-2 in the following. The class (a)-1 corresponds to classical r-matrices associated with various kinds of TsT–transformations. Let α and β be dual coordinates for the generators a and b in the sense of the Lie algebra. Concretely for $r \sim p_3 \wedge n_{12}$, we take $\alpha = x^3$ and $\beta = \theta$ is the angular coordinate in the x^1-x^2 plane. Then the corresponding TsT–transformation is the following sequence of T–dualities and shifts: 1. T–dualize along α to $\tilde{\alpha}$; 2. Replace $\tilde{\alpha}$ with $\tilde{\alpha} + \eta\,\beta$; 3. Finally, T–dualize along the $\tilde{\alpha}$ direction. Of course the coordinates α and β are compactified with periodic boundary conditions under this sequence. Since we have succeeded to construct Lax pairs for these r-matrices, Yang–Baxter sigma models for the class (a)-1 certainly describe integrable deformations of 4D-Minkowski spacetime.

In the case of the class (b)-2, T-duals of dS_4 and AdS_4 are realized from $r \sim p_0 \wedge \hat{d}$ and $r \sim p_3 \wedge \hat{d}$, respectively. These backgrounds cannot be reproduced by applying TsT–transformations to flat space like for the class (a)-1. Although two-dimensional non-linear sigma models with these target spaces are classically integrable, we do not have a formal proof or explicit construction of the Lax pairs. Hence it is a conjecture that Yang–Baxter sigma models in the class (b)-2 are classically integrable. For a more comprehensive list of classical r-matrices, see [47].

4. Deformed backgrounds

4.1. TsT–duals of flat space

In this subsection, we present deformed backgrounds associated with classical r-matrices in the class (a)-1. Here we assume the extra six dimensions to be flat (as resulting from a T^6-compactification) and we introduce only a dilaton field. All backgrounds are identified with a TsT–transformation of flat space.

T-dual of Melvin background First, let us consider the r-matrix

$$r = \frac{1}{2} p_3 \wedge n_{12}\,. \tag{29}$$

The associated deformed metric and B–field are T–dual to a Melvin background:

$$\begin{aligned} ds^2 &= -(dx^0)^2 + dr^2 + \frac{r^2 d\theta^2 + (dx^3)^2}{1+\eta^2 r^2}\,, \\ B &= \frac{\eta r^2}{1+\eta^2 r^2}\, d\theta \wedge dx^3\,, \end{aligned} \tag{30}$$

where we have performed a coordinate transformation,

$$x^1 = r\,\cos\theta\,, \qquad\qquad x^2 = r\,\sin\theta\,. \tag{31}$$

It should be remarked that the Yang–Baxter deformations cannot reproduce the associated dilaton, although it may be possible to perform a supercoset construction in principle. However we can embed the background into string theory by observing that the one-loop beta function vanishes when adding the dilaton [31–33]

$$\Phi = -\frac{1}{2}\log(1+\eta^2 r^2)\,. \tag{32}$$

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This background can also be constructed by applying a sequence of a TsT–transformation to flat space as noted above. More concretely, the sequence in this case is 1.T–duality in x^3; 2. shift $\theta \to \theta + \eta\, \tilde{x}^3$; 3.T–duality in $\hat{x}^3$. Consistency requires $\tilde{x}^3$ to be periodic with period $\tilde{x}^3 \simeq \tilde{x}^3 + 2\pi/\eta$ and x^3 with period[2] $x^3 \simeq x^3 + \alpha'\eta/(2\pi)$.

pp-wave background Let us consider the classical r-matrix

$$r = \frac{1}{2\sqrt{2}}(p_0 - p_3) \wedge n_{12}\,. \tag{33}$$

The associated deformed background is

$$\begin{aligned} ds^2 &= -2dx^+ dx^- - \eta^2 r^2 (dx^+)^2 + (dr)^2 + r^2 d\theta^2\,, \\ B &= \eta r^2\, dx^+ \wedge d\theta\,, \end{aligned} \tag{34}$$

where we have introduced the polar coordinate system for x^1 and x^2 given by

$$x^1 = r\cos\theta\,, \qquad x^2 = r\sin\theta\,. \tag{35}$$

This is a pp-wave background which can also be understood as a generalization of a (null) TsT transformation obtained as 1. a T–duality from θ to $\tilde{\theta}$, followed by 2. the shifts $x^0 \to \eta\tilde{\theta} + x^0$, $x^3 \to -\eta\tilde{\theta} + x^3$ and the final 3. T–duality from $\tilde{\theta}$ to θ.

Note that this requires the identifications $x^0 \simeq x^0 + \alpha'\eta/(2\pi)$ and $x^3 \simeq x^3 + \alpha'\eta/(2\pi)$.

Hashimoto–Sethi background Next we consider the following abelian classical r-matrix

$$r = \frac{1}{2\sqrt{2}} p_2 \wedge (n_{01} + n_{13})\,. \tag{36}$$

The resulting metric and B–field are given by

$$\begin{aligned} ds^2 &= -2dx^- dx^+ + \frac{1}{1+\eta^2(x^+)^2}\left[(dx^1)^2 + (dx^2)^2 + \eta^2 x^1 dx^+ (2x^+ dx^1 - x^1 dx^+)\right], \\ B &= \frac{\eta}{1+\eta^2(x^+)^2}(x^1 dx^+ - x^+ dx^1) \wedge dx^2\,. \end{aligned} \tag{37}$$

Note that this background depends on the light-cone time x^+ explicitly. The associated dilaton to complete the string embedding is taken to be

$$\Phi = -\frac{1}{2}\log(1 + \eta^2 (x^+)^2)\,. \tag{38}$$

The metric and B–field (37) agree with those of the Hashimoto–Sethi background. To show this agreement, one has to introduce new coordinates,

$$x^+ = y^+\,, x^1 = y^+ y\,, x^- = y^- + \frac{1}{2}y^+(y)^2, x^2 = -z\,. \tag{39}$$

Under this coordinate transformation, the above deformed background (37) becomes

$$\begin{aligned} ds^2 &= -2dy^+ dy^- + \frac{(y^+)^2 (dy)^2 + dz^2}{1+\eta^2(y^+)^2}\,, \\ B &= \frac{\eta\,(y^+)^2}{1+\eta^2(y^+)^2} dy \wedge dz\,. \end{aligned} \tag{40}$$

This reproduces the expression in Eq. (25) of [34] where the background is shown to be the result of a TsT–transformation for $U(1) \times U(1)$ isometries which are shifts of the y and z directions.

[2] We reintroduce the explicit parameter α' in the identifications to manifestly illustrate the dimensions of the variables.

Spradlin–Volovoich–Takayanagi background Let us here consider the classical r-matrix

$$r = \frac{1}{2} n_{12} \wedge n_{03} \,. \tag{41}$$

Then the associated metric and B–field are given by

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + \frac{r^2 d\theta^2 + t^2 d\phi^2}{1 + \eta^2 r^2 t^2} \,, \\ B &= \frac{\eta r^2 t^2}{1 + \eta^2 r^2 t^2} d\phi \wedge d\theta \,, \end{aligned} \tag{42}$$

where we have introduced new coordinates

$$x^0 = t\, \cosh\phi \,, \qquad x^3 = t\, \sinh\phi \,. \tag{43}$$

Note that the coordinates in (43) do not cover the whole x^0-x^3 plane and the background (42) contains no singularity. Then the metric and B–field in (42) agree with those of (6.1) in [35]. This is a time-dependent background realized by a TsT–transformation of Minkowski spacetime on the torus generated by ϕ and θ. The associated dilaton is

$$\Phi = -\frac{1}{2} \log(1 + \eta^2 r^2 t^2) \,. \tag{44}$$

4.2. Non-Twist cases

We consider Yang-Baxter deformations of type (b)-1 in the following. In this article, we present two examples, the T–duals of dS_4 and AdS_4, associated with classical r-matrices that contain the dilation generator $\hat{d}$. One can find more examples for this type in [29].

T–dual of dS_4 Let us consider the non-abelian r-matrix

$$r = \frac{1}{2} p_0 \wedge \hat{d} \quad \text{with} \quad [\hat{d}\,, p_0] = p_0 \,. \tag{45}$$

This r-matrix was used to deform the conformal algebra $\mathfrak{so}(2,4)$ in [41]. The resulting background is

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + dr^2}{1 - \eta^2 r^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \,, \\ B &= \frac{\eta r}{1 - \eta^2 r^2} dx^0 \wedge dr \,, \end{aligned} \tag{46}$$

where we have introduced new coordinates r , θ and ϕ through

$$x^1 = r \cos\phi\, \sin\theta \,, \qquad x^2 = r \sin\phi \sin\theta \,, \qquad x^3 = r\cos\theta \,. \tag{47}$$

Note here that the above B–field can be rewritten in the form of a total derivative. One can understand this background as T–dual of four dimensional de Sitter space by following transformation. Firstly, we take a timelike T–duality along the x^0 direction [40]. The deformed metric becomes

$$ds^2 = (dr + \eta r\, dx^0)^2 - (dx^0)^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \,. \tag{48}$$

Note that the B–field has disappeared due to the T–duality along the x^0 direction. Next we introduce the new coordinate

$$x^0 = t - \frac{1}{2\eta} \log(\eta^2 r^2 - 1) \,. \tag{49}$$

In this way, we get the well-known metric of dS_4 in static coordinates,

$$ds^2 = -(1 - \eta^2 r^2) dt^2 + \frac{dr^2}{1 - \eta^2 r^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{50}$$

Note that there is a cosmological horizon at $r = 1/\eta$.

T–dual of AdS_4 As another example, let us consider the classical r-matrix

$$r = \frac{1}{2}\hat{d}\wedge p_1 \quad \text{with} \quad [\hat{d}\,, p_1] = p_1\,. \tag{51}$$

This r-matrix also contains the dilatation $\hat{d}$. The associated metric and B–field are given by

$$\begin{aligned} ds^2 &= \frac{dt^2 + (dx^1)^2}{1+\eta^2 t^2} + t^2\cosh^2\phi d\theta^2 - t^2 d\phi^2\,, \\ B &= \frac{\eta t}{1+\eta^2 t^2} dt\wedge dx^1\,, \end{aligned} \tag{52}$$

where we have introduced new coordinates t, θ and ϕ through

$$x^0 = t\,\sinh\phi\,, \qquad x^2 = t\,\cos\theta\,\cosh\phi\,, \qquad x^3 = t\,\sin\theta\,\cosh\phi\,. \tag{53}$$

Note here that the B–field can be recast as a total derivative.

As in the previous case, it is nice to perform a T–duality along the x^1-direction. Then the resulting background is given by[3]

$$ds^2 = (dt - \eta t\,dx^1)^2 + (dx^1)^2 + t^2(-d\phi^2 + \cosh^2\phi\,d\theta^2)\,. \tag{54}$$

Now the B–field has disappeared. Let us perform a coordinate transformation,

$$x^1 = y + \frac{1}{2\eta}\log(\eta^2 t^2 + 1)\,. \tag{55}$$

Then the resulting metric is given by

$$ds^2 = (1+\eta^2 t^2)dy^2 + \frac{dt^2}{1+\eta^2 t^2} + t^2(-d\phi^2 + \cosh^2\phi\,d\theta^2)\,. \tag{56}$$

By replacing the coordinates (with a double Wick rotation) by

$$y \to it\,, \quad t\to r\,, \quad \phi \to i\theta\,, \quad \theta\to\phi\,, \tag{57}$$

one can obtain the standard metric of AdS_4 with global coordinates

$$ds^2 = -(1+\eta^2 r^2)dt^2 + \frac{dr^2}{1+\eta^2 r^2} + r^2(d\theta^2 + \cos^2\theta\,d\phi^2)\,. \tag{58}$$

Note that η^2 measures the curvature.

5. A Yang–Baxter deformation with mCYBE

So far, we have considered classical r-matrices satisfying the classical Yang–Baxter equation (25) (or (22)). Here, as an exceptional case, let us consider a classical r-matrix of Drinfeld–Jimbo (DJ) type,

$$r_{\mathrm{DJ}} = -i\sum_{1\le i<j\le 4} E_{ij}\wedge E_{ji}\,, \qquad (E_{ij})_{kl} \equiv \delta_{ik}\delta_{jl}\,, \tag{59}$$

[3] Note that, at this stage, one can see that this metric describes AdS_4 by explicitly computing the scalar curvature and the Ricci tensor.

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which satisfies the modified Yang–Baxter equation,

$$\big[R(M), R(N)\big] - R\left([R(M), N] + [M, R(N)]\right) = [M, N]\,, \tag{60}$$

where $M, N \in \mathfrak{so}(2,4)$. In comparison to the CYBE in (22), the right-hand side of (60) is modified. The overall factor of the classical r-matrix is determined by a reality condition of the deformed B–field. The r-matrix (59) was used to construct an integrable deformation of the $\mathrm{AdS}_5 \times S^5$ superstring [6]. The deformed metric and B–field were explicitly computed in [42].

Since we are only interested in the associated background, the computation scheme is identical to the one of the CYBE case.

The resulting metric and B–field are given by

$$\begin{aligned} ds^2 &= -r^2 \sin^2\theta\, dt^2 + dr^2 + \frac{r^2}{1+\eta^2 r^4 \sin^2\theta}\left(d\theta^2 + \cos^2\theta\, d\phi^2\right)\,, \\ B &= -\frac{\eta\, r^4 \sin\theta\cos\theta}{1+\eta^2 r^4\sin^2\theta}\, d\theta\wedge d\phi\,. \end{aligned} \tag{61}$$

Here we have performed a coordinate transformation,

$$\begin{aligned} x^0 &= r\sin\theta\,\sinh t\,, \qquad x^1 = r\cos\theta\,\cos\phi\,, \\ x^2 &= r\cos\theta\,\sin\phi\,, \qquad x^3 = r\,\sin\theta\,\cosh t\,, \end{aligned} \tag{62}$$

and rescaled $\eta \to \eta/2$. It is worth noting that the metric in (61) is be regular as opposed to the case of the q-deformed AdS_5 superstring. The scalar curvature has no singularity. The singular surface of the metric identified in [42] has not appeared due to the fact that we are now working on a slice of the Poincaré AdS_5 at $z = 1$.

6. Conclusion and Discussion

In this article, we have presented a brief review of Yang–Baxter deformations of 4D Minkowski spacetime. The essential point is to realize 4D Minkowski spacetime by embedding the Poincaré AdS_5 to avoid the degeneracy of the Killing form for a standard coset $ISO(1,3)/SO(1,3)$. Following this prescription, we can reproduce the metric and B–field of well-known backgrounds such as T–duals of Melvin backgrounds, Hashimoto–Sethi backgrounds, time-dependent backgrounds of Spradlin–Takayanagi–Volovich, pp-wave backgrounds, and T–duals of dS_4 and AdS_4. Finally, we have considered the Yang–Baxter deformation for Drinfeld–Jimbo r-matrix and present the associated deformed background explicitly. The list of correspondences between deformed backgrounds and classical r-matices is given in Table 1.

There are many open questions. Since the Lax pairs associated with classical r-matrices for the class (a)-1 have been constructed explicitly, our Yang–Baxter sigma model describe integrable deformations in this class at least. However we have not yet constructed Lax pairs for the class (b)-2, although the sigma models with the resulting deformed backgrounds are classically integrable. It would be interesting to study the form of the Lax pairs and the associated deformed algebra for these cases.

In particular, deformed Poincaré algebras are studied in [45] in terms of classical r-matrices. These r-matrices are contained in the class (a). As a first step to clarify the relations between the list [47] and our results, the associated backgrounds and corresponding Lax pairs for special cases of the class (a)-2 that describe so-called κ-deformations of the 4D Poincaré algebra are given in a forthcoming paper [49].

r-matrix	Type of Twist	Background
$p_i \wedge p_j \ (i,j=1,2,3)$	Melvin Shift Twist	Seiberg-Witten
$p_0 \wedge p_i$	Melvin Shift Twist	NCOS
$(p_0+p_i)\wedge p_j \ (i \neq j)$	Null Melvin Shift Twist	light-like NC
$\frac{1}{2}p_3 \wedge n_{12}$	Melvin Twist	T–dual Melvin
$\frac{1}{2\sqrt{2}}p_2 \wedge (n_{01}+n_{13})$	Melvin Null Twist	Hashimoto–Sethi
$\frac{1}{2}n_{12} \wedge n_{03}$	R Melvin R Twist	Spradlin–Takayanagi–Volovich
$\frac{1}{2}p_1 \wedge n_{03}$	Melvin Boost Twist	T–dual of Grant space
$\frac{1}{2\sqrt{2}}(p_0-p_3)\wedge n_{12}$	Null Melvin Twist	pp-wave
$\frac{1}{2\sqrt{2}}(\hat{d}-n_{03})\wedge(p_0-p_3)$	Non-Twist	pp-wave
$\frac{1}{2}\hat{d}\wedge p_0$	Non-Twist	T–dual of dS_4
$\frac{1}{2}\hat{d}\wedge p_1$	Non-Twist	T–dual of AdS_4
DJ-type (mCYBE)	Non-Twist	q-deformation (?)

Table 1. A catalog of classical r-matrices and the associated backgrounds.

It would be most important to generalize our argument to 10D Minkowski spacetime in order to extend our argument to a consistent string theory. For this purpose, we have to consider the 10D conformal group $SO(2,10)$ and realize 10D Minkowski spacetime as a slice of 11D AdS space in Poincaré coordinates. We expect that in this case, 10D supersymmetric configurations like the fluxtrap backgrounds [46] could be reproduced as Yang–Baxter deformations.

Our discussion has been constrained to the bosonic sector. As opposed to deformations of $\mathrm{AdS}_5 \times S^5$, it is an easier task to generalize our computations to a supercoset construction than the $\mathrm{AdS}_5 \times S^5$ case. In recent work, deformations of the $\mathrm{AdS}_5 \times S^5$ superstring for the class (a)-1 have been reinterpreted as twisted boundary conditions of the undeformed $\mathrm{AdS}_5 \times S^5$ [44]. We can apply a similar analysis to the flat space case. Thus since for the class (a)-1 deformed sigma models on Yang–Baxter deformed Minkowski spacetime are equivalent to undeformed sigma models up to boundary conditions, one can quantize it and get its quantum spectrum. It would be very interesting to investigate the relation between the string spectrum and Yang–Baxter deformations including the fermionic sector.

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