

# Quantum affine algebras and universal functional relations

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**Abstract.** By the universal integrability objects we mean certain monodromy-type and transfer-type operators, where the representation in the auxiliary space is properly fixed, while the representation in the quantum space is not. This notion is actually determined by the structure of the universal R-matrix. We call functional relations between such universal integrability objects, and so, being independent of the representation in the quantum space, the universal functional relations. We present a short review of the universal functional relations for the quantum integrable systems associated with the quantum groups of loop Lie algebras.

## 1. Introduction

The discovery of the so-called *Hidden Grassmann Structure* in the state space of the XXZ model [1–5] opened principally novel perspectives in the study of correlation functions in quantum integrable systems. At the same time, it raised fundamental questions about the algebraic origin of the fermionic basis in the XXZ model and its possible generalizations. Trying to give any adequate response to this and other related problems, we came to the conclusion that we need a better understanding of Baxter’s *TQ*-relations [6,7], with all their ingredients and possible generalizations. Then, at certain point of our study, we came across an earlier series of remarkable papers on integrable structure of conformal field theory [8–11]. For us, the main idea of these pioneer works was that the objects describing the model and related to its integrability should be constructed on the basis of the *universal R-matrix*, which is an element of the tensor product of two copies of the quantum group [12]. Traditionally, the first factor in this tensor product is regarded as the *auxiliary space*, and the second one as the *quantum space*. But their roles can certainly be interchanged, giving rise to various interesting relations between different integrable structures. The functional relations, supposed to be a substitute and generalization of the famous Bethe Ansatz [13], should follow from the characteristics of the appropriate representations of the underlying quantum group. Altogether, our deliberations over the contents of these papers made clear that, in the theory of quantum integrable systems, the functional relations are too important to treat them superficially. And so, we have carefully reconsidered the whole quantum group approach to quantum integrable systems. Essential part of this revision is actually what we are presenting here in a short review based mainly on our publications [14–19].



By  $\mathcal{L}(\mathfrak{g})$  we denote the loop Lie algebra of a finite-dimensional Lie algebra  $\mathfrak{g}$  and by  $\tilde{\mathcal{L}}(\mathfrak{g})$  its standard central extension [20]. The corresponding quantum group is denoted by adding the symbol  $U_q$ , as appropriate for the  $q$ -deformation of the universal enveloping algebra of a given Lie algebra. It is assumed here that  $q$  is not a root of unity. To construct integrability objects one uses spectral parameters. They are introduced by means of a  $\mathbb{Z}$ -gradation of the quantum group under consideration. In the case of our interest, the quantum group  $U_q(\mathcal{L}(\mathfrak{sl}_n))$ , a  $\mathbb{Z}$ -gradation is determined by  $n$  integers  $s_i, i = 0, \dots, n-1$ , and we use the notation  $s = s_0 + \dots + s_{n-1}$  for their total sum. Hence, the functional relations are certain difference equations for the integrability objects considered as functions of the spectral parameters. When the Bethe Ansatz is not applicable anymore, the functional relations can serve to diagonalize the transfer matrix of such models. However, they can also be studied independently to elucidate certain algebraic structures of quantum integrable systems. In particular, they can help in understanding the group-theoretic background of the Hidden Grassmann Structure and see ways to extend it to other models. It turns out here that it is justified to fix the auxiliary space by an appropriate representation, while keeping the quantum space free. Possible advantage of such an approach was mentioned earlier in different contexts, see, for example, [21, 22]. In our recent works [16–19], we have developed and elaborated this idea and have given the complete proof of the functional relations in the form independent of the representation of the quantum group in the quantum space. We have also described the specialization of the universal functional relations to the case when the quantum space is the state space of a discrete spin chain. In this presentation we consider only the first part of the program, the universal integrability objects and the corresponding universal functional relations without their specialization in the quantum space.

We use the notation  $\kappa_q = q - q^{-1}$ , so that the definition of the  $q$ -number can be written as

$$[v]_q = \frac{q^v - q^{-v}}{q - q^{-1}} = \kappa_q^{-1}(q^v - q^{-v}), \quad v \in \mathbb{C}.$$

## 2. Quantum groups in general

Here we recollect basic definitions of a well-known object and fix the corresponding notations. The object in question, denoted here by  $\mathcal{A}$ , is nothing but the *quantum group*, being actually the main tool for the whole construction. We generally follow the definitions proposed by Drinfeld [23, 24] and Jimbo [25]; for a review see [26]. Hence, it is a unital associative algebra obtained by a deformation of the universal enveloping algebra of a given Lie algebra. The quantum group is thus an algebra, and if the initial Lie algebra is an affine Kac–Moody Lie algebra, then one deals with a *quantum affine algebra*. This is the case of our consideration.

It should be noted here that, depending on the nature of the deformation parameter  $q$ , the quantum group can be interpreted differently. If  $q = \exp \hbar$ , where  $\hbar$  is an indeterminate, then the quantum group is a  $\mathbb{C}[[\hbar]]$ -algebra. If  $q$  itself is an indeterminate, then the quantum group is a  $\mathbb{C}(q)$ -algebra. If  $q = \exp \hbar$ , where  $\hbar$  is a complex number, then the quantum group is a  $\mathbb{C}$ -algebra. We deal with the quantum group defined according to the last way in order to have meaningful traces, see, for example, [27, 28].

The quantum group is a Hopf algebra with respect to appropriate co-multiplication, antipode and co-unit. Moreover, as any Hopf algebras, it also has an opposite co-multiplication  $\Delta^{\text{op}}$ , defined by the initial co-multiplication  $\Delta$  as

$$\Delta^{\text{op}} = \Pi \circ \Delta, \quad \Pi(a \otimes b) = b \otimes a, \quad a, b \in \mathcal{A}.$$

Now, assuming that there exists an invertible element  $\mathcal{R}$  of the tensor product of two copies of the quantum group, such that

$$\Delta^{\text{op}}(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1}, \quad \mathcal{R} \in \mathcal{A} \otimes \mathcal{A}, \quad (2.1)$$

for any element  $a$  of the quantum group, and the following relations

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}, \quad (2.2)$$

with the standard convention about the indices, are fulfilled, we restrict ourselves by the so-called *quasi-triangular* Hopf algebras.<sup>1</sup> The element  $\mathcal{R}$  is called the *universal R-matrix*. It follows from (2.1) and (2.2) that it satisfies the *master equation*

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} \quad (2.3)$$

defined on the tensor product of three copies of the quantum group. However, it is important to note that in the case under consideration the universal  $R$ -matrix belongs actually to a smaller subset of the tensor square of the full quantum group; to be precise, to the tensor product of the Borel subalgebras  $\mathcal{B}_+$  and  $\mathcal{B}_-$  of the quantum group  $\mathcal{A}$ , i. e.  $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_- \subset \mathcal{A} \otimes \mathcal{A}$ . Therefore, and in particular, the master equation is defined on the tensor product  $\mathcal{B}_+ \otimes \mathcal{A} \otimes \mathcal{B}_-$ .

### 3. Universal integrability objects

Let us introduce a group-like element, that is an invertible element of the quantum group with the co-multiplication property given by

$$\Delta(t) = t \otimes t, \quad t \in \mathcal{A}.$$

With the help of the universal  $R$ -matrix we define the *universal integrability objects*. These are the monodromy- and transfer-type operators defined as follows.

First, let  $\varphi$  be a representation of the quantum group  $\mathcal{A}$  in a vector space  $V^2$

$$\varphi : \mathcal{A} \rightarrow \text{End}(V).$$

Then, the corresponding universal monodromy operator  $\mathcal{M}_\varphi$  is defined as

$$\mathcal{M}_\varphi = (\varphi \otimes \text{id})(\mathcal{R}) \in \text{End}(V) \otimes \mathcal{A}. \quad (3.1)$$

The corresponding universal transfer operator is defined as

$$\mathcal{T}_\varphi = (\text{tr}_V \otimes \text{id})(\mathcal{M}_\varphi(\varphi(t) \otimes 1)) = ((\text{tr}_V \circ \varphi) \otimes \text{id})(\mathcal{R}(t \otimes 1)), \quad (3.2)$$

where 1 means the unit element of  $\mathcal{A}$ , and now  $t$  makes sense as a *twist element*. Using appropriate  $t$ , in general, we can obtain meaningful traces over infinite-dimensional representation spaces, see, for example, [17]. The representation  $\varphi$  can depend on certain parameters, which we collectively denote by  $\zeta$ . In such a case we write  $\varphi_\zeta$  and  $\mathcal{M}_\varphi(\zeta)$ ,  $\mathcal{T}_\varphi(\zeta)$ , respectively.

The other universal integrability objects are introduced in a similar way, but the representation  $\rho$  used for them is essentially different from the representation  $\varphi$  used earlier. Indeed, here we recall that, in the case under consideration, the universal  $R$ -matrix is an element of the tensor product of two principal Borel subalgebras of  $\mathcal{A}$ , that is  $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ . Then, let  $\rho$  be such a representation of  $\mathcal{B}_+$  that it cannot be extended to a representation of the whole quantum group  $\mathcal{A}$ . Therefore, it cannot be obtained simply by the restriction of  $\varphi$  (from  $\mathcal{A}$  to  $\mathcal{B}_+$ ). We should understand, however, that to produce nontrivial relations between the

<sup>1</sup> In fact, there is a subtle conflict between the definition of the quantum group as a  $\mathbb{C}$ -algebra and the last requirement. However, this problem is resolvable in our case, see, for example, the discussions in [17–19, 29].

<sup>2</sup> In other words,  $V$  is an  $\mathcal{A}$ -module.

integrability objects, these representations,  $\varphi$  and  $\rho$ , should be somehow connected. Now, with such a representation of  $\mathcal{B}_+$  in a vector space  $W$ ,<sup>3</sup>

$$\rho : \mathcal{B}_+ \rightarrow \text{End}(W),$$

we introduce the universal  $L$ -operator

$$\mathcal{L}_\rho = (\rho \otimes \text{id})(\mathcal{R}) \in \text{End}(W) \otimes \mathcal{B}_- \tag{3.3}$$

and the corresponding universal  $Q$ -operator

$$\mathcal{Q}_\rho = (\text{tr}_W \otimes \text{id})(\mathcal{L}_\rho(\rho(t) \otimes 1)) = ((\text{tr}_W \circ \rho) \otimes \text{id})(\mathcal{R}(t \otimes 1)), \tag{3.4}$$

where the group-like element  $t$  is the same as for the universal transfer-operator, and 1 again means the unit of the quantum group.

Further, we can rewrite the master equation (2.3) as follows:

$$(\mathcal{R}^{13}t^1)(\mathcal{R}^{23}t^2) = (\mathcal{R}^{12})^{-1}(\mathcal{R}^{23}t^2)(\mathcal{R}^{13}t^1)\mathcal{R}^{12}. \tag{3.5}$$

Using the tensor products of the maps  $(\text{tr} \circ \varphi_\zeta)$ ,  $(\text{tr} \circ \rho_\zeta)$  for various  $\varphi$  and  $\zeta$  in the providently rewritten master equation (3.5), we obtain the first functional relations. Applying  $(\text{tr} \circ \varphi_{1\zeta_1}) \otimes (\text{tr} \circ \rho_{2\zeta_2})$  to both sides of (3.5), we obtain

$$\mathcal{T}_{\varphi_1}(\zeta_1)\mathcal{T}_{\varphi_2}(\zeta_2) = \mathcal{T}_{\varphi_2}(\zeta_2)\mathcal{T}_{\varphi_1}(\zeta_1). \tag{3.6}$$

Applying  $(\text{tr} \circ \rho_{\zeta_1}) \otimes (\text{tr} \circ \varphi_{\zeta_2})$  to both sides of (3.5), we come to the relation

$$\mathcal{Q}_\rho(\zeta_1)\mathcal{T}_\varphi(\zeta_2) = \mathcal{T}_\varphi(\zeta_2)\mathcal{Q}_\rho(\zeta_1). \tag{3.7}$$

These are the commutativity relations for the universal integrability objects.

The commutativity between two universal  $Q$ -operators is not obtained in this way due to the specific nature of the representation  $\rho$ , as just described above. Nevertheless, we have a useful formula for the product of two (and more) universal  $Q$ -operators presented here:

$$\mathcal{Q}_{\rho_1}(\zeta_1)\mathcal{Q}_{\rho_2}(\zeta_2) = ((\text{tr}_{W_1 \otimes W_2} \circ (\rho_{1\zeta_1} \otimes_\Delta \rho_{2\zeta_2}) \otimes \text{id})(\mathcal{R}(t \otimes 1))), \tag{3.8}$$

where we have used the equation

$$\mathcal{R}^{13}t^1\mathcal{R}^{23}t^2 = [(\Delta \otimes \text{id})(\mathcal{R})][(\Delta \otimes \text{id})(t \otimes 1)] = (\Delta \otimes \text{id})(\mathcal{R}(t \otimes 1)). \tag{3.9}$$

Equation (3.8) proves to be useful in establishing the commutativity between the universal  $Q$ -operators. However, the very proof of this fact, and also further functional relations, require more details on the quantum group and its representations. This is the subject of the next section.

<sup>3</sup> Similarly as before,  $W$  is a  $\mathcal{B}_+$ -module.

**4. Quantum groups  $U_q(\mathfrak{gl}_n)$  and  $U_q(\mathcal{L}(\mathfrak{sl}_n))$**

We need two quantum groups, the quantum group of the general linear Lie group  $\mathfrak{gl}_n$ , and the quantum group of the loop Lie algebra  $\mathcal{L}(\mathfrak{sl}_n)$ , the latter justifying the term *quantum affine algebra*.<sup>4</sup>

Recall that there are  $2(n - 1)$  generators  $E_i, F_i, i = 1, \dots, n - 1$ , of the Lie algebra  $\mathfrak{gl}_n$ , and  $n$  basis elements  $G_i, i = 1, \dots, n$ , of its standard Cartan subalgebra  $\mathfrak{k}_n$ . They are subject to well-known commutation relations and Serre relations. Then, the quantum group  $U_q(\mathfrak{gl}_n)$  is generated by the elements<sup>5</sup>

$$E_i, F_i, i = 1, \dots, n - 1, q^X, X \in \mathfrak{k}_n,$$

satisfying the corresponding  $q$ -deformed defining relations

$$q^0 = 1, q^{X_1} q^{X_2} = q^{X_1 + X_2}, \tag{4.1}$$

$$q^X E_i q^{-X} = q^{\alpha_i(X)} E_i, q^X F_i q^{-X} = q^{-\alpha_i(X)} F_i, \tag{4.2}$$

$$[E_i, F_j] = \delta_{ij} \kappa_q^{-1} (q^{G_i - G_{i+1}} - q^{G_{i+1} - G_i}), \tag{4.3}$$

where  $\alpha_i \in \mathfrak{k}_n^*$  denote the respective simple positive roots, such that  $\alpha_j(G_i) = c_{ij}$ , where  $c_{ij}$  are the entries of an  $n \times (n - 1)$  matrix with  $c_{ii} = 1, c_{i+1,i} = -1, i = 1, \dots, n - 1$ , and  $c_{ij} = 0$  otherwise (i. e. for  $|j - i| \geq 2$ ). It means, in particular, that  $\alpha_j(G_i) - \alpha_j(G_{i+1}) = a_{ij}$  are the entries of the Cartan matrix of the Lie algebra  $\mathfrak{sl}_n$ . Besides, we have the  $q$ -deformed Serre relations

$$E_i E_j = E_j E_i, F_i F_j = F_j F_i, |i - j| \geq 2, \tag{4.4}$$

$$E_i^2 E_{i\pm 1} - [2]_q E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0, \tag{4.5}$$

$$F_i^2 F_{i\pm 1} - [2]_q F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0. \tag{4.6}$$

Remember that the deformation parameter  $q$  here is the exponential of a complex number  $\hbar$ , therefore, the quantum group is just a complex algebra. Here we assume that

$$q^{X+v} = q^v q^X, [X + v]_q = \kappa_q^{-1} (q^{X+v} - q^{-X-v}) = \kappa_q^{-1} (q^v q^X - q^{-v} q^{-X})$$

for any complex number  $v$  and element  $X$  of  $\mathfrak{k}_n$ . It is important that  $U_q(\mathfrak{gl}_n)$  is a Hopf algebra with respect to appropriately defined co-multiplication, antipode and counit; their explicit form can be omitted, though.

The generators  $E_i$  and  $F_i, i = 1, \dots, n - 1$ , are certainly the root vectors corresponding to the simple positive and simple negative roots  $\alpha_i$  and  $-\alpha_i$ . There are also non-simple roots for  $n > 2$ , and one constructs more root vectors corresponding to these non-simple roots. All these root vectors are used in constructing the basis vectors of the highest-weight  $U_q(\mathfrak{gl}_n)$ -modules (see Appendix). Actually, we will need infinite-dimensional highest-weight representations  $\tilde{\pi}^\lambda$  defined by

$$E_i v^\lambda = 0, i = 1, \dots, n - 1, q^X v^\lambda = q^{\lambda(X)} v^\lambda, X \in \mathfrak{k}_n, \lambda \in \mathfrak{k}_n^*, \tag{4.7}$$

where  $v^\lambda$  is the highest-weight vector with the weight  $\lambda$  which can be seen in terms of its  $n$  components  $\lambda_i = \lambda(G_i)$ . In particular, denoting the highest-weight vector by  $v_0$ , one usually chooses the basis vectors for  $n = 2$  as

$$v_k = F^k v_0, E v_0 = 0, q^{v G_i} v_0 = q^{v \lambda_i} v_0, i = 1, 2,$$

<sup>4</sup> The term *quantum loop algebra* is also used.

<sup>5</sup> We use the same notation for the generators of the quantum group as for the corresponding Lie algebra. The notation  $q^X, X \in \mathfrak{k}_n$ , is used to emphasize that  $\mathfrak{k}_n$  parametrizes the corresponding set of elements of  $U_q(\mathfrak{gl}_n)$ .

and for  $n = 3$  one chooses

$$v_k = F_1^{k_1} F_3^{k_3} F_2^{k_2} v_0, \quad E_i v_0 = 0, \quad q^{\nu G_i} v_0 = q^{\lambda_i} v_0, \quad i = 1, 2, 3,$$

where  $F_3 = F_2 F_1 - q F_1 F_2$  is the root vector corresponding to the root  $-\alpha_1 - \alpha_2$ . The infinite-dimensional  $U_q(\mathfrak{gl}_n)$ -modules corresponding to the representations  $\tilde{\pi}^\lambda$  are denoted by  $\tilde{V}^\lambda$ . We will also need the corresponding finite-dimensional representations  $\pi^\lambda$  that can be obtained from  $\tilde{\pi}^\lambda$  as the quotient representations over infinite-dimensional maximal sub-representations in the case when all the differences  $\lambda_i - \lambda_{i+1}$  are non-negative integers. The finite-dimensional  $U_q(\mathfrak{gl}_n)$ -modules corresponding to the representations  $\pi^\lambda$  are denoted by  $V^\lambda$ .

The next object, the quantum affine algebra  $U_q(\mathcal{L}(\mathfrak{sl}_n))$ , is more complicated, because, unlike the preceding case, it is the quantum group of an infinite-dimensional Lie algebra. Again, it is a unital associative complex algebra which can be defined in terms of its generators and Cartan subalgebra. We start with a reminder that the Lie algebra  $\tilde{\mathcal{L}}(\mathfrak{sl}_n)$  has  $2n$  generators  $e_i, f_i, i = 0, 1, \dots, n-1$ , its Cartan subalgebra  $\tilde{\mathfrak{h}}_n$  can be described by  $n$  basis elements  $h_i, i = 0, 1, \dots, n-1$ , and there is a nontrivial center generated by the element  $c = \sum_{i=0}^{n-1} h_i$ . All these elements are subject to well-known commutation relations and Serre relations. The quantum group  $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_n))$  is generated by the elements

$$e_i, f_i, \quad i = 0, 1, \dots, n-1, \quad q^x, \quad x \in \tilde{\mathfrak{h}}_n,$$

satisfying certain defining relations. These are the following  $q$ -deformed commutation

$$q^0 = 1, \quad q^{x_1} q^{x_2} = q^{x_1+x_2}, \tag{4.8}$$

$$q^x e_i q^{-x} = q^{\alpha_i(x)} e_i, \quad q^x f_i q^{-x} = q^{-\alpha_i(x)} f_i, \tag{4.9}$$

$$[e_i, f_j] = \delta_{ij} \kappa_q^{-1} (q^{h_i} - q^{-h_i}) \tag{4.10}$$

and Serre relations

$$\sum_{k=0}^{1-\tilde{a}_{ij}} (-1)^k \left[ \begin{matrix} 1-\tilde{a}_{ij} \\ k \end{matrix} \right]_q (e_i)^{1-\tilde{a}_{ij}-k} e_j (e_i)^k = 0, \tag{4.11}$$

$$\sum_{k=0}^{1-\tilde{a}_{ij}} (-1)^k \left[ \begin{matrix} 1-\tilde{a}_{ij} \\ k \end{matrix} \right]_q (f_i)^{1-\tilde{a}_{ij}-k} f_j (f_i)^k = 0, \tag{4.12}$$

where  $\tilde{a}_{ij}$  are the entries of the generalized Cartan matrix of type  $A_{n-1}^{(1)}$ , and the set of simple positive roots is extended by the additional root  $\alpha_0$ .

Since we need finite-dimensional representations, we should first note that there is no finite-dimensional representation of  $\tilde{\mathcal{L}}(\mathfrak{sl}_n)$  with  $c \neq 0$ , hence we consider the loop Lie algebra  $\mathcal{L}(\mathfrak{sl}_n)$  defined as the quotient

$$\mathcal{L}(\mathfrak{sl}_n) = \tilde{\mathcal{L}}(\mathfrak{sl}_n) / \mathbb{C}c.$$

Moreover, also the quantum group  $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_n))$  does not have any finite-dimensional representations with  $q^{\nu c} \neq 1$  for any  $\nu \in \mathbb{C}^\times$ . Therefore, we consider the quantum group  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  defined as the quotient

$$U_q(\mathcal{L}(\mathfrak{sl}_n)) = U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_n)) / \langle q^{\nu c} - 1 \rangle_{\nu \in \mathbb{C}}.$$

The quantum group  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  can be considered in terms of the same generators and defining relations as  $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_n))$ , where the additional relation  $q^{\nu c} = 1, \nu \in \mathbb{C}$ , is taken into account.

The restriction  $q^{v_c} = 1$  makes it possible to construct finite-dimensional representations of the quantum affine algebras under consideration.

The Hopf algebra structure of  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  can be defined by the relations with the co-multiplication  $\Delta$ ,

$$\Delta(q^x) = q^x \otimes q^x, \quad \Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i, \quad (4.13)$$

antipode  $S$ ,

$$S(q^x) = q^{-x}, \quad S(e_i) = -q^{h_i} e_i, \quad S(f_i) = -f_i q^{-h_i}, \quad (4.14)$$

and counit  $\varepsilon$ ,

$$\varepsilon(q^x) = 1, \quad \varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0, \quad (4.15)$$

acting on the generators explicitly as given.

### 5. Jimbo's homomorphism and representations of $U_q(\mathcal{L}(\mathfrak{sl}_n))$

As the representation  $\varphi$  mentioned in section 3 we use a representation  $\tilde{\varphi}_\zeta^\lambda$  constructed as follows.<sup>6</sup> We first define a general grading of the quantum affine algebra  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  with the help of a map  $\Gamma_\zeta$  which acts on the generators as

$$\Gamma_\zeta(q^x) = q^x, \quad x \in \tilde{\mathfrak{h}}_n, \quad \Gamma_\zeta(e_i) = \zeta^{s_i} e_i, \quad \Gamma_\zeta(f_i) = \zeta^{-s_i} f_i, \quad (5.1)$$

where  $\zeta \in \mathbb{C}^\times$  is called the *spectral parameter*, and  $s_i$  are arbitrary integers. We denote the total sum of these integers by  $s$ . Secondly, we use Jimbo's homomorphism from the quantum affine algebra  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  to the quantum group  $U_q(\mathfrak{gl}_n)$  [30],

$$\varphi : U_q(\mathcal{L}(\mathfrak{sl}_n)) \rightarrow U_q(\mathfrak{gl}_n). \quad (5.2)$$

For some values of  $n$  of our specific interest, we write it down explicitly. Thus, Jimbo's homomorphism  $\varphi : U_q(\mathcal{L}(\mathfrak{sl}_2)) \rightarrow U_q(\mathfrak{gl}_2)$  is given by the relations

$$\varphi(q^{v_{h_0}}) = q^{v(G_2 - G_1)}, \quad \varphi(q^{v_{h_1}}) = q^{v(G_1 - G_2)}, \quad (5.3)$$

$$\varphi(e_0) = F q^{-G_1 - G_2}, \quad \varphi(e_1) = E, \quad (5.4)$$

$$\varphi(f_0) = E q^{G_1 + G_2}, \quad \varphi(f_1) = F, \quad (5.5)$$

and for  $n = 3$ , we have  $\varphi : U_q(\mathcal{L}(\mathfrak{sl}_3)) \rightarrow U_q(\mathfrak{gl}_3)$  defined as

$$\varphi(q^{v_{h_0}}) = q^{v(G_3 - G_1)}, \quad \varphi(q^{v_{h_1}}) = q^{v(G_1 - G_2)}, \quad \varphi(q^{v_{h_2}}) = q^{v(G_2 - G_3)}, \quad (5.6)$$

$$\varphi(e_0) = F_3 q^{-G_1 - G_3}, \quad \varphi(e_1) = E_1, \quad \varphi(e_2) = E_2, \quad (5.7)$$

$$\varphi(f_0) = E_3 q^{G_1 + G_3}, \quad \varphi(f_1) = F_1, \quad \varphi(f_2) = F_2, \quad (5.8)$$

where  $E_3 = E_1 E_2 - q^{-1} E_2 E_1$  and  $F_3 = F_2 F_1 - q F_1 F_2$ . Thirdly, and finally, we use the highest-weight representation  $\tilde{\pi}^\lambda$  of  $U_q(\mathfrak{gl}_n)$  briefly described in section 4, see equation (4.7) and around. We understand that Jimbo's homomorphism allows us to use representations of  $U_q(\mathfrak{gl}_n)$  to construct representations of  $U_q(\mathcal{L}(\mathfrak{sl}_n))$ . Hence, our basic representation  $\tilde{\varphi}_\zeta^\lambda$  is constructed as the superposition of the above three maps,

$$\tilde{\varphi}_\zeta^\lambda = \tilde{\pi}^\lambda \circ \varphi \circ \Gamma_\zeta.$$

<sup>6</sup> In our notation, the tilde corresponds to infinite-dimensional representations; for the corresponding finite-dimensional representations we use the same symbols simply omitting the tilde.

Here, using the finite-dimensional representations  $\pi^\lambda$  of  $U_q(\mathfrak{gl}_n)$ , we will obtain finite-dimensional representations of  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  with  $\varphi_\zeta^\lambda = \pi^\lambda \circ \varphi \circ \Gamma_\zeta$ .

Recall again that the universal  $R$ -matrix is an element of the tensor product of two Borel subalgebras of the quantum group, in the case under consideration,  $\mathcal{R} \in U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)$ , where the Borel subalgebra  $U_q(\mathfrak{b}_+)$  of  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  is generated by the elements  $e_i$ ,  $i = 0, 1, \dots, n - 1$ , and  $q^x$ , and the Borel subalgebra  $U_q(\mathfrak{b}_-)$  of  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  is generated by the elements  $f_i$ ,  $i = 0, 1, \dots, n - 1$ , and  $q^x$ . One can try to define the representation  $\rho$  of  $U_q(\mathfrak{b}_+)$  introduced in section 3 in the following way. Let  $\tilde{\varphi}_\zeta^\lambda$  be a representation of  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  constructed in accordance with the above prescription, and  $\zeta \in \tilde{\mathfrak{h}}_n^*$ . Then the relations

$$\tilde{\varphi}_\zeta^\lambda[\zeta](e_i) = \tilde{\varphi}_\zeta^\lambda(e_i), \quad \tilde{\varphi}_\zeta^\lambda[\zeta](q^x) = q^{\zeta(x)} \tilde{\varphi}_\zeta^\lambda(q^x) \tag{5.9}$$

define a representation of  $U_q(\mathfrak{b}_+)$  called a *shifted representation*. We see that the only difference between the shifted and initial representations appears simply in the factor  $q^{\zeta(x)}$  in the second relation of (5.9). Moreover, choosing the twist element  $t$  explicitly in the form

$$t = q^{\sum_{i=0}^{n-1} \phi_i h_i / n}$$

for some complex numbers  $\phi_i$ , which are subject to the condition  $\sum_i \phi_i = 0$  to respect the restriction  $q^{vc} = 1$ , we obtain that the universal integrability objects based on the representation  $\tilde{\varphi}_\zeta^\lambda$  are related to the universal integrability objects based on its shifted counterpart  $\tilde{\varphi}_\zeta^\lambda[\zeta]$  simply as

$$\mathcal{T}_{\tilde{\varphi}_\zeta^\lambda[\zeta]}(\zeta) = \mathcal{T}_{\tilde{\varphi}^\lambda}(\zeta) q^{\sum_{i=0}^{n-1} \zeta(h_i) h'_i / n}, \tag{5.10}$$

where we use the notation  $h'_i = h_i + \phi_i$ .

It is not difficult to see that the shifted representation  $\tilde{\varphi}_\zeta^\lambda[\zeta]$  for a nonzero  $\zeta$  cannot be extended to a representation of the full quantum affine algebra. Therefore, the shifted representation is what we actually need to construct analogs of the representation  $\rho$  introduced in section 3. It is clear that the universal  $Q$ -operators constructed with the help of the shifted representations  $\tilde{\varphi}_\zeta^\lambda[\zeta]$  will be connected with the corresponding universal transfer operators based on  $\tilde{\varphi}_\zeta^\lambda$  according to (5.10). Hence, we do not obtain a really new object. However, considering all the  $U_q(\mathfrak{b}_+)$ -modules with general nonzero shifts  $\zeta$ , we can demonstrate that there are interesting limits of the corresponding representations when the differences  $\lambda_i - \lambda_{i+1} = \mu_i$ ,  $i = 1, \dots, n - 1$ , go to infinity and  $\zeta$  is chosen appropriately. To be precise, we note that for  $n > 2$  the obtained representation is reducible and we take the corresponding irreducible subrepresentation. The final formulas for the  $Q$ -operators look for  $n = 2$  as

$$\mathcal{Q}(\zeta) = \lim_{\mu \rightarrow \infty} \left( \tilde{\mathcal{T}}^{(\mu, 0)}(q^{-1/s} \zeta) q^{(\mu h'_0 - \mu h'_1) / 2} \right),$$

and for  $n = 3$  as

$$\mathcal{Q}(\zeta) = (1 - q^{(h'_0 - 2h'_1 + h'_2) / 3}) \lim_{\mu_1, \mu_2 \rightarrow \infty} \left( \tilde{\mathcal{T}}^{(\mu_1 + \mu_2, \mu_2, 0)}(q^{-2/s} \zeta) q^{((\mu_1 + \mu_2)h'_0 - \mu_1 h'_1 - \mu_2 h'_2) / 3} \right).$$

Here, we use the notation  $\tilde{\mathcal{T}}^\lambda(\zeta)$  for  $\mathcal{T}_{\tilde{\varphi}^\lambda}(\zeta)$ , and we do not write explicitly the representation  $\rho_\zeta$  in  $\mathcal{Q}(\zeta)$ , simply keeping in mind that it is obtained from  $\tilde{\varphi}_\zeta^\lambda[\zeta]$  by the procedure shortly described above.

The above two (lower and higher rank) basic examples can be directly generalized to the case of general  $n$ . Remarkably, it follows from the limit relation between the universal

integrability objects that also the universal  $Q$ -operators commute as well as the universal transfer operators in (3.6) and (3.7),

$$\mathcal{Q}_{\rho_1}(\zeta_1)\mathcal{Q}_{\rho_2}(\zeta_2) = \mathcal{Q}_{\rho_2}(\zeta_2)\mathcal{Q}_{\rho_1}(\zeta_1). \quad (5.11)$$

The representations  $\rho_\zeta$  have a useful interpretation in terms of the so-called  $q$ -oscillators, see e. g. [9–11, 31]. We define them by a natural deformation of the usual oscillators with the deformation parameter  $q = \exp \hbar$ , where  $\hbar$  is a complex number, such that  $q$  is not a root of unity. The  $q$ -oscillator algebra  $\text{Osc}_q$  is a unital associative  $\mathbb{C}$ -algebra with generators  $b^\dagger, b, q^{\nu N}$ ,  $\nu \in \mathbb{C}$ , and relations

$$\begin{aligned} q^0 &= 1, & q^{\nu_1 N} q^{\nu_2 N} &= q^{(\nu_1 + \nu_2)N}, \\ q^{\nu N} b^\dagger q^{-\nu N} &= q^\nu b^\dagger, & q^{\nu N} b q^{-\nu N} &= q^{-\nu} b, \\ b^\dagger b &= [N]_q, & b b^\dagger &= [N + 1]_q. \end{aligned}$$

The monomials  $(b^\dagger)^{k+1} q^{\nu N}$ ,  $b^{k+1} q^{\nu N}$  and  $q^{\nu N}$  for  $k \in \mathbb{Z}_+$  and  $\nu \in \mathbb{C}$  form a basis of  $\text{Osc}_q$ .

The representations of the  $q$ -oscillator algebra are constructed as follows. One can see that the relations

$$q^{\nu N} v_k = q^{\nu k} v_k, \quad (5.12)$$

$$b^\dagger v_k = v_{k+1}, \quad b v_k = [k]_q v_{k-1}, \quad (5.13)$$

supplied with the assumption  $v_{-1} = 0$ , endow free vector space generated by the set  $\{v_0, v_1, \dots\}$  with the structure of an  $\text{Osc}_q$ -module. We denote the corresponding representations by  $\chi$ .<sup>7</sup>

In the case of  $n = 2$  for the representation  $\rho_\zeta$  we have

$$\begin{aligned} q^{\nu h_0} v_k &= q^{2\nu k} v_k, & q^{\nu h_1} v_k &= q^{-2\nu k} v_k, \\ e_0 v_k &= v_{k+1}, & e_1 v_k &= \kappa_q^{-1} q^{-k} [k]_q v_{k-1}. \end{aligned}$$

Here  $v_k, k \in \mathbb{Z}_+$ , are the basis vectors in the representation space. Comparing these relations with (5.12) and (5.13), we see that it is natural to define the mapping  $\theta$  from  $U_q(\mathfrak{b}_+)$  to  $\text{Osc}_q$  as

$$\begin{aligned} \theta(q^{\nu h_0}) &= q^{2\nu N}, & \theta(q^{\nu h_1}) &= q^{-2\nu N}, \\ \theta(e_0) &= b^\dagger, & \theta(e_1) &= \kappa_q^{-1} b q^{-N}. \end{aligned}$$

It can be shown that  $\theta$  is a homomorphism. Now, for the representation  $\rho_\zeta$  we have

$$\rho_\zeta = \chi \circ \theta \circ \Gamma_\zeta. \quad (5.14)$$

In the case of  $n = 3$  we need two copies of the algebra  $\text{Osc}_q$ . The corresponding homomorphism from  $U_q(\mathfrak{b}_+)$  to  $\text{Osc}_q \otimes \text{Osc}_q$  has here the form

$$\begin{aligned} \theta(q^{\nu h_0}) &= q^{\nu(2N_1 + N_2)}, & \theta(q^{\nu h_1}) &= q^{\nu(-N_1 + N_2)}, & \theta(q^{\nu h_2}) &= q^{\nu(-N_1 - 2N_2)}, \\ \theta(e_0) &= b_1^\dagger q^{-N_2}, & \theta(e_1) &= -b_1 b_2^\dagger q^{-N_1 + N_2 + 1}, & \theta(e_2) &= \kappa_q^{-1} b_2 q^{-N_2}. \end{aligned}$$

One can produce more universal integrability objects with the help of the automorphisms of the quantum group  $U_q(\mathcal{L}(\mathfrak{sl}_n))$ . These are the automorphism  $\sigma$  corresponding to the cyclic

<sup>7</sup> There is one more useful representation, but we do not use it here.

permutations of the Dynkin diagram of the Kac–Moody Lie algebra of type  $A_{n-1}^{(1)}$  transforming the simple positive roots as  $\alpha_i \rightarrow \alpha_{i+1}$ , and the automorphism  $\tau$  acting as  $\alpha_i \rightarrow \alpha_{n-i}$  while leaving  $\alpha_0$  alone. Here,  $\sigma^n = \text{id}$  and  $\tau^2 = \text{id}$ . If  $\theta : U_q(\mathfrak{b}_+) \rightarrow \underbrace{\text{Osc}_q \otimes \cdots \otimes \text{Osc}_q}_{n-1}$  is the initial

homomorphism for the basic representation  $\rho_\zeta$ , we define

$$\theta_i = \theta \circ \sigma^{-i}, \quad \bar{\theta}_i = \theta \circ \tau \circ \sigma^{-i+1}, \quad i = 1, \dots, n,$$

and use the formulas of type (5.14) to define the representations  $\rho_{i\zeta}$  and  $\bar{\rho}_{i\zeta}$  and then the set of  $2n$  universal  $Q$ -operators. Similarly, one can produce more representations  $\tilde{\varphi}_{i\zeta}^\lambda$  with the help of the automorphisms  $\sigma$  and  $\tau$  starting from a basic one, and construct the respective universal transfer operators. However, not all of them will be independent. In fact, there is only one universal transfer operator for  $n = 2$ , and there are two independent universal transfer operators in the higher-rank case.

## 6. The universal functional relations

### 6.1. The key relations

Thus, we have  $2n$  different universal  $Q$ -operators.<sup>8</sup> Specifying the irreducible representations  $\rho_\zeta$  and constructing the corresponding universal  $Q$ -operators, we can use the formula for their products given in section 3. We see here that to analyze these products, we have to consider tensor products  $(\rho_{\ell\zeta_\ell} \otimes_\Delta \cdots \otimes_\Delta \rho_{1\zeta_1})$  of the representations  $\rho_{i\zeta}$ , with  $\ell = 2, 3, \dots, n$ , and go similarly with  $\bar{\rho}_{i\zeta}$ . Then, choosing appropriate bases for the corresponding  $U_q(\mathfrak{b}_+)$ -modules  $(W_\ell)_{\zeta_\ell} \otimes_\Delta \cdots \otimes_\Delta (W_1)_{\zeta_1}$ , we should consider their defining module relations. In this way we obtain the whole set of the universal functional relations.

In particular, putting  $\ell = 2$  in a higher-rank case ( $n = 3$ ), we see that the  $U_q(\mathfrak{b}_+)$ -module  $(W_2)_{\zeta_2} \otimes_\Delta (W_1)_{\zeta_1}$  for some special choice of  $\zeta_1$  and  $\zeta_2$  is reducible and the corresponding quotient module is isomorphic to  $(\bar{W}_3)_\zeta[\xi]$  with a new spectral parameter  $\zeta$  expressed in terms of  $\zeta_1$  and  $\zeta_2$ , and a certain shift  $\xi$  of the corresponding representation  $\bar{\rho}_{3\zeta}$ . Similarly, one can obtain the  $U_q(\mathfrak{b}_+)$ -module  $(W_3)_\zeta[\xi]$  for some  $\zeta$  and  $\xi$  as a quotient of  $(\bar{W}_2)_{\zeta_2} \otimes_\Delta (\bar{W}_1)_{\zeta_1}$ . This consideration allows us to write down the universal double- $Q$  functional relations in the determinant form,

$$\mathcal{C}_i \bar{\mathcal{Q}}_i(\zeta) = \mathcal{Q}_j(q^{1/s}\zeta) \mathcal{Q}_k(q^{-1/s}\zeta) - \mathcal{Q}_j(q^{-1/s}\zeta) \mathcal{Q}_k(q^{1/s}\zeta), \quad (6.1)$$

$$\mathcal{C}_i \mathcal{Q}_i(\zeta) = \bar{\mathcal{Q}}_j(q^{-1/s}\zeta) \bar{\mathcal{Q}}_k(q^{1/s}\zeta) - \bar{\mathcal{Q}}_j(q^{1/s}\zeta) \bar{\mathcal{Q}}_k(q^{-1/s}\zeta), \quad (6.2)$$

where  $(i, j, k)$  runs over all cyclic permutations of the set  $(1, 2, 3)$ , and we use the notation

$$\mathcal{C}_i = q^{-\mathcal{D}_i/s} (q^{2\mathcal{D}_j/s} - q^{2\mathcal{D}_k/s})^{-1}, \quad \mathcal{D}_i = (h'_{i-1} - h'_i)s/6. \quad (6.3)$$

Note that such relations are absent in the lower-rank case.

To derive the major functional relations, we have to put  $\ell = n$  and analyze the  $n$ -tuple product representation  $(\rho_{n\zeta_n} \otimes_\Delta \cdots \otimes_\Delta \rho_{1\zeta_1})$  corresponding to the tensor product of  $n$   $U_q(\mathfrak{b}_+)$ -modules  $(W_n)_{\zeta_n} \otimes_\Delta \cdots \otimes_\Delta (W_1)_{\zeta_1}$ . It allows one to obtain the key relation between the universal transfer operator and universal  $Q$ -operators,

$$\mathcal{C} \tilde{\mathcal{T}}^\lambda(\zeta) = \mathcal{Q}_1(q^{-2(\lambda+\rho)_1/s}\zeta) \cdots \mathcal{Q}_n(q^{-2(\lambda+\rho)_n/s}\zeta), \quad \mathcal{C} = \mathcal{C}_1 \cdots \mathcal{C}_n, \quad (6.4)$$

where  $\rho \in \mathfrak{k}_n^*$  is the half-sum of the positive roots,

$$\rho = ((n-1)/2, (n-3)/2, \dots, -(n-1)/2).$$

<sup>8</sup> One has only two universal  $Q$ -operators for the lower-rank case (for  $n = 2$ ).

This is the central equation from which follow all other functional relations. One can see that relation (6.4) contains only the universal transfer operator for the infinite-dimensional representation  $\tilde{\varphi}_\zeta^\lambda$ . To obtain the corresponding relations for the universal transfer operator based on the finite-dimensional representation  $\varphi_\zeta^\lambda$ , one uses the quantum version of the so-called *B.G.G. resolution* [18, 19, 32], which implies an exact sequence of  $U_q(\mathfrak{gl}_n)$ -modules and  $U_q(\mathfrak{gl}_n)$ -homomorphisms,

$$U_k = \bigoplus_{\substack{w \in W \\ \ell(w)=k}} \tilde{V}^{w \cdot \lambda}, \quad w \cdot \lambda = w(\lambda + \rho) - \rho,$$

$$0 \longrightarrow U_n \xrightarrow{\varphi_n} U_{n-1} \xrightarrow{\varphi_{n-1}} \dots \xrightarrow{\varphi_1} U_0 \xrightarrow{\varphi_0} U_{-1} \longrightarrow 0, \quad U_{-1} = V^\lambda,$$

where  $w$  means any element of the Weyl group  $W$  of the root system of  $\mathfrak{gl}_n$ , with  $\ell(w)$  being its length, and  $w \cdot \lambda$  stands for the affine action of  $w$  defined explicitly as above. The B.G.G. resolution allows one to express the trace over the finite-dimensional  $U_q(\mathfrak{gl}_n)$ -module  $V^\lambda$  as certain linear combination of traces over the infinite-dimensional  $U_q(\mathfrak{gl}_n)$ -modules  $\tilde{V}^{w \cdot \lambda}$ ,

$$\text{tr}^\lambda = \sum_{w \in W} (-1)^{\ell(w)} \tilde{\text{tr}}^{w \cdot \lambda}. \tag{6.5}$$

Then, using our definition of the universal transfer operator (3.2), we immediately come to a remarkable relation between the universal transfer operator based on the finite-dimensional representation  $\varphi_\zeta^\lambda$  and universal transfer operators based on the infinite-dimensional representations  $\tilde{\varphi}_\zeta^{w \cdot \lambda}$ ,

$$\mathcal{T}^\lambda(\zeta) = \sum_{p \in S_n} \text{sgn}(p) \tilde{\mathcal{T}}^{p(\lambda+\rho)-\rho}(\zeta). \tag{6.6}$$

It is taken into account here that  $W$  can be identified with the symmetric group  $S_n$ , and so,  $p$  is an element of  $S_n$  acting on  $\mathfrak{k}_n$  by appropriate permutations. Relation (6.6) leads to the following representation of the universal transfer operator in terms of the universal  $Q$ -operators:

$$\mathcal{C} \mathcal{T}^{\lambda-\rho}(\zeta) = \det \left( Q_i(q^{-2\lambda_j/s} \zeta) \right)_{i,j=1,\dots,n}. \tag{6.7}$$

Similar consideration holds as well when also the automorphism  $\tau$  is involved giving rise to the barred universal integrability objects. In this case we come to the determinant representation

$$\mathcal{C} \bar{\mathcal{T}}^{\lambda-\rho}(\zeta) = \det \left( \bar{Q}_i(q^{2\lambda_j/s} \zeta) \right)_{i,j=1,\dots,n} \tag{6.8}$$

It is worthwhile noting that for the lower-rank case,  $n = 2$ , we would be able to obtain the universal functional relations in the determinant form even if we started with Jimbo's homomorphism from  $U_q(\mathcal{L}(\mathfrak{sl}_2))$  to  $U_q(\mathfrak{sl}_2)$ . However, for higher-rank cases,  $n \geq 3$ , it is most convenient to use Jimbo's homomorphism from  $U_q(\mathcal{L}(\mathfrak{sl}_n))$  to  $U_q(\mathfrak{gl}_n)$ .

### 6.2. Universal TQ- and TT-relations

Further universal functional relations can be obtained from the vanishing of the determinants of certain  $(n + 1) \times (n + 1)$  matrices with one dependent row using equations (6.7), (6.8). The

universal  $TQ$ -relations follow from the identity

$$\det \begin{pmatrix} Q_1(q^{-2\lambda_1/s}\zeta) & \dots & Q_1(q^{-2\lambda_n/s}\zeta) & Q_1(q^{-2\lambda_{n+1}/s}\zeta) \\ Q_2(q^{-2\lambda_1/s}\zeta) & \dots & Q_2(q^{-2\lambda_n/s}\zeta) & Q_2(q^{-2\lambda_{n+1}/s}\zeta) \\ \vdots & \ddots & \vdots & \vdots \\ Q_n(q^{-2\lambda_1/s}\zeta) & \dots & Q_n(q^{-2\lambda_n/s}\zeta) & Q_n(q^{-2\lambda_{n+1}/s}\zeta) \\ Q_j(q^{-2\lambda_1/s}\zeta) & \dots & Q_j(q^{-2\lambda_n/s}\zeta) & Q_j(q^{-2\lambda_{n+1}/s}\zeta) \end{pmatrix} = 0$$

obviously satisfied for any  $j = 1, \dots, n$ . Such matrices can be constructed also for the barred universal integrability objects. In the lower-rank case, putting  $n = 2$ , we obtain

$$\begin{aligned} \mathcal{T}^{(\lambda_1-1/2, \lambda_2+1/2)}(\zeta)Q_j(q^{-2\lambda_3/s}\zeta) \\ - \mathcal{T}^{(\lambda_1-1/2, \lambda_3+1/2)}(\zeta)Q_j(q^{-2\lambda_2/s}\zeta) \\ + \mathcal{T}^{(\lambda_2-1/2, \lambda_3+1/2)}(\zeta)Q_j(q^{-2\lambda_1/s}\zeta) = 0. \end{aligned} \quad (6.9)$$

We call this equation the universal  $TQ$ -relation. For example, choosing in (6.9) the components of  $\lambda$  as  $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$ , after some transformations based on obvious symmetry properties of  $\mathcal{T}^{(\lambda_1, \lambda_2)}(\zeta)$ , we derive

$$\mathcal{T}^{(1/2, -1/2)}(\zeta)Q_j(\zeta) = Q_j(q^{2/s}\zeta) + Q_j(q^{-2/s}\zeta).$$

This is the universal analog of the famous Baxter's  $TQ$ -relation, that is given in the form independent of the representation of the quantum group in the quantum space.

In a higher-rank case,  $n = 3$ , we obtain the  $U_q(\mathcal{L}(\mathfrak{sl}_3))$  universal  $TQ$ -relations

$$\begin{aligned} \mathcal{T}^{(\lambda_1-1, \lambda_2, \lambda_3+1)}(\zeta)Q_j(q^{-2\lambda_4/s}\zeta) - \mathcal{T}^{(\lambda_1-1, \lambda_2, \lambda_4+1)}(\zeta)Q_j(q^{-2\lambda_3/s}\zeta) \\ + \mathcal{T}^{(\lambda_1-1, \lambda_3, \lambda_4+1)}(\zeta)Q_j(q^{-2\lambda_2/s}\zeta) - \mathcal{T}^{(\lambda_2-1, \lambda_3, \lambda_4+1)}(\zeta)Q_j(q^{-2\lambda_1/s}\zeta) = 0. \end{aligned}$$

Now, with the choice  $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = -1$ , also using symmetry properties of  $\mathcal{T}^{(\lambda_1, \lambda_2, \lambda_3)}(\zeta)$ , we derive

$$\mathcal{T}^{(1, 1, 0)}(\zeta)Q_j(\zeta) - \mathcal{T}^{(1, 0, 0)}(\zeta)Q_j(q^{-2/s}\zeta) = Q_j(q^{2/s}\zeta) - Q_j(q^{-4/s}\zeta). \quad (6.10)$$

In a similar way we obtain

$$\bar{\mathcal{T}}^{(1, 1, 0)}(\zeta)\bar{Q}_j(\zeta) - \bar{\mathcal{T}}^{(1, 0, 0)}(\zeta)\bar{Q}_j(q^{2/s}\zeta) = \bar{Q}_j(q^{-2/s}\zeta) - \bar{Q}_j(q^{4/s}\zeta). \quad (6.11)$$

Unlike the lower-rank case, each of the above equations involves different universal transfer operators. However, one can derive functional relations containing only one universal transfer operator,  $\mathcal{T}^{(1, 0, 0)}(\zeta)$  or  $\mathcal{T}^{(1, 1, 0)}(\zeta)$ , or their barred analog, but having mixed  $Q_i$  and  $\bar{Q}_j$  for distinct  $i$  and  $j$  in one same equation. To this end, one can use the Jacobi identity for determinants [33]. Then, in the simplest higher-rank case  $n = 3$ , we obtain the corresponding  $TQ\bar{Q}$ -relations

$$\begin{aligned} \mathcal{T}^{(1, 0, 0)}(\zeta)Q_i(q^{-2/s}\zeta)\bar{Q}_j(q^{-1/s}\zeta) = Q_i(q^{-4/s}\zeta)\bar{Q}_j(q^{-1/s}\zeta) \\ + Q_i(\zeta)\bar{Q}_j(q^{-3/s}\zeta) + Q_i(q^{-2/s}\zeta)\bar{Q}_j(q^{1/s}\zeta) \end{aligned}$$

and

$$\mathcal{T}^{(1,1,0)}(\zeta) \mathcal{Q}_i(\zeta) \bar{\mathcal{Q}}_j(q^{-1/s}\zeta) = \mathcal{Q}_i(q^{2/s}\zeta) \bar{\mathcal{Q}}_j(q^{-1/s}\zeta) + \mathcal{Q}_i(q^{-2/s}\zeta) \bar{\mathcal{Q}}_j(q^{1/s}\zeta) + \mathcal{Q}_i(\zeta) \bar{\mathcal{Q}}_j(q^{-3/s}\zeta).$$

The universal  $TT$ -relations can be derived from the equation

$$\det \begin{pmatrix} \mathcal{Q}_1(q^{-2\lambda_1/s}\zeta) & \dots & \mathcal{Q}_1(q^{-2\lambda_n/s}\zeta) & \mathcal{Q}_1(q^{-2\lambda_{n+1}/s}\zeta) \\ \mathcal{Q}_2(q^{-2\lambda_1/s}\zeta) & \dots & \mathcal{Q}_2(q^{-2\lambda_n/s}\zeta) & \mathcal{Q}_2(q^{-2\lambda_{n+1}/s}\zeta) \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{Q}_n(q^{-2\lambda_1/s}\zeta) & \dots & \mathcal{Q}_n(q^{-2\lambda_n/s}\zeta) & \mathcal{Q}_n(q^{-2\lambda_{n+1}/s}\zeta) \\ \mathcal{T}^{(\lambda'_1, \lambda'_{n+2}, \dots, \lambda'_{2n})}(\zeta) & \dots & \mathcal{T}^{(\lambda'_n, \lambda'_{n+2}, \dots, \lambda'_{2n})}(\zeta) & \mathcal{T}^{(\lambda'_{n+1}, \lambda'_{n+2}, \dots, \lambda'_{2n})}(\zeta) \end{pmatrix} = 0,$$

where  $\lambda'_j = \lambda_j - (n - 1)/2$ ,  $j = 1, \dots, n + 1$ , and  $\lambda'_{n+1+k} = \lambda_{n+1+k} - (n - 2k - 1)/2$ ,  $k = 1, \dots, n - 1$ . Using equation (6.7) to express the operators  $\mathcal{T}^{(\lambda'_j, \lambda'_{n+2}, \dots, \lambda'_{2n})}(\zeta)$  in terms of the universal  $Q$ -operators for all  $j = 1, \dots, n + 1$ , one can see that the last row of the above  $(n + 1) \times (n + 1)$  matrix is a linear combination of the first  $n$  rows. Therefore, its determinant identically vanishes. Expanding this determinant over the last row and using again (6.7), we obtain the universal  $TT$ -relations. The case with the barred universal integrability objects can be considered in the same way. Actually, the universal  $TT$ -relations for the barred integrability objects can be obtained directly from the relations for the unbarred quantities changing  $q$  by  $q^{-1}$  there [18].

In the simplest lower-rank case we obtain

$$\begin{aligned} &\mathcal{T}^{(\lambda_1-1/2, \lambda_2+1/2)}(\zeta) \mathcal{T}^{(\lambda_3-1/2, \lambda_4+1/2)}(\zeta) \\ &\quad - \mathcal{T}^{(\lambda_1-1/2, \lambda_3+1/2)}(\zeta) \mathcal{T}^{(\lambda_2-1/2, \lambda_4+1/2)}(\zeta) \\ &\quad + \mathcal{T}^{(\lambda_2-1/2, \lambda_3+1/2)}(\zeta) \mathcal{T}^{(\lambda_1-1/2, \lambda_4+1/2)}(\zeta) = 0. \end{aligned} \quad (6.12)$$

Choosing the weights subsequently as  $\lambda_1 = \ell + 1$ ,  $\lambda_2 = \ell$ ,  $\lambda_3 = 0$ ,  $\lambda_4 = -1$  and  $\lambda_1 = \ell + 1$ ,  $\lambda_2 = \ell$ ,  $\lambda_3 = \ell - 1$ ,  $\lambda_4 = -1$ , we derive the universal analog of the  $TT$ -relations of particular interest,

$$\mathcal{T}^{(\ell,0)}(q^{-1/s}\zeta) \mathcal{T}^{(\ell,0)}(q^{1/s}\zeta) = 1 + \mathcal{T}^{(\ell-1,0)}(q^{-1/s}\zeta) \mathcal{T}^{(\ell+1,0)}(q^{1/s}\zeta) \quad (6.13)$$

and

$$\mathcal{T}^{(1,0)}(q^{-2\ell/s}\zeta) \mathcal{T}^{(\ell,0)}(\zeta) = \mathcal{T}^{(\ell+1,0)}(\zeta) + \mathcal{T}^{(\ell-1,0)}(\zeta), \quad (6.14)$$

respectively.

For a more complicated higher-rank case,  $n = 3$ , we obtain

$$\begin{aligned} &\mathcal{T}^{(\lambda_1-1, \lambda_2, \lambda_3+1)}(\zeta) \mathcal{T}^{(\lambda_4-1, \lambda_5, \lambda_6+1)}(\zeta) - \mathcal{T}^{(\lambda_1-1, \lambda_2, \lambda_4+1)}(\zeta) \mathcal{T}^{(\lambda_3-1, \lambda_5, \lambda_6+1)}(\zeta) \\ &\quad + \mathcal{T}^{(\lambda_1-1, \lambda_3, \lambda_4+1)}(\zeta) \mathcal{T}^{(\lambda_2-1, \lambda_5, \lambda_6+1)}(\zeta) - \mathcal{T}^{(\lambda_2-1, \lambda_3, \lambda_4+1)}(\zeta) \mathcal{T}^{(\lambda_1-1, \lambda_5, \lambda_6+1)}(\zeta) = 0. \end{aligned} \quad (6.15)$$

Putting here  $\lambda_1 = \ell + 2$ ,  $\lambda_2 = \ell + 1$ ,  $\lambda_3 = 1$ ,  $\lambda_4 = 0$ ,  $\lambda_5 = 0$ ,  $\lambda_6 = -1$ , we derive one  $TT$ -relation of particular interest in the universal form,

$$\mathcal{T}^{(\ell-1,0,0)}(\zeta) \mathcal{T}^{(\ell+1,0,0)}(q^{2/s}\zeta) = \mathcal{T}^{(\ell,0,0)}(\zeta) \mathcal{T}^{(\ell,0,0)}(q^{2/s}\zeta) - \mathcal{T}^{(\ell,\ell,0)}(\zeta), \quad (6.16)$$

while the choice  $\lambda_1 = \ell + 2$ ,  $\lambda_2 = \ell$ ,  $\lambda_3 = 0$ ,  $\lambda_4 = \ell + 1$ ,  $\lambda_5 = \ell + 1$ ,  $\lambda_6 = -1$  produces another interesting universal  $TT$ -relation,

$$\mathcal{T}^{(\ell-1, \ell-1, 0)}(q^{-2/s}\zeta) \mathcal{T}^{(\ell+1, \ell+1, 0)}(\zeta) = \mathcal{T}^{(\ell, \ell, 0)}(q^{-2/s}\zeta) \mathcal{T}^{(\ell, \ell, 0)}(\zeta) - \mathcal{T}^{(\ell, 0, 0)}(\zeta). \quad (6.17)$$

Relations of the type (6.13), (6.14) and (6.16), (6.17) are usually called fusion relations, see [11, 34, 35]. They are given here in the form independent of the representation of the quantum group in the quantum space, that is, they are the *universal fusion relations*.

### 6.3. Quantum Jacobi–Trudi identity

Equation (6.14) allows one to express  $\mathcal{T}^{(\ell,0)}$  through  $\mathcal{T}^{(1,0)}$ . In the higher-rank case the situation is more intricate. Considering  $n = 3$ , we first note that equation (6.16) can be generalized as follows. Let  $\ell_1, \ell_2$  be positive integers, such that  $\ell_1 \geq \ell_2$ . Choosing the weight components  $\lambda_i$  in (6.15) as  $\lambda_1 = \ell_1 + 2, \lambda_2 = \ell_2 + 1, \lambda_3 = 1, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = -1$ , we obtain

$$\mathcal{T}^{(\ell_1, \ell_2, 0)}(\zeta) = \mathcal{T}^{(\ell_1, 0, 0)}(\zeta)\mathcal{T}^{(\ell_2, 0, 0)}(q^{2/s}\zeta) - \mathcal{T}^{(\ell_1+1, 0, 0)}(q^{2/s}\zeta)\mathcal{T}^{(\ell_2-1, 0, 0)}(\zeta). \quad (6.18)$$

Equation (6.18) implies that  $\mathcal{T}^{(\ell_1, \ell_2, 0)}(\zeta)$  can be expressed via  $\mathcal{T}^{(\ell, 0, 0)}(\zeta)$ , and subsequently via  $\mathcal{T}^{(1, 0, 0)}(\zeta), \mathcal{T}^{(1, 1, 0)}(\zeta)$ . Explicitly, the result of this procedure is given by the determinant

$$\mathcal{T}^{(\ell_1, \ell_2, 0)}(\zeta) = \det \left( \varepsilon_{\ell_i^t - i + j}(q^{-2(j-1)/s}\zeta) \right)_{1 \leq i, j \leq \ell_1}, \quad (6.19)$$

where we use the notation

$$\begin{aligned} \varepsilon_0(\zeta) = \varepsilon_3(\zeta) = 1, \quad \varepsilon_1(\zeta) = \mathcal{T}^{(1, 0, 0)}(\zeta), \quad \varepsilon_2(\zeta) = \mathcal{T}^{(1, 1, 0)}(\zeta), \\ \varepsilon_k(\zeta) = 0 \quad \forall k < 0, k > 3, \\ \ell_i^t = 2, \quad 1 \leq i \leq \ell_2, \quad \ell_i^t = 1, \quad \ell_2 < i \leq \ell_1. \end{aligned}$$

Here, one connects the integers  $\ell_1$  and  $\ell_2$  with the Young diagram with the rows of the length  $\ell_1$  and  $\ell_2$ , then the numbers  $\ell_i^t$  describe the rows of the transposed diagram. Equation (6.19) can thus be regarded as the universal quantum analog of the Jacobi–Trudi identity from the theory of symmetric polynomials [11, 36, 37]. To prove this identity in the universal form, we use  $(n + 1)$ -term universal functional relations and certain symmetries of the universal transfer operators [18].

Remarkably, also the barred universal transfer operators  $\bar{\mathcal{T}}^{(\ell_1, \ell_2, 0)}(\zeta)$  can be expressed through the same basic universal transfer operators  $\mathcal{T}^{(1, 0, 0)}(\zeta)$  and  $\mathcal{T}^{(1, 1, 0)}(\zeta)$ . To come to this conclusion, one can write the Jacobi–Trudi identity for  $\bar{\mathcal{T}}^{(\ell_1, \ell_2, 0)}(\zeta)$  down and use the relations

$$\bar{\mathcal{T}}^{(1, 0, 0)}(\zeta) = \mathcal{T}^{(1, 0, 0)}(q^{3/s}\zeta), \quad \bar{\mathcal{T}}^{(1, 1, 0)}(\zeta) = \mathcal{T}^{(1, 1, 0)}(q^{1/s}\zeta),$$

following actually from the comparison of the representations  $\varphi_\zeta^\lambda$  and  $\bar{\varphi}_\zeta^\lambda$ .

## 7. Conclusions

We have presented a short review of the universal functional  $TQ$ - and  $TT$ -relations for the quantum integrable systems associated with the quantum affine algebra  $U_q(\mathcal{L}(\mathfrak{sl}_n))$ , emphasizing the lower-rank ( $n = 2$ ) and a higher-rank ( $n = 3$ ) cases as basic examples. We have also given the quantum analog of the Jacobi–Trudi identity allowing one to express ‘higher-weight’ universal transfer operators by means of two non-trivial basic operators with lowest weight. Here, the representation of the quantum group in the auxiliary space is properly specified, giving rise to this or another universal integrability object, while the representation in the quantum space is not.

To consider concrete physical models, one should make the corresponding specialization of the quantum group also in the quantum space. Thus, choosing an appropriate infinite- or finite-dimensional representation of the quantum group in the quantum space, one can consider either a low-dimensional quantum field theory or a lattice model with the corresponding spin-chain counterpart. Upon such a specialization, one can use the remarkable Khoroshkin–Tolstoy formula [38–41] for the universal  $R$ -matrix to carry out explicit calculations of the monodromy operators [19, 42],  $R$ -operators [14, 15, 19, 43–46] and  $L$ -operators [14, 15]. Another possibility is offered by a 3D approach to the Yang–Baxter equation [47].

### Acknowledgments

This work was supported in part by the German Research Foundation grant KL 645/10-1, the Russian Foundation for Basic Research grants # 13-01-00217, # 14-01-91335, and by the Volkswagen Foundation. We thank our colleagues and coauthors H. Boos, F. Göhmann and A. Klümper for discussions. Kh.S.N. thanks the organizers of the ISQS'23 in Prague (June 23–27, 2015) for invitation and hospitality.

### Appendix. Defining $U_q(\mathfrak{gl}_n)$ -module relations

The root system of type  $A_{n-1}$  consists of  $n(n-1)$  roots. We introduce for the system of positive roots the *normal ordering* [48,49]

$$(\alpha_1), (\alpha_1 + \alpha_2, \alpha_2), (\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3), \dots \\
 \dots (\alpha_1 + \alpha_2 + \dots + \alpha_i, \alpha_2 + \dots + \alpha_i, \dots, \alpha_i), \dots \\
 \dots (\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}, \alpha_2 + \dots + \alpha_{n-1}, \dots, \alpha_{n-1}),$$

where  $\alpha_j, j = 1, \dots, n-1$  are the simple positive roots. As usual, we define the root vectors corresponding to the negative composite roots as follows. Let  $\gamma = \alpha + \beta$  be a composite positive root. Then the relation

$$F_\gamma = F_\beta F_\alpha - q F_\alpha F_\beta$$

gives the root vector  $F_\gamma$  corresponding to the negative root  $-\gamma$ . The whole set of negative root vectors ordered in accordance with the above given normal ordering of the roots is used to define the basis vectors of the  $U_q(\mathfrak{gl}_n)$  modules. We write

$$v_k = (F_1^{k_1})(F_{12}^{k_{12}} F_2^{k_2})(F_{13}^{k_{13}} F_{23}^{k_{23}} F_3^{k_3}) \dots (F_{1i}^{k_{1i}} F_{2i}^{k_{2i}} \dots F_i^{k_i}) \dots (F_{1,n-1}^{k_{1,n-1}} F_{2,n-1}^{k_{2,n-1}} \dots F_{n-1}^{k_{n-1}}) v_0$$

where  $k$  means the set of non-negative integers  $(k_1; k_{12}, k_2; k_{13}, k_{23}, k_3; \dots k_{1i}, k_{2i}, \dots, k_i; \dots k_{1,n-1}, k_{2,n-1}, \dots, k_{n-1})$  being the powers of the root vectors acting on the highest-weight vector  $v_0$ . Here we use the notation  $F_i$  for the root vector corresponding to the simple negative root  $-\alpha_i$ , and  $F_{ij}$  means the root vector corresponding to the composite negative root  $-\alpha_i - \dots - \alpha_j$  which is defined according to the above relation.

Acting on the basis vectors  $v_k$  by the generators  $q^{v_{G_i}}, E_i$  and  $F_i$  of  $U_q(\mathfrak{gl}_n)$ , we obtain the defining relations of the  $U_q(\mathfrak{gl}_n)$  modules. For the simplest lower-rank case  $n = 2$  the basis vectors are  $v_k = F^k v_0$ , and we obtain

$$q^{v_{G_1}} v_k = q^{v(\lambda_1 - k)} v_k, \quad q^{v_{G_2}} v_k = q^{v(\lambda_2 + k)} v_k, \\
 F v_k = v_{k+1}, \quad E v_k = [k]_q [\lambda_1 - \lambda_2 - k + 1]_q v_{k-1}.$$

For a more complicated and rich higher-rank case  $n = 3$  we act on  $v_k = F_1^{k_1} F_{12}^{k_{12}} F_2^{k_2} v_0$  and obtain

$$q^{v_{G_1}} v_k = q^{v(\lambda_1 - k_1 - k_3)} v_k, \quad q^{v_{G_2}} v_k = q^{v(\lambda_2 + k_1 - k_2)} v_k, \quad q^{v_{G_3}} v_k = q^{v(\lambda_3 + k_2 + k_3)} v_k, \\
 F_1 v_k = v_{k+\varepsilon_1}, \quad F_2 v_k = q^{k_1 - k_3} v_{k+\varepsilon_3} + [k_1]_q v_{k-\varepsilon_1 + \varepsilon_2}, \quad F_3 v_k = q^{-k_1} v_{k+\varepsilon_2}, \\
 E_1 v_k = [\lambda_1 - \lambda_2 - k_1 + k_2 - k_3 + 1]_q [k_1]_q v_{k-\varepsilon_1} - q^{\lambda_1 - \lambda_2 + k_2 - k_3 + 2} [k_2]_q v_{k-\varepsilon_2 + \varepsilon_3}, \\
 E_2 v_k = [\lambda_2 - \lambda_3 - k_2 + 1]_q [k_2]_q v_{k-\varepsilon_3} + q^{-(\lambda_2 - \lambda_3) + 2k_2} [k_3]_q v_{k+\varepsilon_1 - \varepsilon_2}, \\
 E_3 v_k = q^{k_1} [\lambda_1 - \lambda_3 - k_1 - k_2 - k_3 + 1]_q [k_3]_q v_{k-\varepsilon_2}, \\
 - q^{-(\lambda_1 - \lambda_2) + k_1 - k_2 + k_3 - 1} [\lambda_2 - \lambda_3 - k_2 + 1]_q [k_1]_q [k_2]_q v_{k-\varepsilon_1 - \varepsilon_3}.$$

Here, for brevity, we have denoted  $F_{12}$  and  $k_{12}$  by  $F_3$  and  $k_3$ , respectively, and use the notation  $k - \varepsilon_i, i = 1, 2, 3$ , for the sets of three integers  $(k_1 - 1, k_3, k_2), (k_1, k_3 - 1, k_2)$  and  $(k_1, k_3, k_2 - 1)$ , respectively.

Thorough consideration of these module relations and appropriate interpretation of their consequences is the actual basis for the complete proof of the universal functional relations.

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