

First results on the representation theory of the Ultrahyperbolic BMS group $\mathcal{UHB}(2, 2)$

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Abstract. The Bondi–Metzner–Sachs (BMS) group B is the common asymptotic group of all asymptotically flat (lorentzian) space–times, and is the best candidate for the universal symmetry group of General Relativity (G.R.). B admits generalizations to real space–times of any signature, to complex space–times, and supersymmetric generalizations for any space–time dimension. With this motivation McCarthy constructed the strongly continuous unitary irreducible representations (IRs) of B some time ago, and he identified $B(2, 2)$ as the generalization of B appropriate to the to the ultrahyperbolic signature $(+, +, -, -)$ and asymptotic flatness in null directions. We continue this programme by introducing a new group $\mathcal{UHB}(2, 2)$ in the group theoretical study of ultrahyperbolic G.R. which happens to be a proper subgroup of $B(2, 2)$. We report on the first general results on the representation theory of $\mathcal{UHB}(2, 2)$. In particular the main general results are that the all little groups of $\mathcal{UHB}(2, 2)$ are compact and that the Wigner–Mackeys inducing construction is exhaustive despite the fact that $\mathcal{UHB}(2, 2)$ is not locally compact in the employed Hilbert topology.

1. Introduction

In 1939 Wigner laid the foundations of special relativistic quantum mechanics and relativistic quantum field theory by constructing the Hilbert space strongly continuous unitary irreducible representations (IRs) of the (universal cover) of the Poincare group P . The Bondi–Metzner–Sachs (BMS) group B is the common asymptotic group of all curved real lorentzian space–times which are asymptotically flat in future null directions [1, 2], and is the best candidate for the universal symmetry group of G.R.. In a quantum setting the universal property of B for G.R. make it reasonable to attempt to lay a similarly firm foundation for quantum gravity by following through the analogue of Wigner’s programme with B replacing P . Some years ago McCarthy constructed explicitly [3, 4, 5, 6, 7, 8, 9, 10] the IRs of B for exactly this purpose. This work was based on G.W.Mackey’s pioneering work on group representations [11, 12, 13, 14, 15]; in particular an extension to the relevant infinite–dimensional case of his semi–direct product theory.

It is difficult to overemphasize the importance of Piard’s results [16, 17] who soon afterwards proved that *all* the IRs of B , when this is equipped with the Hilbert topology, are derivable by the inducing construction. This proves the exhaustivity of McCarthy’s list of representations and renders his results even more important.

However, in quantum gravity, complexified or euclidean versions of G.R. are frequently considered and the question arises: Are there similar symmetry groups for these versions of the theory? McCarthy constructed [18], in abstract form, all possible analogues of B , both real



and in any signature, or complex, with all possible notions of asymptotic flatness ‘near infinity’. There are, in fact, forty–one such groups. These abstract constructions were given in a quantum setting; the paper was concerned with finding the IRs of the groups \mathcal{G} in Hilbert spaces (especially for the complexification \mathcal{CB} of B itself). It was argued that these Hilbert space representations were related to elementary particles and quantum gravity (via gravitational instantons).

Let $B(2, 2)$ be the BMS group \mathcal{G} appropriate to the ‘ultrahyperbolic signature’ and asymptotic flatness in null directions. $B(2, 2)$, is like B itself, based on a null cone [18], and it is given by

$$B(2, 2) = C_e^\infty(T^2, R) \otimes_T G^2 \tag{1}$$

,i.e., it is the semi–direct product of the group G^2 , where $G = SL(2, R)$, times the abelian normal subgroup $C_e^\infty(T^2, R)$ of so called supertranslations; $C_e^\infty(T^2, R)$ is the set of *even* real–valued infinitely–differentiable functions defined on the 2–Torus $T^2 = S^1 \times S^1$, S^1 being the set of vectors of unit length in $R^2 - \{0\}$. That is the functions $\alpha(m, n) \in C_e^\infty(T^2, R)$ satisfy the even–ness condition

$$\alpha(m, n) = \alpha(-m, -n),$$

where, $m = \frac{x}{|x|}$, $x = (x_1, x_2) \in R^2 - \{0\}$, and, similarly, $n = \frac{y}{|y|}$, $y = (y_1, y_2) \in R^2 - \{0\}$. The representation theory of $B(2, 2)$ has been initiated elsewhere [19, 20].

The present paper reports the first general results on the representation theory of

$$\mathcal{UHB}(2, 2) = C^\infty(P_1(R) \times P_1(R), R) \otimes_T G^2, \tag{2}$$

$P_1(R) = S^1/Z_2$ being the one–dimensional real projective space (the circle quotient the antipodal map). $\mathcal{UHB}(2, 2)$ arises naturally in the construction of the generalizations of B given in [18] but it remained unnoticed in [18]. The crucial difference between $\mathcal{UHB}(2, 2)$ and $B(2, 2)$ is that for $\mathcal{UHB}(2, 2)$ the supertranslations are *completely unconstrained*, whereas, for $B(2, 2)$ they are described by *even* functions on the torus T^2 . The representation theory of $\mathcal{UHB}(2, 2)$ has been initiated in [21, 22, 23].

2. The group $\mathcal{UHB}(2, 2)$

Recall that the ultrahyperbolic version of Minkowski space is the vector space R^4 of row vectors with 4 real components, with scalar product defined as follows. Let $x, y \in R^4$ have components x^μ and y^μ respectively, where $\mu = 0, 1, 2, 3$. Define the scalar product $x.y$ between x and y by

$$x.y = x^0y^0 + x^2y^2 - x^1y^1 - x^3y^3. \tag{3}$$

Then the ultrahyperbolic version of Minkowski space, sometimes written $R^{2,2}$, is just R^4 with this scalar product.

In [21] it was shown that

Theorem 1 *The group $\mathcal{UHB}(2, 2)$ can be realised as*

$$\mathcal{UHB}(2, 2) = L^2(\mathcal{P}, \lambda, R) \otimes_T G^2 \tag{4}$$

with semi–direct product specified by

$$(T(g, h)\alpha)(x, y) = k_g(x)s_g(x)k_h(w)s_h(w)\alpha(xg, yh), \tag{5}$$

where $\alpha \in L^2(\mathcal{P}, \lambda, R)$ and $(x, y) \in \mathcal{P}$. For ease of notation, we write \mathcal{P} for the torus $T \simeq P_1(R) \times P_1(R)$, $P_1(R)$ is the one–dimensional real projective space, and \mathcal{G} for $G \times G$, $G = SL(2, R)$. In analogy to B , it is natural to choose a measure λ on \mathcal{P} which is invariant

under the maximal compact subgroup $SO(2) \times SO(2)$ of \mathcal{G} . $L^2(\mathcal{P}, \lambda, R)$ is the separable Hilbert space of real-valued functions defined on \mathcal{P} .

Moreover, if $g \in G$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{6}$$

then the components x_1, x_2 of $\mathbf{x} \in R^2$ transform linearly, so that the ratio $x = x_1/x_2$ transforms fraction linearly. Writing xg for the transformed ratio,

$$xg = \frac{(\mathbf{x}g)_1}{(\mathbf{x}g)_2} = \frac{x_1a + x_2c}{x_1b + x_2d} = \frac{xa + c}{xb + d}. \tag{7}$$

The factors $k_g(x)$ and $s_g(x)$ on the right hand side of (5) are defined by

$$k_g(x) = \left\{ \frac{(xb + d)^2 + (xa + c)^2}{1 + x^2} \right\}^{\frac{1}{2}}, \tag{8}$$

$$s_g(x) = \frac{xb + d}{|xb + d|}, \tag{9}$$

with similar formulae for $yh, k_h(y)$ and $s_h(y)$.

It is well known that the topological dual of a Hilbert space can be identified with the Hilbert space itself, so that we have $L^2'(\mathcal{P}, \lambda, R) \simeq L^2(\mathcal{P}, \lambda, R)$. In fact, given a continuous linear functional $\phi \in L^2'(\mathcal{P}, \lambda, R)$, we can write, for $\alpha \in L^2(\mathcal{P}, \lambda, R)$

$$(\phi, \alpha) = \langle \phi, \alpha \rangle \tag{10}$$

where the function $\phi \in L^2(\mathcal{P}, \lambda, R)$ on the right is uniquely determined by (and denoted by the same symbol as) the linear functional $\phi \in L^2'(\mathcal{P}, \lambda, R)$ on the left. The representation theory of $\mathcal{UHB}(2, 2)$ is governed by the dual action T' of \mathcal{G} on the topological dual $L^2'(\mathcal{P}, \lambda, R)$ of $L^2(\mathcal{P}, \lambda, R)$. The dual action T' is defined by:

$$\langle T'(g, h)\phi, \alpha \rangle = \langle \phi, T(g^{-1}, h^{-1})\alpha \rangle. \tag{11}$$

A short calculation gives

$$(T'(g, h)\phi)(x, y) = k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y)\phi(xg, yh). \tag{12}$$

Now, this action T' of \mathcal{G} on $L^2'(\mathcal{P}, \lambda, R)$, given explicitly above, is like the action T of \mathcal{G} on $L^2(\mathcal{P}, \lambda, R)$, continuous. The ‘little group’ L_ϕ of any $\phi \in L^2'(\mathcal{P}, \lambda, R)$ is the stabilizer

$$L_\phi = \{(g, h) \in \mathcal{G} \mid T'(g, h)\phi = \phi\}. \tag{13}$$

By continuity, $L_\phi \subset \mathcal{G}$ is a closed subgroup.

3. Representation theory

Let A and \mathcal{G} be topological groups, and let T be a given homomorphism from \mathcal{G} into the group of automorphisms $\text{Aut}(A)$ of A . Suppose A is abelian and $\mathcal{H} = A \circledast_T \mathcal{G}$ is the semi-direct product of A and \mathcal{G} , specified by the continuous action $T : \mathcal{G} \rightarrow \text{Aut}(A)$. In the product topology of $A \times \mathcal{G}$, \mathcal{H} then becomes a topological group. It is assumed that it becomes a separable locally compact topological group.

In order to give the operators of the induced representations explicitly it is necessary ([11], [12], [13], [14], [15] and references therein) to give the following information

- (i) An irreducible unitary representation U of L_{ϕ_o} on a Hilbert space D for each L_{ϕ_o} .
- (ii) A \mathcal{G} –quasi–invariant measure μ on each orbit $\mathcal{G}\phi \approx \mathcal{G}/L_{\phi_o}$; where L_{ϕ_o} denotes the little group of the base point $\phi_o \in A'$ of the orbit $\mathcal{G}\phi_o$; A' is the topological dual of A .

Let D_μ be the space of functions $\psi : \mathcal{G} \rightarrow D$ which satisfy the conditions

$$(a) \quad \psi(gl) = U(l^{-1})\psi(g) \quad (g \in \mathcal{G}, l \in L_\phi)$$

$$(b) \quad \int_{\mathcal{G}\phi_o} \langle \psi(q), \psi(q) \rangle d\mu(q) < \infty,$$

where the scalar product under the integral sign is that of D . Note, that the constraint (a) implies that $\langle \psi(gl), \psi(gl) \rangle = \langle \psi(g), \psi(g) \rangle$, and therefore the inner product $\langle \psi(g), \psi(g) \rangle$, $g \in \mathcal{G}$, is constant along every element q of the coset space $\mathcal{G}/L_{\phi_o} \approx \mathcal{G}\phi_o$. This allows to assign a meaning to $\langle \psi(q), \psi(q) \rangle$, where $q = gL_{\phi_o}$, by defining $\langle \psi(q), \psi(q) \rangle := \langle \psi(g), \psi(g) \rangle$. Thus the integrand in (b) becomes meaningful due to the condition (a). A pre–Hilbert space structure can now be given to D_μ by defining the scalar product

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathcal{G}\phi_o} \langle \psi_1(q), \psi_2(q) \rangle d\mu(q), \tag{14}$$

where $\psi_1, \psi_2 \in D_\mu$. It is convenient to complete the space D_μ with respect to the norm defined by the scalar product (14). In the resulting Hilbert space, functions are identified whenever they differ, at most, on a set of μ –measure zero. Thus our Hilbert space is

$$D_\mu = L^2(\mathcal{G}\phi_o, \mu, D). \tag{15}$$

Define now an action of $\mathcal{H} = A \otimes_T \mathcal{G}$ on D_μ by

$$(g_o\psi)(q) = \sqrt{\frac{d\mu_{g_o}}{d\mu}}(q)\psi(g_o^{-1}q), \tag{16}$$

$$\alpha\psi(q) = e^{i\langle g_o\phi_o, \alpha \rangle} \psi(q) \tag{17}$$

where, $g_o \in \mathcal{G}$, $q \in \mathcal{G}\phi_o$, and $\alpha \in A$. Eqs. (16) and (17) define the IRs of \mathcal{HB} induced for each $\phi_o \in A'$ and each irreducible representation U of L_{ϕ_o} . The ‘Jacobian’ $\frac{d\mu_{g_o}}{d\mu}$ of the group transformation is known as the *Radon–Nikodym* derivative of μ_{g_o} with respect to μ and ensures that the resulting IRs of \mathcal{HB} are unitary.

The central results of induced representation theory ([11], [12], [13], [14], [15] and references therein) are the following

- (i) Given the topological restrictions on $\mathcal{H} = A \otimes_T \mathcal{G}$ (separability and local compactness), any representation of \mathcal{H} , constructed by the method above, is irreducible if the representation U of L_{ϕ_o} on D is irreducible. Thus an irreducible representation of \mathcal{H} is obtained for each $\phi_o \in A'$ and each irreducible representation U of L_{ϕ_o} .
- (ii) If $\mathcal{H} = A \otimes_T \mathcal{G}$ is a regular semi–direct product (i.e., A' contains a Borel subset which meets each orbit in A' under \mathcal{H} in just one point) then *all* of its irreducible representations can be obtained in this way.

4. Obstructions and resolutions

Two remarks are in order regarding the representations of $\mathcal{UHB}(2, 2)$ obtained by the above construction

- (i) As it is explained in [21] the subgroup $L^2(\mathcal{P}, \lambda, R)$ of $\mathcal{UHB}(2, 2) = L^2(\mathcal{P}, \lambda, R) \otimes_T \mathcal{G}$ is topologised as a (pre) Hilbert space by using a natural measure on $\mathcal{P} = P_1(R) \times P_1(R)$ and by introducing a scalar product into $L^2(\mathcal{P}, \lambda, R)$. If R^8 is endowed with the natural metric topology then the group $\mathcal{G} = SL(2, R) \times SL(2, R)$, considered as a subset of R^8 , inherits the induced topology on \mathcal{G} . In the product topology of $L^2(\mathcal{P}, \lambda, R) \times \mathcal{G}$ $\mathcal{UHB}(2, 2)$ is a non-locally compact group (the proof follows without substantial change Cantoni's proof [24], see also [3]). (In fact the subgroup $L^2(\mathcal{P}, \lambda, R)$, and therefore the group $\mathcal{UHB}(2, 2)$ can be employed with many different topologies. The Hilbert type topology employed here appears to describe quantum mechanical systems in asymptotically flat space-times [9]). Since in the Hilbert type topology $\mathcal{UHB}(2, 2) = L^2(\mathcal{P}, \lambda, R) \otimes_T \mathcal{G}$ is not locally compact the theorems dealing with the irreducibility of the representations obtained by the above construction no longer apply (see e.g. [13]). However, it can be proved that the induced representations obtained above *are* irreducible. The proof follows very closely the one given in [6] for the case of the original BMS group B .
- (ii) Here it is assumed that $\mathcal{UHB}(2, 2)$ is equipped with the Hilbert topology. It is of outmost significance that it can be proved [21] that in this topology $\mathcal{UHB}(2, 2)$ is a regular semi-direct-product. The proof follows the corresponding proof [16, 17] for the group B . Regularity amounts to the fact that [12] $L^2(\mathcal{P}, \lambda, R)$ can have no equivalent classes of quasi-invariant measures μ such that the action of \mathcal{G} is strictly ergodic with respect to μ . When such measures μ do exist it can be proved [12] that an irreducible representation of the group, with the semi-direct-product structure at hand, may be associated with each that is not equivalent to any of the IRs constructed by the Wigner-Mackey's inducing method. In a different topology it is not known if $\mathcal{UHB}(2, 2)$ is a regular or irregular semi-direct-product. Irregularity of $\mathcal{UHB}(2, 2)$ in a topology different from the Hilbert topology would imply that there are IRs of $\mathcal{UHB}(2, 2)$ that are not equivalent to any of the IRs obtained above by the inducing construction. Strictly ergodic actions are notoriously hard to deal with even in the locally compact case. Indeed, for locally compact non-regular semi-direct products, there is no known example for which all inequivalent irreducibles arising from strictly ergodic actions have been found. For the other 41 groups defined in [18] regularity has only been proved for B [16, 17] when B is equipped with the Hilbert topology. Similar remarks apply to all of them regarding IRs arising from strictly ergodic actions in a given topology.

5. Results

The new results are: A new group $\mathcal{UHB}(2, 2)$ is introduced for the group theoretical study of ultrahyperbolic G.R.. $\mathcal{UHB}(2, 2)$ is a proper subgroup of the group $B(2, 2)$ initially proposed in [18] as appropriate to the 'ultrahyperbolic signature' and asymptotic flatness in null directions. Both $\mathcal{UHB}(2, 2)$ and $B(2, 2)$ are based on the null cone \mathcal{N} of $R^{2,2}$. The crucial difference between $\mathcal{UHB}(2, 2)$ and $B(2, 2)$ is that the supertranslations in the case of $\mathcal{UHB}(2, 2)$ are *free* functions defined on $P_1(R) \times P_1(R)$, whereas in the case of $B(2, 2)$ the supertranslations are *even* functions defined on $S^1 \times S^1$. Remarkably, it is the nature of the supertranslations of $\mathcal{UHB}(2, 2)$ which allows to establish contact [21] of the representation theory of $\mathcal{UHB}(2, 2)$ with standard representation theory; something which is not feasible for the representation theory of $B(2, 2)$. $\mathcal{UHB}(2, 2)$ captures more efficiently, via its subgroup $L^2(\mathcal{P}, \lambda, R)$, the fundamental characteristic of $R^{2,2}$, namely, that there is no clear-cut distinction of past and future in $R^{2,2}$. In [21] it is proved that when $\mathcal{UHB}(2, 2)$ is employed with the Hilbert topology *all* little groups

of $\mathcal{UHB}(2, 2)$ are compact. Moreover in [21] it is shown that the Wigner–Mackey’s inducing construction is exhaustive despite the fact that $\mathcal{UHB}(2, 2)$ is not locally compact in the employed Hilbert topology. This result is rather important because other group theoretical approaches to quantum gravity which invoke Wigner–Mackey’s inducing construction (see e.g. [25]) are typically plagued by the non–exhaustiveness of the inducing construction which results precisely from the fact that the group in question is not locally compact in the prescribed topology. Exhaustiveness is not just a mathematical nicety: If the inducing construction is not exhaustive one cannot simply know if the most interesting information or part of it is coded in the irreducibles which cannot be found by the Wigner–Mackey’s inducing procedure. These results, compactness of the little groups and exhaustiveness of the inducing construction, not only are they significant for the group theoretical approach to quantum gravity advocated here, but also they have repercussions [21] for the other approaches to quantum gravity.

In comparing the representation theory of $\mathcal{UHB}(2, 2)$ [21, 22, 23] with the representation theory of $B(2, 2)$ [19, 20], we find both similarities and differences. The key difference between $\mathcal{UHB}(2, 2)$ and $B(2, 2)$, the supertranslations in the case of $\mathcal{UHB}(2, 2)$ are *free* functions defined on $P_1(R) \times P_1(R)$, whereas in the case of $B(2, 2)$ the supertranslations are *even* functions defined on $S^1 \times S^1$, leaves its trace on the representation theory: the proof [21] of compactness for little groups of $\mathcal{UHB}(2, 2)$ is similar to, but subtly different from, the corresponding proof [19] for $B(2, 2)$. On the other hand, an interesting similarity between $\mathcal{UHB}(2, 2)$ and $B(2, 2)$ lies in the structure of their little groups: Their one–dimensional little groups form an unexpected family of continuous/discrete groups (with many connected components). Also, their finite little groups involve subgroups of direct products of the symmetry groups of the regular polygons only; the regular polyhedra do not appear at all here. The regular polyhedra appear [5] in the representation theory of the ordinary Bondi–Metzner–Sachs group.

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