

Hamiltonian action of spinning particle with gravimagnetic moment

Alexei A Deriglazov^{1,2}, and W Guzmán Ramírez¹

¹ Depto. de Matemática, ICE, Universidade Federal de Juiz de Fora, MG, Brazil

² Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin Ave. 30, Russian Federation

E-mail: alexei.deriglazov@ufjf.edu.br

Abstract. We develop Hamiltonian variational problem for spinning particle non-minimally interacting with gravity through the gravimagnetic moment κ . For $\kappa = 0$ our model yields Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations, the latter show unsatisfactory behavior of MPTD-particle in ultra-relativistic regime: its longitudinal acceleration increases with velocity. $\kappa = 1$ yields a modification of MPTD-equations with the reasonable behavior: in the homogeneous fields, both longitudinal acceleration and (covariant) precession of spin-tensor vanish as $v \rightarrow c$.

Evolution of a rotating body in curved background usually described on the base of equations formulated by Mathisson [1], Papapetrou [2], Tulczyjew [3], and Dixon [4]. In this work we will refer to MPTD-equations as those (6.31)-(6.33) in [4]. Detailed analysis and interpretation of these equations and their generalizations is an actual task since they are widely used now to account spin effects in compact binaries and rotating black holes, see [5, 6] and references therein.

It was recently observed that MPTD-equations have unsatisfactory behavior in ultra-relativistic limit: longitudinal acceleration of the particle grows up to infinity in this limit [7]. Here we briefly describe the origin for such behavior and then obtain Hamiltonian variational problem which yields equations with improved behavior. The improved equations correspond to the spinning particle non-minimally interacting with gravity through the gravimagnetic moment κ . Our main conclusion is that spinning particle with gravimagnetic moment $\kappa = 1$ seems to be more promising for description of a rotating body in a gravitational field than MPTD-equations (the latter correspond to the spinning particle with vanishing gravimagnetic moment).

We first obtain MPTD-equations as the Hamiltonian system starting from the vector model of spin. This allows us to exclude, in unambiguous way, the momenta and to obtain second-order equation for trajectory, which is necessary for analysis of ultra-relativistic behavior of the particle. In the vector model of relativistic spin the configuration-space variables¹ are the position $x^\mu(\tau)$ and non Grassmann vector $\omega^\mu(\tau)$ attached to the point x^μ . The Frenkel spin-tensor [8] in our model is a composite quantity constructed from ω^μ and its conjugated

¹ The dynamical variables are taken in arbitrary parametrization τ , then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Covariant derivative is $\nabla P^\mu = \frac{dP^\mu}{d\tau} + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha P^\beta$ and curvature is $R^\sigma{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\sigma{}_{\lambda\nu} - \partial_\nu \Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\beta\mu} \Gamma^\beta{}_{\lambda\nu} - \Gamma^\sigma{}_{\beta\nu} \Gamma^\beta{}_{\lambda\mu}$. The square brackets mean antisymmetrization, $\omega^{[\mu} \pi^{\nu]} = \omega^\mu \pi^\nu - \omega^\nu \pi^\mu$. We use the condensed notation $\dot{x}^\mu G_{\mu\nu} \dot{x}^\nu = \dot{x} G \dot{x}$, $N^\mu{}_\nu \dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = g_{\mu\nu} \omega^\mu \omega^\nu$, $\mu, \nu = 0, 1, 2, 3$, and so on. Notation for the scalar functions constructed from second-rank tensors are $\theta S = \theta^{\mu\nu} S_{\mu\nu}$, $S^2 = S^{\mu\nu} S_{\mu\nu}$.



momentum $\pi_\mu = \frac{\partial L}{\partial \dot{\omega}^\mu}$ as follows: $S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu)$. In contrast to its constituents ω^μ and π^ν , the spin-tensor is invariant under local spin-plane symmetry presented in the model [9], and thus represents an observable quantity. The spinning particle in flat space is described by the Lagrangian action [10, 7]

$$S = -\frac{1}{\sqrt{2}} \int d\tau \sqrt{m^2 c^2 - \frac{\alpha}{\omega^2}} \sqrt{-\dot{x} N \dot{x} - \dot{\omega} N \dot{\omega} + \sqrt{[\dot{x} N \dot{x} + \dot{\omega} N \dot{\omega}]^2 - 4(\dot{x} N \dot{\omega})^2}}. \quad (1)$$

The matrix $N_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega_\mu \omega_\nu}{\omega^2}$ is the projector on the plane orthogonal to ω^ν : $N_{\mu\nu} \omega^\nu = 0$, the parameter α determines the value of spin. The value $\alpha = \frac{3\hbar^2}{4}$ corresponds to the elementary spin one-half particle. In the spinless limit, $\alpha = 0$ and $\omega^\mu = 0$, the expression (1) reduces to the standard Lagrangian of relativistic particle, $-mc\sqrt{-\dot{x}^\mu \dot{x}_\mu}$.

The minimal interaction with gravity is achieved by covariantization of the formulation (1), that is we replace $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$, and usual derivative by the covariant one, $\dot{\omega}^\mu \rightarrow \nabla \omega^\mu = \frac{d\omega^\mu}{d\tau} + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \omega^\beta$. With the resulting expression at hands, we pass to the Hamiltonian formulation. This gives the Hamiltonian variational problem which is equivalent to (1)

$$S_H = \int d\tau \left[p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu - \left[\frac{\lambda_1}{2} \left(P^2 + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi) \right] \right]. \quad (2)$$

By λ_a we denoted the Lagrangian multipliers, they are written in front of the corresponding primary constraints. For the general-covariant and spin-plane invariant phase-space quantities $\dot{x}^\mu$, $P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} - \Gamma_{\alpha\mu}^\beta \omega^\alpha \pi_\beta$ and $S^{\mu\nu}$, the variational problem (1) (or, equivalently, the variational problem (2)) yields the Hamiltonian equations which can be written as follows:

$$P^\mu = \frac{mc}{\sqrt{-\dot{x} G \dot{x}}} \tilde{T}^\mu{}_\nu \dot{x}^\nu, \quad \nabla P^\mu = -\frac{1}{4} \theta^\mu{}_\nu \dot{x}^\nu, \quad \nabla S^{\mu\nu} = 2P^{[\mu} \dot{x}^{\nu]}, \quad (3)$$

as well as the constraints

$$S^{\mu\nu} P_\nu = 0, \quad (4)$$

$$P^2 + (mc)^2 = 0, \quad S^2 = 8\alpha. \quad (5)$$

We have denoted

$$\theta^\mu{}_\nu \equiv R^\mu{}_{\nu\alpha\beta} S^{\alpha\beta}. \quad (6)$$

In the equations appeared the matrix G constructed with help of "tetrad field" $\tilde{T}$

$$G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \equiv \tilde{T}^\alpha{}_\mu g_{\alpha\beta} \tilde{T}^\beta{}_\nu, \quad \tilde{T}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{1}{8m^2 c^2} S^{\mu\sigma} \theta_{\sigma\nu}. \quad (7)$$

The matrix G is composed from the original metric $g_{\mu\nu}$ plus (spin and field-dependent) contribution $h_{\mu\nu}$. So we call G the effective metric produced along the world-line by interaction of spin with gravity.

Comparing (3)-(5) with MPTD-equations², we concluded [7] that all the trajectories of MPTD-equations with given integrals of motion $P^2 = k$ and $S^2 = \beta$ are described by our

² In [11] we noticed that MPTD-equations have an expected spinless limit being completed with mass-shell type condition. We also point out that the Lagrangian considered above yields the fixed value of spin, that is this corresponds to an elementary particle. In [11] we presented the modification which leads to the theory with unfixed spin, and, similarly to Hanson-Regge approach [12], with a mass-spin trajectory constraint. The modified model has four physical degrees of freedom in the spin-sector.

spinning particle with mass $m = \frac{k}{c}$ and spin $\alpha = \frac{\beta}{8}$. By the way, we demonstrated that MPTD-equations correspond to the minimal interaction of the spinning particle with gravity.

Inverting the first equation from (3), we can exclude momenta P^μ from the Hamiltonian equations. This yields second-order equation for the particle's position $x^\mu(\tau)$ (we refer the resulting equations as Lagrangian form of MPTD-equations)

$$\nabla \left[\frac{\tilde{T}^\mu{}_\nu \dot{x}^\nu}{\sqrt{-\dot{x}G\dot{x}}} \right] = -\frac{1}{4mc} \theta^\mu{}_\nu \dot{x}^\nu, \quad (8)$$

$$\nabla S^{\mu\nu} = \frac{1}{4mc\sqrt{-\dot{x}G\dot{x}}} \dot{x}^{[\mu} (S\theta\dot{x})^{\nu]}, \quad (9)$$

$$S^\mu{}_\nu \dot{x}^\nu - \frac{1}{8(mc)^2} (S\theta\dot{x})^\mu = 0. \quad (10)$$

The last equation is the Lagrangian counterpart of the Frenkel condition (4).

In the spinless limit the equation (8) turn into the geodesic equation. Spin causes deviations from the geodesic motion due to right hand side of this equation, as well as due to presence of the tetrad field $\tilde{T}$ and the effective metric G in the left hand side.

The equations (8) and (9) became singular at critical velocity which obeys the equation $\dot{x}G\dot{x} = 0$. The singularity determines behavior of the particle in ultra-relativistic limit. To clarify this point, let us consider the standard equations of spinless particle interacting with electromagnetic field in the physical-time parametrization $x^\mu(t) = (ct, \mathbf{x}(t))$

$$\left(\frac{\dot{x}^\mu}{\sqrt{c^2 - \mathbf{v}^2}} \right)' = \frac{e}{mc^2} F^\mu{}_\nu \dot{x}^\nu, \quad (11)$$

then the factor is just $c^2 - \mathbf{v}^2$. Rewriting the equations in the form of second law of Newton we find an acceleration. For the case, the longitudinal acceleration reads $a_{||} = \mathbf{v}\mathbf{a} = \frac{e(c^2 - \mathbf{v}^2)^{\frac{3}{2}}}{mc^3} (\mathbf{E}\mathbf{v})$, that is the factor, elevated in some degree, appears on the right hand side of the equation, and thus determines the value of velocity at which the longitudinal acceleration vanishes, $a_{||} \xrightarrow{v \rightarrow c} 0$. In resume, for the present case³ the singularity implies that during its evolution in external background, the particle can not exceed the speed of light c .

We can repeat this computation for our case (8)-(10). Due to the dependence of the tetrad field in Eq. (8) on the spin-tensor S , the singularity presented in (9) causes the appearance of the term proportional to $\frac{1}{\sqrt{\dot{x}G\dot{x}}}$ in the expression for longitudinal acceleration. In the result, the acceleration grows up to infinity as the particle's speed approximates to the critical speed $\sim c$, see [7].

We have seen above that in the Lagrangian formalism MPTD-equations correspond to the minimal interaction of spinning particle with gravity. Let us demonstrate that MPTD-particle admits also a non-minimal interaction. We add the term $\frac{\lambda_1}{2} \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu \equiv \frac{\lambda_1}{2} \frac{\kappa}{16} \theta S$ into the Hamiltonian action (2). By analogy with the magnetic moment, the interaction constant κ is called gravimagnetic moment [14]. Thus we consider the variational problem

$$S_\kappa = \int d\tau \left(p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu - \left[\frac{\lambda_1}{2} \left(P^2 + \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi) \right] \right) \quad (12)$$

on the space of independent variables $x^\mu, p_\nu, \omega^\mu, \pi_\nu$ and λ_a . We have denoted $P_\mu \equiv p_\mu - \Gamma_{\alpha\mu}^\beta \omega^\alpha \pi_\beta$, $P^2 = g^{\mu\nu} P_\mu P_\nu$ and so on. Note also that the first two terms can be identically rewritten in the

³ In a curved space we need to be more careful since the three-dimensional geometry should respect the coordinate independence of the speed of light, see [13, 7].

general-covariant form $p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu = P_\mu \dot{x}^\mu + \pi_\mu \nabla \omega^\mu$. Variation of the action with respect to λ_a gives the algebraic equations

$$P^2 + \kappa R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu \pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0, \quad (13)$$

$$\omega\pi = 0, \quad P\omega = 0, \quad P\pi = 0, \quad (14)$$

while variations with respect to the remaining variables yield dynamical equations which can be written in the covariant form as follows⁴

$$\frac{\partial S_\kappa}{\partial p_\mu} = 0 \quad \Leftrightarrow \quad \dot{x}^\mu = \lambda_1 P^\mu + \lambda_3 \omega^\mu + \lambda_4 \pi^\mu, \quad (15)$$

$$\frac{\partial S_\kappa}{\partial x^\mu} = 0 \quad \Leftrightarrow \quad \nabla P_\mu = -R_{\mu\nu\alpha\beta} \dot{x}^\nu \omega^\alpha \pi^\beta - \frac{1}{2} \lambda_1 \kappa \nabla_\mu R_{\sigma\nu\alpha\beta} \omega^\sigma \pi^\nu \omega^\alpha \pi^\beta, \quad (16)$$

$$\frac{\partial S_\kappa}{\partial \pi_\mu} = 0 \quad \Leftrightarrow \quad \nabla \omega^\mu = \lambda_1 \pi^\mu - \lambda_1 \kappa R^\mu_{\alpha\beta\nu} \omega^\alpha \omega^\beta \pi^\nu + \lambda_2 \omega^\mu + \lambda_4 P^\mu, \quad (17)$$

$$\frac{\partial S_\kappa}{\partial \omega^\mu} = 0 \quad \Leftrightarrow \quad \nabla \pi_\mu = -\frac{\lambda_1 \alpha}{\omega^4} \omega_\mu - \lambda_1 \kappa R_{\mu\nu\alpha\beta} \pi^\nu \omega^\alpha \pi^\beta - \lambda_2 \pi_\mu - \lambda_3 P_\mu. \quad (18)$$

Eq. (15) has been repeatedly used to obtain the final form (16)-(18) of the equations $\frac{\partial S_\kappa}{\partial x^\mu} = 0$, $\frac{\partial S_\kappa}{\partial \pi_\mu} = 0$ and $\frac{\partial S_\kappa}{\partial \omega^\mu} = 0$. Computing time-derivative of algebraic equations (14) and using (15)-(18) we obtain the consequences

$$\pi^2 - \frac{\alpha}{\omega^2} = 0, \quad (19)$$

$$\begin{aligned} \lambda_3 &= 4a\lambda_1 \left[2(1 - \kappa) R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \pi^\mu P^\nu + \kappa \pi^\sigma (\nabla_\sigma R_{\mu\nu\alpha\beta}) \omega^\mu \pi^\nu \omega^\alpha \pi^\beta \right], \\ \lambda_4 &= -4a\lambda_1 \left[2(1 - \kappa) R_{\alpha\beta\mu\nu} \omega^\alpha \pi^\beta \omega^\mu P^\nu + \kappa \omega^\sigma (\nabla_\sigma R_{\mu\nu\alpha\beta}) \omega^\mu \pi^\nu \omega^\alpha \pi^\beta \right], \end{aligned} \quad (20)$$

Here and below we use the following notation. In the equation which relates velocity and momentum will appear the matrix $\mathcal{T}$

$$\mathcal{T}^\alpha{}_\nu \equiv \delta^\alpha{}_\nu - (\kappa - 1) a S^{\alpha\sigma} \theta_{\sigma\nu}, \quad a = \frac{2}{16m^2 c^2 + (\kappa + 1)(S\theta)}. \quad (21)$$

The matrix has an inverse given by

$$\tilde{\mathcal{T}}^\alpha{}_\nu \equiv \delta^\alpha{}_\nu + (\kappa - 1) b S^{\alpha\sigma} \theta_{\sigma\nu}, \quad b = \frac{1}{8m^2 c^2 + \kappa(S\theta)}. \quad (22)$$

The vector Z^μ is defined by

$$Z^\mu = \frac{b}{8} S^{\mu\sigma} (\nabla_\sigma R_{\alpha\beta\rho\delta}) S^{\alpha\beta} S^{\rho\delta} \equiv \frac{b}{8} S^{\mu\sigma} \nabla_\sigma (S\theta). \quad (23)$$

⁴ Note that by construction of S_κ , the variational equation $\frac{\partial S_\kappa}{\partial p_\mu} = 0$ is equivalent to $\dot{x}^\mu = \{x^\mu, H\}$, and so on.

This vanishes in a space with homogeneous curvature, $\nabla R = 0$.

The time-derivatives of (13), (19) and (20) do not yield new algebraic equations. Due to (19) we can replace the constraint (13) on $P^2 + \kappa R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu\pi^\nu + (mc)^2 = 0$. The obtained expressions for λ_3 and λ_4 can be used to exclude these variables from the equations (15)-(18).

Neither constraints nor equations of motion do not determine the functions λ_1 and λ_2 , that is the non-minimal interaction preserves both reparametrization and spin-plane symmetries of the theory. The presence of λ_1 and λ_2 in the equations of motion implies that evolution of our basic variables is ambiguous. According to general theory [15, 16, 17], variables with ambiguous dynamics do not represent observable quantities, so we need to search for the variables that can be candidates for observables. We note that (17) and (18) imply an equation for $S^{\mu\nu}$ which does not contain λ_2 . So we rewrite (15) and (16) in terms of spin-tensor and add to them the equation for $S^{\mu\nu}$, this gives the system

$$\dot{x}^\mu = \lambda_1 \left[\mathcal{T}^\mu{}_\nu P^\nu + \kappa \frac{a}{b} Z^\mu \right], \quad (24)$$

$$\nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\lambda_1\kappa}{32}\nabla_\mu(S\theta), \quad (25)$$

$$\nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]} - \frac{\lambda_1\kappa}{4}\theta^{[\mu}{}_\sigma S^{\nu]\sigma}. \quad (26)$$

Besides, the constraints (13), (14) and (19) imply

$$P^2 + \frac{\kappa}{16}\theta S + (mc)^2 = 0, \quad (27)$$

$$S^{\mu\nu}P_\nu = 0, \quad S^{\mu\nu}S_{\mu\nu} = 8\alpha. \quad (28)$$

The equations (28) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle. Eq. (26), contrary to the equations for ω and π , does not depend on λ_2 . This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ_1 is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Equations (24)-(26), together with (27) and (28), form a closed system which determines evolution of a spinning particle.

The gravimagnetic moment κ can generally take any value. When $\kappa = 0$ we recover the MPTD equations.

Let us exclude P^μ and λ_1 from the equations (25) and (26). Using (22) we solve (24) with respect to P^μ . Using the resulting expression in the constraint (27) we obtain λ_1

$$\lambda_1 = \frac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{m_rc}, \quad \text{with} \quad m_r^2 \equiv m^2 + \frac{\kappa}{16c^2}(S\theta) - \left(\frac{\kappa a}{cb}\right)^2 Z^2, \quad (29)$$

where the effective metric now is given by

$$\mathcal{G}_{\mu\nu} = \tilde{\mathcal{T}}^\alpha{}_\mu g_{\alpha\beta} \tilde{\mathcal{T}}^\beta{}_\nu. \quad (30)$$

Then the expression for momentum in terms of velocities implied by (24) is

$$P^\mu = \frac{m_rc}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \tilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu - \kappa Z^\mu. \quad (31)$$

We substitute this P_μ into (25), (26)

$$\nabla \left[\frac{m_rc}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \tilde{\mathcal{T}}_{\mu\nu} \dot{x}^\nu \right] = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \kappa \frac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{32m_rc} \nabla^\mu(S\theta) + \kappa \nabla Z^\mu, \quad (32)$$

$$\nabla S^{\mu\nu} = \frac{2m_rc(1-\kappa)b}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \dot{x}^{[\mu} S^{\nu]\sigma} \theta_{\sigma\alpha} \dot{x}^\alpha + \frac{\kappa\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{4m_rc} \theta^{[\mu}{}_\alpha S^{\nu]\sigma} - 2\kappa Z^{[\mu} \dot{x}^{\nu]}. \quad (33)$$

Together with (28), this gives us the Lagrangian equations for the spinning particle with gravimagnetic moment.

In resume, we have formulated Hamiltonian variational problem for spinning particle with gravimagnetic moment κ in an arbitrary gravitational background. The model is consistent for any value of κ . When $\kappa = 0$, our equations of motion (24)-(26) coincide with MPTD-equations (3)-(5). The latter also follow from the Lagrangian variational problem without auxiliary variables (1), and correspond to the minimal interaction of our spinning particle with gravity.

Comparing our equations for arbitrary κ to the equations of spinning particle on an arbitrary electromagnetic background (see Eqs. (28)-(29) in [10]), we see that the two systems have the same structure after the identification $\kappa \sim \mu$ and $\theta_{\mu\nu} \equiv R_{\mu\nu\alpha\beta}S^{\alpha\beta} \sim F_{\mu\nu}$, where μ is the magnetic moment. That is curvature influences trajectory of spinning particle in the same way as the electromagnetic field strength $\theta_{\mu\nu}$.

Now we are ready to discuss the main subject of the work. As we have pointed out above, the MPTD-equations, which correspond to the value $\kappa = 0$ of our model (12), have non satisfactory behavior in ultra-relativistic limit. The reason is that variation rate of spin-tensor diverges in this limit and contributes into longitudinal acceleration of the particle through the tetrad field $\tilde{T}^\mu{}_\nu(S)$, see also [7]. Consider now our spinning particle with gravimagnetic moment $\kappa = 1$. This implies $\tilde{T}^\mu{}_\nu = \delta^\mu{}_\nu$, $\mathcal{G}_{\mu\nu} = g_{\mu\nu}$, and crucially simplifies the equations of motion. The Hamiltonian equations (24)-(26) read

$$\frac{m_r c}{\sqrt{-\dot{x}g\dot{x}}} \dot{x}^\mu = P^\mu + \frac{a}{b} Z^\mu, \quad (34)$$

$$\nabla P_\mu = -\frac{1}{4} \theta_{\mu\nu} \dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32m_r c} \nabla_\mu (S\theta), \quad (35)$$

$$\nabla S^{\mu\nu} = 2P^{[\mu} \dot{x}^{\nu]} - \frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c} \theta^{[\mu}{}_\sigma S^{\nu]\sigma}. \quad (36)$$

while the Lagrangian equations are composed now by the equation for trajectory

$$\nabla \left[\frac{m_r c \dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}} \right] = -\frac{1}{4} \theta^\mu{}_\nu \dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32m_r c} \nabla^\mu (S\theta) + \nabla Z^\mu, \quad (37)$$

and by the equation for precession of spin-tensor

$$\nabla S^{\mu\nu} = \frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c} \theta^{[\mu}{}_\sigma S^{\nu]\sigma} - 2Z^{[\mu} \dot{x}^{\nu]}. \quad (38)$$

These equations can be compared with (8)-(9). Even in homogeneous field, $\nabla R = 0$, we have modified dynamics for both x and S . Covariant precession of spin (38) has a smooth behavior $\nabla S \sim (\sqrt{-\dot{x}g\dot{x}})^0$ contrary to $\nabla S \sim \frac{1}{\sqrt{-\dot{x}g\dot{x}}}$ in the equation (9). The equation (34) in space with homogeneous curvature has the structure similar to (11), hence we expect that longitudinal acceleration vanishes as $v \rightarrow c$. This has been confirmed by direct computations in [7].

Acknowledgments

The work of AAD has been supported by the Brazilian foundations CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil) and FAPEMIG (Fundação de Amparo à Pesquisa do estado de Minas Gerais- Brasil). WGR thanks CAPES for the financial support (Programa PNPd/2011).

References

- [1] Mathisson M 1937 *Acta Phys. Polon.* **6** 163
- [2] Papapetrou A 1951 *Proc. Roy. Soc. Lond. A* **209** 248
- [3] Tulczyjew W M 1959 *Acta Phys. Polon.* **18** 393
- [4] Dixon W G 1964 *Nuovo Cimento* **34** 317
- [5] Pomeransky A A, Sen'kov R A and Khriplovich I B 2000 *Physics-Uspekhi* **43** (10) 1055
- [6] Balakin A B and Popov V A 2015 Spin-axion coupling *Preprint* arXiv:1509.06058 (gr-qc)
- [7] Deriglazov A A and Ramirez W G 2015 Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations in ultra-relativistic regime and gravimagnetic moment *Preprint* arXiv:1509.05357 (gr-qc)
- [8] Frenkel J 1926 *Z. Phys.* **37** 243
- [9] Deriglazov A A and Pupasov-Maksimov A M 2014 *SIGMA* **10** 012
- [10] Deriglazov A A 2014 *Phys. Lett. B* **736** 278
- [11] Ramirez W G and Deriglazov A A 2015 Lagrangian formulation for Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations *Preprint* arXiv:1509.04926 (gr-qc)
- [12] Hanson A J and Regge T 1974 *Annals of Physics* **87** (2) 498
- [13] Deriglazov A A and Ramirez W G 2015 *Mod. Phys. Lett. A* **30** 1550101 (*Preprint* arXiv:1409.4756 (physics.gen-ph))
- [14] Khriplovich I B 1989 *Sov. Phys. JETP* **69** 217
- [15] Dirac P A M 1964 *Lectures on Quantum Mechanics* (Yeshiva University, New York)
- [16] Gitman D M and Tyutin I 1990 *Quantization of fields with constraints* (Springer-Verlag, Berlin)
- [17] Deriglazov A 2010 *Classical Mechanics: Hamiltonian and Lagrangian Formalism* (Springer-Verlag)