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Unifying Ancient and Modern Geometries Through Octonions

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Abstract.

We show the first unified description of some of the oldest known geometries such as the Pappus' theorem with more modern ones like Desargues' theorem, Monge's theorem and Ceva's theorem, through octonions, the highest normed division algebra in eight dimensions. We also show important applications in hadronic physics, giving a full description of the algebra of color applicable to quark physics, and comment on further applications.

1. Introduction

In mid sixties Miyazawa, in a series of papers[1], extended the $SU(6)$ group to the supergroup $SU(6/21)$ that could be generated by constituent quarks and diquarks that could be transformed to each other. In particular, he found the following: (a) A general definition of $SU(m/n)$ superalgebras, expressing the symmetry between m bosons and n fermions, with Grassman-valued parameters. (b) A derivation of the super-Jacobi identity. (c) The relation of the baryon mass splitting to the meson mass splitting through the new mass formulae.

This work contained the first classification of superalgebras (later rediscovered by mathematicians in the seventies). Because of the field-theoretic prejudice against $SU(6)$, Miyazawa's work was generally ignored. Supersymmetry was, of course, rediscovered in the seventies within the dual resonance model by Ramond[2], and Neveu and Schwarz[3]. Golfand and Likhtman[4], and independently Volkov and Akulov[5], proposed the extension of the Poincaré group to the super-Poincaré group. Examples of supersymmetric field theories were given and the general method based on the super-Poincaré group was discovered by Wess and Zumino[6]. The super-Poincaré group allowed transformations between fields associated with different spins 0, $\frac{1}{2}$ and 1. The Coleman-Mandula theorem was amended in 1975 by Haag, Lopuszanski and Sohnius to allow super-Poincaré e group $\times G_{int}$ as the maximum symmetry of the S-matrix. Unfortunately, $SU(6)$ symmetry was still forbidden.

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2. SU(6) and Hadronic Supersymmetry

How do we interpret the symmetries of the QCD spectrum in this light? In the ultraviolet, the running coupling constant tends to zero and quarks behave like free point particles. Thus an approximate conformal symmetry exists, allowing spin to be conserved separately from orbital angular momentum. Thus spin behaves as an internal quantum number; this makes a $SU(6)$ symmetry possible, since the quarks are almost free Dirac particles. Single vector-gluon exchange breaks this symmetry; thus, as shown by Glashow, Georgi and deRujula[7], the mass-degeneracy of hadrons of different spins is lifted by a hyperfine-interaction term.

Here is the main point. In the infrared we expect confinement to set in. The quark-antiquark potential becomes proportional to the distance. Careful studies of quarkonium spectra and lattice-gauge calculations show that at large separation the quark forces become spin-independent. QCD is also flavor independent. We therefore find approximate spin- and flavor-independent quark binding forces; these are completely consistent with $SU(6)$ symmetry. This is not an exact symmetry, but is a good starting point, before spin and flavor effects are included.

There is good phenomenological evidence that in a rotationally excited baryon a quark-diquark $(q - D)$ structure is favored over a three-quark (qqq) structure[8],[9],[10]. Eguchi[11] had shown that it is energetically favorable for the three quarks in a baryon to form a linear structure with a quark on one end and bilocal structure qq at the other end. Similarly, Bars and Hanson[12], and independently Johnson and Thorn[13] had shown that the string-like hadrons may be pictured as vortices of color flux lines which terminate on concentration of color at the end points. A baryon with three valence quarks would be arranged as a linear chain of molecule where the largest angular momentum for a state of a given mass is expected when two quarks are at one end, and the third is at the other: At large spin, two of the quarks form a diquark at one end of the string, the remaining quark being at the other. Regge trajectories for mesons and baryons are closely parallel; both have a slope of about $0.9(GeV)^{-2}$. If the quarks are light, the underlying quark-diquark symmetry leads to a Miyazawa symmetry between mesons and baryons. Thus we studied QCD with a weakly broken supergroup $SU(6/21)$. Note that the fundamental theory is not supersymmetric. For quarks, the generators of the Poincaré group and those of the color group $SU(3)^c$ commute. It is only the effective Hamiltonian which exhibits an approximate supersymmetry among the bound states $q\bar{q}$ and qD.

Under the color group $SU(3)^c$, meson $q\bar{q}$ and diquark $(D = qq)$ states transform as[10],[14]

$$qq: \quad \mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \;\; ; \quad q\bar{q}: \quad \mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \tag{1}$$

and under the spin flavor $SU(6)$ they transform as

$$qq: \quad \mathbf{6} \times \mathbf{6} = \mathbf{15} + \mathbf{21} \;\; ; \quad q\bar{q}: \quad \mathbf{6} \times \bar{\mathbf{6}} = \mathbf{1} + \mathbf{35} \tag{2}$$

Dimensions of internal degrees of quarks and diquarks are shown in the following table:

	$SU_f(3)$	$SU_s(2)$	$dim.$
q	□	$s = 1/2$	$3 \times 2 = 6$
D	□ □	$s = 1$	$6 \times 3 = 18$
	□ □	$s = 0$	$3 \times 1 = 3$

For a more general case of the above table we refer the reader to the table with the flavor-spin content at the end of the paper.

If one writes qqq as qD, then the quantum numbers of D are $\bar{\mathbf{3}}$ for color since when combined with q must give a color singlet, and $\mathbf{21}$ for spin-flavor since combined with color must give antisymmetric wavefunctions. The quantum numbers for $\bar{q}$ are for color, $\bar{\mathbf{3}}$, and for spin-flavor, $\bar{\mathbf{6}}$. Thus $\bar{q}$ and D have the same quantum numbers (color forces can not distinguish between $\bar{q}$ and D). Therefore there is a dynamic supersymmetry in hadrons with supersymmetric partners

$$\psi = \begin{pmatrix} \bar{q} \\ D \end{pmatrix}, \quad \bar{\psi} = (q \quad \bar{D}) \tag{3}$$

We can obtain all hadrons by combining ψ and $\bar{\psi}$: mesons are $q\bar{q}$, baryons are qD, and exotics are $D\bar{D}$ states. Inside rotationally excited baryons, QCD leads to the formation of diquarks well separated from the remaining quark. At this separation the scalar, spin-independent, confining part of the effective QCD potential is dominant. Since QCD forces are also flavor-independent, the force between the quark q and the diquark D inside an excited baryon is essentially the same as the one between q and the antiquark $\bar{q}$ inside an excited meson. Thus the approximate spin-flavor independence of hadronic physics expressed by $SU(6)$ symmetry is extended to $SU(6/21)$ supersymmetry[10],[14] through a symmetry between $\bar{q}$ and D, resulting in parallelism of mesonic and baryonic Regge trajectories.

3. Color Algebra and Octonions

We shall now give an algebraic justification to our remarks above. We will find all the answers in an algebra we build in terms of octonions and their split basis. The exact, unbroken color group $SU(3)^c$ is the backbone of the strong interaction. It is worthwhile to understand its role in the diquark picture more clearly.

In what follows we first give a simple description of octonion algebra (also known as Cayley algebra). Later we'll show how to build split octonion algebra that will close into a fermionic Heisenberg algebra. Split octonion algebra will then be shown to produce algebra of color forces in QCD in application to hadronic supersymmetry when the split units and their conjugates become associated with quark and antiquark fields, respectively.

An octonion x is a set of eight real numbers

$$x = (x_0, x_1, \ldots, x_7) = x_0e_0 + x_1e_1 + \ldots + x_7e_7 \tag{4}$$

that are added like vectors and multiplied according to the rules

$$e_0 = 1, \quad e_0e_i = e_ie_0 = e_i, \quad i = 0, 1, \ldots, 7 \tag{5}$$

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma}e_\gamma. \quad \alpha, \beta, \gamma = 1, 2, \ldots, 7 \tag{6}$$

where e_0 is the multiplicative unit element and e_α's are the imaginary octonion units. The structure constants $\epsilon_{\alpha\beta\gamma}$ are completely antisymmetric and take the value 1 for combinations

$$\epsilon_{\alpha\beta\gamma} = (165), (257), (312), (471), (543), (624), (736) \tag{7}$$

Note that summation convention is used for repeated indices.

The octonion algebra $\mathcal{C}$ is an algebra defined over the field $\mathbf{Q}$ of rational numbers, which as a vector space over $\mathbf{Q}$ has dimension 8.

We shall now give reasons for incorporation of the octonion algebra for hadronic physics, showing only they through their split octonionic parts one can provide the correct description of the color algebra in hadrons. Later in another publication we shall show[15] a previously unknown multiplication rules for octonions by producing a wheel that allows generalized multiplication rules for doublets and triplets of octonionic units.

First, the reasons: Two of the colored quarks in the baryon combine into an anti-triplet $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + (\mathbf{6})$, and in a nucleon $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + (\mathbf{8})$. The $(\mathbf{6})$ partner of the diquark and the $(\mathbf{8})$ partner

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of the nucleon do not exist. In hadron dynamics the only color combinations to consider are $\mathbf{3} \times \mathbf{3} \to \bar{\mathbf{3}}$ and $\bar{\mathbf{3}} \times \mathbf{3} \to \mathbf{1}$. These relations imply the existence of split octonion units u_i defined below through a representation of the Grassmann algebra $\{u_i, u_j\} = 0$, $i = 1, 2, 3$. What is a bit strange is that operators u_i, unlike ordinary fermionic operators, are not associative. We also have $\frac{1}{2}[u_i, u_j] = \epsilon_{ijk}\, u_k^*$. The Jacobi identity does not hold since $[u_i, [u_j, u_k]] = -ie_7 \neq 0$, where e_7, anticommute with u_i and u_i^*.

The behavior of various states under the color group are best seen if we use split octonion units defined by[16]

$$u_0 = \frac{1}{2}(1 + ie_7), \qquad u_0^* = \frac{1}{2}(1 - ie_7) \tag{8}$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad j = 1, 2, 3 \tag{9}$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units $e_\alpha (\alpha = 1, ..., 7)$ fall into its 7-dimensional representation.

Under the $SU(3)^c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* are singlets, while u_j and u_j^* correspond, respectively, to the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner (together with the complex conjugate equations):

$$u_0^2 = u_0, \quad u_0 u_0^* = 0 \tag{10}$$

$$u_0 u_j = u_j u_0^* = u_j, \quad u_0^* u_j = u_j u_0 = 0 \tag{11}$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^* \tag{12}$$

$$u_i u_j^* = -\delta_{ij} u_0 \tag{13}$$

where ϵ_{ijk} is completely antisymmetric with $\epsilon_{ijk} = 1$ for ijk = 123, 246, 435, 651, 572, 714, 367; and zero otherwise. Here, one sees the virtue of octonion multiplication. If we consider the direct products

$$C: \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \tag{14}$$

$$G: \quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \tag{15}$$

for $SU(3)^c$, then these equations show that octonion multiplication gets rid of $\mathbf{8}$ in $\mathbf{3} \otimes \bar{\mathbf{3}}$, while it gets rid of $\mathbf{6}$ in $\mathbf{3} \otimes \mathbf{3}$. Combining Eq.(12) and Eq.(13) we find

$$(u_i u_j) u_k = -\epsilon_{ijk} u_0^* \tag{16}$$

Thus the octonion product leaves only the color part in $\mathbf{3} \otimes \bar{\mathbf{3}}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$, so that it is a natural algebra for colored quarks.

For convenience we now produce the following multiplication table for the split octonion units:

	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki} u_i^*$	$-\delta_{jk} u_0$
u_j^*	u_j^*	0	$-\delta_{jk} u_0^*$	$\epsilon_{jki} u_i$

Double and Triple Octonionic Multiplication

1 2 3	2 4 6	4 3 5	3 6 7	6 5 1	5 7 2	7 1 4
4 6 5 7	3 5 7 1	6 7 1 2	5 1 2 4	7 2 4 3	1 4 3 6	2 3 6 5

It is worth noting that u_i and u_j^* behave like fermionic annihilation and creation operators:

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ik} \tag{17}$$

For more recent reviews on octonions and nonassociative algebras we refer to papers by Okubo[17], Baez[18] and Catto[19].

What we would like to discuss now is a recent unpublished, taking the usual octonionic multiplication rules from doublets of octonions to triplets.

The usual octonion multiplication once sees in literature is a diagram (as shown by, for example, in Baez's paper[18]). That triangle was replaced by Gürsey and Günaydin. We extended and generalized their work by adding the dashed lines shown below in Figure 1. In Double and Triple Octonionic Multiplication table as shown, we describe the rotation of triangle and dashed lines for all octonionic multiplications. We also show by moving to the left instead of the right, we build a completely new multiplication table for doublets and triplets in the Left Handed Double and Triple Octonionic Multiplication and Figure 2. There is a complete correspondence between those diagrams and figures which will be explained in another publication.

We can now show how one can map some of the old and new geometries into each other.

In the well-known Papus' geometry, one can put the *e*'s into a diagram and show how a center line can be mapped into two arbitrary lines outside, above and below as shown Figure 3. One can perform six more distinct copies of these mappings.

In a similar way, one can see all these mappings into the Desargues' theorem in a beautiful generalized way. One can show that these pictures can be generated in two and three dimensions. These will also be published in another publication.

Finally, we want to mention that these mappings described above can be extended into Monge's and Ceva's theorems as shown in the Figures 4 and 5. These have tremendous applications not only in hadronic physics, but also in solutions of problems in astrophysics.

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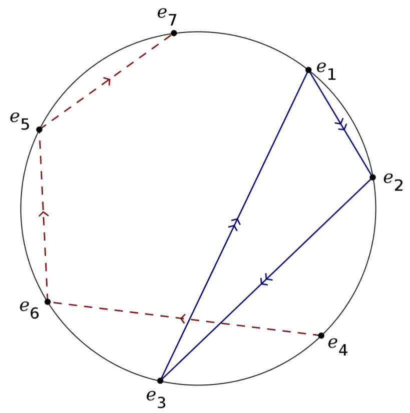

Figure 1.

Left Handed Double and Triple Octonionic Multiplication

1 7 6	7 5 3	5 6 4	6 3 2	3 4 1	4 2 7	2 1 5
5 3 4 2	6 4 2 1	3 2 1 7	4 1 7 5	2 7 5 6	1 5 6 3	7 6 3 4

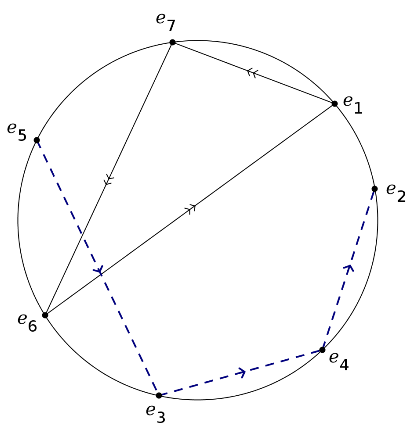

Figure 2.

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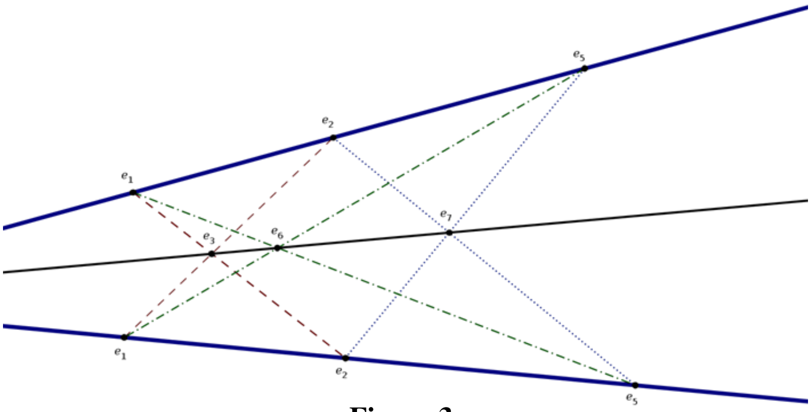

Figure 3.

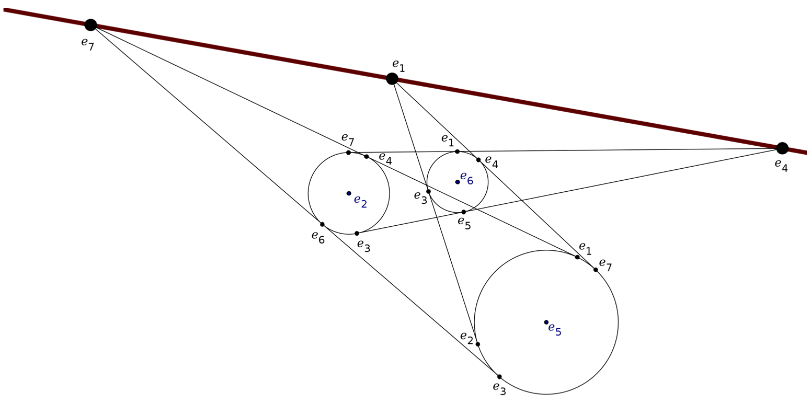

Figure 4.

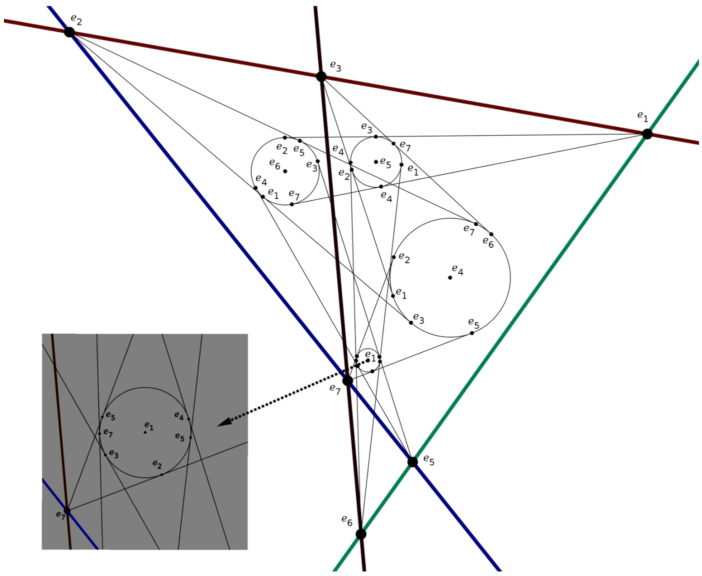

Figure 5.

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Table: Flavor-Spin Content for three quark and quark-diquark baryons

Configuration	$SU_c(3)$	$SU_f(3)$	$SU_s(2)$
q	$\square \equiv \underline{3}$	$\square \equiv \underline{3}$	$S = 1/2$
$D \equiv q^2$	$\begin{array}{l}\square\\ \square\end{array} \equiv \underline{\bar{3}}$	$\square\square \equiv \underline{6}$	$\left\{\begin{array}{l} S = 0 \ \text{not allowed} \\ S = 1 \end{array}\right.$
		$\begin{array}{l}\square\\ \square\end{array} \equiv \underline{\bar{3}}$	$\left\{\begin{array}{l} S = 0 \\ S = 1 \ \text{not allowed} \end{array}\right.$
q^3	$\begin{array}{l}\square\\ \square\\ \square\end{array} \equiv \underline{1}$	$\square\square\square \equiv \underline{10}$	$S = 3/2$
		$\begin{array}{l}\square\square\\ \square\end{array} \equiv \underline{8}$	$S = 1/2$
$q - D$	$\begin{array}{l}\square\\ \square\\ \square\end{array} \equiv \underline{1}$	$\underline{3} \times \underline{\bar{3}}$: $\begin{array}{l}\square\square\\ \square\end{array} \equiv \underline{8}$	$S = 1/2$
		$\underline{3} \times \underline{\bar{3}}$: $\begin{array}{l}\square\\ \square\\ \square\end{array} \equiv \underline{1}$	$S = 1/2$
		$\underline{3} \times \underline{6}$: $\square\square\square \equiv \underline{10}$	$S = 1/2$
		$\underline{3} \times \underline{6}$: $\square\square\square \equiv \underline{10}$	$S = 3/2$
		$\underline{3} \times \underline{6}$: $\begin{array}{l}\square\square\\ \square\end{array} \equiv \underline{8}$	$S = 1/2$
		$\underline{3} \times \underline{6}$: $\begin{array}{l}\square\square\\ \square\end{array} \equiv \underline{8}$	$S = 3/2$

References

[1] H. Miyazawa, *Prog. Theor. Phys.* **36** *(1966) 1266; Phys. Rev.* **170** *(1968) 1586.*
[2] P. Ramond, *Phys. Rev.* **D3** *(1971)2415.*
[3] A. Neveu and J. Schwarz, *Nucl.Phys.* **B31** *(1971) 86.*
[4] T.A. Golfand and E.P. Likhtman, *JETP Lett.***13** *(1971) 323.*
[5] D.V. Volkov and V.P. Akulov, *Phys. Lett.* **46b***(1973) 109.*
[6] J. Wess and B. Zumino, *Nucl.Phys.* **B70** *(1974) 39; Phys. Lett.* **49b** *(1974) 52.*
[7] A. de Rujula, H. Georgi and S.L. Glashow, *Phys. Rev.* **D12** *(1975) 147.*
[8] D.B. Lichtenberg, W. Namgung, E. Predazzi and J.G. Wills, *Phys. Lett.* **48** *(1982) 1653*; W. Namgung and D.B. Lichtenberg, *Lett. Nuovo Cimento* **41** *(1984) 597*; A. Martin, *Z. Phys.* **C32** *(1986) 359* and in "Symmetry in Nature" vol.II, p. 523 (Scuola Normale Superiore, Pisa, 1989).
[9] M. Anselmino, E. Predazzi, S. Ekelin, S. Fredriksson and D.B. Lichtenberg, *Rev. Mod. Phys.* **65** *(1993) 1199*.
[10] S. Catto and F. Gürsey, *Nuovo Cimento* **A 86** *(1985) 201.*
[11] T. Eguchi, *Phys. Lett. B,* **59** *(1975) 457*.
[12] I. Bars and A.J. Hanson, *Phys. Rev.***D13** *(1976) 1744*.
[13] For strings as elongated bags see K. Johnson and C.B. Thorn, *Phys. Rev.* **D13** *(1976) 1934*.
[14] S. Catto and F. Gürsey, *Nuovo Cimento* **A99** *(1988) 685.*
[15] S. Catto, *To be published.*
[16] M. Günaydin and F. Gürsey, *J. Math. Phys.* **14** *(1973) 1651.*
[17] S. Okubo, *"Introduction to Octonion and other Nonassociative Algebras." Montroll Memorial Lecture Series in Mathematical Physics, 2. Cambridge Univ. Press (1995).*
[18] J. Baez, *Bull. Am. Math. Soc.* **39(2)** *(2002) 145-205.*
[19] S. Catto, *"Exceptional Projective Geometries and Internal Symmetries" ArXiv: Hep-th/0302079*

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