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On the uniqueness of the octonionic instanton solution on conformally flat 8-manifolds

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Abstract. Let M be an 8-manifold and E be an $SO(8)$ bundle on M. In a previous paper [F. Ozdemir and A.H. Bilge, "Self-duality in dimensions $2n > 4$: equivalence of various definitions and the derivation of the octonionic instanton solution", ARI (1999) 51:247-253], we have shown that if the second Pontrjagin number p_2 of the bundle E is minimal, then the components of the curvature 2-form matrix F with respect to a local orthonormal frame are $F_{ij} = c_{ij}\omega_{ij}$, where c_{ij}'s are certain functions and the ω_{ij}'s are strong self-dual 2-forms such that for all distinct i, j, k, l, the products $\omega_{ij}\omega_{jk}$ are self dual and $\omega_{ij}\omega_{kl}$ are anti self-dual. We prove that if the c_{ij}'s are equal to each other and the manifold M is conformally flat, then the octonionic instanton solution given in [B.Grossman, T.W.Kephart, J.D.Stasheff, *Commun. Math. Phys.*, **96**, 431-437, (1984)] is unique in this class

1. Introduction

The set-up for gauge theory is based on vector bundles over differentiable manifolds [1]. Let M be a differentiable manifold with a Riemannian connection and E be a vector bundle on M with a structure group G. Let g be the Lie algebra of G. The connection on the vector bundle is defined locally by a g valued connection 1-form A. If F is the curvature of this connection, then the invariant polynomials of F are local representatives of the characteristic classes of the bundle E. Action integrals are given in terms of the inner products of the components of the curvature of the bundle; these integrals are bounded below by the integrals of the characteristic classes of vector bundles. Solutions for which the action integrals reach these topological lower bounds are minimizers of the action integrals. In our approach, we start with a topological lower bound and we use algebraic methods to characterize those solutions that saturate various inequalities among exterior products and inner products of forms. This procedure results in an action with the topological lower bound we started with.

We use the notion of "Strong self-duality" of a 2-form ω in even dimensions. These are building blocks for the solutions that saturate various inequalities. In this approach, we identify 2-forms and skew-symmetric matrices. A form is called "strong self-dual", if the minimal polynomial of the corresponding skew-symmetric matrix A is $A^2 + \lambda^2 I = 0$ [2]. Equivalently, in $4n$ dimensions, ω^n is Hodge self-dual and in $2n$ dimensions, $\omega^n = k * \omega$ [3].

The octonionic instanton solution [4] is the minimizer of the the action $\int_M |F^2, F^2| dvol$ on S^8. In [3], we have derived this solution by the maximality of (minus) the second Pontrjagin class by the procedure described above. In the present work, we show that the octonionic instanton solution is unique in the class on solutions on conformally flat 8-manifolds that admit an $SO(8)$ vector bundle with an action that maximize $-p_2$.

2. Preliminaries

The base manifold M is a Riemannian manifold, equipped with a torsion free metric connection B. Local sections of the cotangent bundle are denoted as $\{e_i\}$. The connection is given by

$$de_i = \sum B_{ij} \wedge e_j$$

where $B_{ij} = -B_{ji}$ are 1-forms. The curvature of the manifold is given by $R = dB - B \wedge B$.

The vector bundle E is an n-plane bundle on M. Local sections of E are denoted as s_i. The connection on E is given by

$$ds_i = \sum A_{ij} s_j$$

where A_{ij}'s are 1-forms; A takes vales in the Lie algebra of the structure group of E. The curvature of E is given by

$$F = dA - A \wedge A$$

and F satisfies the Bianchi identities

$$dF + F \wedge A - A \wedge F = 0.$$

Local representatives of the characteristic classes are computed as follows. If s_i, $i = 1, \ldots, N$ is a basis for local sections of E, then with respect to this basis, F is represented by an $N \times N$ matrix of 2-forms. As 2-forms belong to a commutative ring, we can compute the determinant

$$det(F + \lambda I) = \lambda^N + \sigma_1 \lambda^{N-1} + \sigma_2 \lambda^{N-2} + \cdots + \sigma_{N-1}\lambda + \sigma_N$$

The σ_i's are $2i$ forms that are proportional to the representatives of the Chern or Pontrjagin classes of E. If E is real, the odd ones, c_{2k+1} are trivial, only the even ones, $c_{2k} \sim p_k$ are nontrivial. The Euler Class is the square root of the determinant of F. Actions that involve the curvature F can be related to Pontrjagin numbers by relating inner products and exterior products.

In our approach, we start with a characteristic class say p_1 on a 4-dimensional manifold or a combination of p_2 and p_1^2 on an 8-dimensional manifold. We use conditions for the saturation of various inequalities to obtain an upper bound for the integrals of the characteristic classes as algebraic equations for the curvature. Then we try to solve for a connection that gives the curvature 2-form we determined by algebraic requirements. The solvability of this connection usually impose conditions on the base manifold hence it may determine the background metric. We will construct the action by maximizing $-p_2$ on conformally flat 8-manifolds by relating $\langle F^2, F^2 \rangle$ and traceF^4. Actually, it will turn out that trace$F^2 = 0$, hence traceF^4 is proportional to the second Pontrjagin class.

3. Strong Self-Dual 2-forms

In 4-dimensions, 2-forms have a number of nice properties; they live in the middle dimension hence their Hodge duality is defined; self-dual 2-forms belong to a linear space; when $F = *F$, Yang-Mills equations are satisfied and finally the Yang-Mills equations form an elliptic system.

We noticed that the matrix of a self-dual 2-form has minimal polynomial $A^2 + \lambda^2 I = 0$, i.e, its eigenvalues are equal in absolute value. "Strong self-duality of 2-forms" in $2n$ dimensions is defined by the equality of the absolute values of the eigenvalues of the matrix of ω with respect to an orthonormal basis [2] We have also shown that it is equivalent to the self-duality in the Hodge sense of $\omega^{n/2}$ (used in [4] and to the equality $*\omega = k\omega^{n-1}$ (used by Trautman, in[5].

In 4-dimensions, the Yang-Mills action $\int \langle F, F \rangle = \int F \wedge *F$, where F is the curvature 2-form of an $SO(N)$ bundle, reaches the topological lower bound $\int F \wedge F$, provided that F is self-dual.

In 4-dimensions self-dual and antiself-dual 2-forms are eigenspaces of the Hodge map and they form linear subspaces. In higher dimensions we look for linear subspaces of the set of strong self-dual 2 forms. In [6], we have shown that the dimension of maximal linear subspaces of strongly self-dual forms on a $2n$ manifold is equal to the number of linearly independent vector fields on S^{2n-1}. In eight dimensions, there are 7 dimensional maximal linear subspaces of strong self-dual 2-forms. We will use these these subspaces to construct the octonionic instanton solution.

4. Strong self-duality and equivalence of various properties.

Let ω_{ij} be the components of a 2-form in $2n$ dimensions with respect to some local orthonormal basis. We denote the 2-form ω and the skew-symmetric matrix consisting of its components with respect to some orthonormal basis by the same symbol. We recall the standard inequalities:

$$(\omega,\eta)^2 \leq (\omega,\omega)(\eta,\eta), \quad 2(\omega,\eta) \leq (\omega,\omega)+(\eta,\eta).$$

The invariant polynomials s_{2i} of ω can be expressed in terms of the elementary symmetric functions of the eigenvalues $\pm\lambda_k^2$'s. The inner products (ω^i,ω^i) and the s_{2i}'s are related as follows.

$$\begin{array}{rcl} s_2 & = & (\omega,\omega)=\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2, \\ s_4 & = & \frac{1}{(2!)^2}(\omega^2,\omega^2)=\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\cdots+\lambda_{n-1}^2\lambda_n^2, \\ s_6 & = & \frac{1}{(3!)^2}(\omega^3,\omega^3)=\lambda_1^2\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_2^2\lambda_4^2+\cdots+\lambda_{n-2}^2\lambda_{n-1}^2\lambda_n^2, \\ \cdots & & \cdots \\ s_{2n} & = & \frac{1}{(n!)^2}(\omega^n,\omega^n)=\frac{1}{(n!)^2}\mid *\omega^n\mid^2=\lambda_1^2\lambda_2^2\ldots\lambda_n^2 \end{array}$$

Defining the weighted elementary symmetric polynomials by $\binom{n}{i}q_i = s_{2i}$, one has the inequalities

$$q_1 \geq q_2^{1/2} \geq q_3^{1/3} \geq \cdots \geq q_n^{1/n}, \qquad q_{r-1}q_{r+1} \leq q_r^2, \quad 1 \leq r < n,$$

and the equalities hold iff all the λ_k's are equal [7]. This is a key result and our definition of strong self-duality,

Definition. Let ω be a 2-form in $2n$ dimensions, $\pm i\lambda_k$, $k=1,\ldots,n$ be its eigenvalues and η be the the square root of the determinant of ω, with a fixed choice of sign. Then ω is called *strongly self-dual* (*strongly anti self-dual*) if $\mid\lambda_1\mid=\mid\lambda_2\mid=\cdots=\mid\lambda_n\mid$, and $\eta>0$ ($\eta<0$).

The strong self-duality condition is equivalent to the matrix equation $\omega^2+\lambda^2 I=0$, where I is the identity matrix, and $\lambda^2=\frac{1}{2n}\mathrm{tr}\omega^2$. This definition gives quadratic equations for the ω_{ij}'s, hence the strong self-duality condition determines a nonlinear set.

In four dimensions, the matrices satisfying $\omega^2+\lambda I=0$ consist of the union of the usual self-dual and anti self-dual forms. In higher dimensions the set $\mathcal{S}_{2n}$ is an n^2-n+1 dimensional submanifold and the dimension of maximal linear submanifolds of $\mathcal{S}_{2n}$ is equal to the number of linearly independent vector fields on S^{2n-1}.

The equivalence of various definitions is given by the following Lemma [3].

Lemma. Let ω be a 2-form in $2n$ dimensions. Then

$$(n-1)(\omega,\omega)^2-\frac{n}{2}(\omega^2,\omega^2)\geq 0, \quad (\omega^{n/2},\omega^{n/2})\geq *\omega^n,$$

and equality holds if and only if all eigenvalues of ω are equal.

From this Lemma , we immediately have

Corollary. The 2-form ω is strongly self-dual iff $\omega^{n/2}$ is self-dual in the Hodge sense.

The strong self duality condition is also equivalent to the self-duality definition used by Trautman.

Proposition. Let ω be a 2-form in $2n$ dimensions. Then $\omega^{n-1} = k * \omega$ where k is a constant, if and only if ω is strongly self-dual and $k = \frac{n!}{n^{n/2}}(\omega,\omega)^{\frac{n}{2}-1}$.

5. Linear subspaces of strongly self-dual forms in eight dimensions.

In eight dimensions,

$$(\omega,\omega)^2 \geq \tfrac{2}{3}(\omega^2,\omega^2) \geq \tfrac{2}{3} * \omega^4$$

For strongly self-dual 2-forms, these inequalities are saturated and we also have

$$\omega^3 = \tfrac{3}{2}(\omega,\omega) * \omega.$$

By applying the equalities above to $\omega \pm \eta$, we obtain

$$\begin{array}{l}(\omega,\omega)^2 + (\eta,\eta)^2 + 2(\omega,\omega)(\eta,\eta) \\ \geq \frac{3}{2}\left[(\omega^2,\omega^2) + (\eta^2,\eta^2)^2 + 2(\omega^2,\eta^2) \pm 4(\omega^2+\eta^2,\omega\eta) + 4(\omega\eta,\omega\eta)\right] \\ \geq \frac{3}{2}\left[\omega^4 + \eta^4 + 6\omega^2\eta^2 \pm 4\omega^3\omega\eta \pm 4\omega\eta^3\right])\end{array}$$

Using these inequalities we obtain a series of results concerning the products involving strongly self-dual forms.

If ω is strongly self-dual and $(\omega,\eta) = 0$. Then $\omega^3\eta = 0$. When ω and η are both strongly self-dual $\omega^2 = *\omega^2$ and $\eta^2 = *\eta^2$, and

$$\begin{array}{rl}2(\omega,\omega)(\eta,\eta) & \geq \frac{3}{2}\left[2\omega^2\eta^2 4(\omega\eta,\omega\eta)\right] \\ & \geq \frac{3}{2}\left[+6\omega^2\eta^2\right]\end{array}$$

If ω, η, and $\omega \pm \eta$ are strongly self-dual and $(\omega,\eta) = 0$. Then

$$(\omega,\omega)(\eta,\eta) = 2(\omega^2,\eta^2) = 2\omega^2\eta^2.$$

Let ω and η be strongly self-dual and $(\omega,\eta) = 0$. Then

$$\omega\eta = *(\omega\eta)$$

if and only if $\omega \pm \eta$ is strongly self-dual.

Let ω and η, and $\omega \pm \eta$ be strongly self-dual and $(\omega,\eta) = 0$. Then

$$(\omega,\omega)(\eta,\eta) = 2(\omega^2,\eta^2) = 2\omega^2\eta^2.$$

Let ω, η and α be mutually orthogonal strongly self-dual 2-forms such that $\omega+\eta+\alpha$ is also strongly self-dual. Then

$$\omega\eta\alpha = 0.$$

Proposition. Let $F = \sum \omega_a E_a$ where ω_a's belong to a linear subspace of strongly self-dual 2-forms and E_a's belong to a basis of the Lie algebra. Then (i) $F^2 = *F^2$ for any Lie algebra, (ii) $*F$ is proportional to kF^3 provided that $\mathrm{tr}E_a^2E_b$ is proportional to E_b.

We note that if k above is constant, then the Yang-Mills equations are automatically satisfied, however this condition means that each ω_a has constant norm.

Another important property of strongly self-dual 2-forms is that the multiplication by a strongly self-dual 2-form is nondegenerate. Let ω be strongly self-dual 2 form and η be any 2-form. Then $\omega\eta = 0$ implies that $\eta = 0$. As a result, the equation $\omega\eta = \alpha$ has a solution unique solution provided that α is in the image of the multiplication by ω, in other words if the equation has a solution, this solution is unique.

Using these results we know show that the solution constructed by Grossman is unique.

6. The Grossman-Kephart-Stasheff solution

In Grossman et.al, F in the form $F = f_0\sigma_{ij}$ where f_0 is a 2 form and the Σ_{ij}'s are a basis for Spin(8). As a result the action $\langle F^2, F^2\rangle$ is equal to the topological term F^4 as consequence of the properties of the Σ_{ij}'s.

We consider real $SO(N)$ bundles, hence locally F takes values in a skew-symmetric matrix. It follows that all principal minors of F are skew-symmetric matrices, their determinants are perfect squares and the σ_{2i}'s are sums of squares of i-fold products of the entries of F. Then, a local representative of the second Pontrjagin class is given by

$$-p_2 = \lambda\sigma_{i<j<k<l} - *(F_{ij}F_{kl} - F_{ik}F_{jl} + F_{il}F_{jk})^2,$$

where λ is a proportionality constant.

In [3] we proved the following theorem.

Theorem Let F be the local curvature 2-form of an 8-plane vector bundle. If the negative of the second Pontrjagin class, $-p_2$ is maximal, then

i. Each F_{ij} is strong self-dual,

ii. For distinct i, j, k, $F_{ij}F_{jk}$ is self-dual,

iii. For distinct i, j, k, l, $F_{ij}F_{kl}$ is anti self dual.

The theorem above implies that

$$F_{ij} = c_{ij}\omega_{ij},$$

where c_{ij}'s are functions and ω_{ij}'s are strong self-dual 2-forms. We use the structure of 7-dimensional linear subspaces to obtain the following set of 2-forms that form a local basis of Λ^2 [3].

$$\begin{aligned}
\omega_{12} &= e_{14} + e_{23} + e_{58} + e_{67}\\
\omega_{13} &= e_{13} - e_{24} - e_{57} + e_{68}\\
\omega_{14} &= e_{16} + e_{25} - e_{38} - e_{47}\\
\omega_{15} &= e_{15} - e_{26} + e_{37} - e_{48}\\
\omega_{16} &= e_{18} + e_{27} + e_{36} + e_{45}\\
\omega_{17} &= e_{17} - e_{28} - e_{35} + e_{46}\\
\omega_{18} &= e_{12} + e_{34} + e_{56} + e_{78}\\
\omega_{23} &= e_{12} + e_{34} - e_{56} - e_{78}\\
\omega_{24} &= -e_{17} - e_{28} - e_{35} - e_{46}\\
\omega_{25} &= -e_{18} + e_{27} + e_{36} - e_{45}\\
\omega_{26} &= e_{15} + e_{26} - e_{37} - e_{48}\\
\omega_{27} &= e_{16} - e_{25} + e_{38} - e_{47}\\
\omega_{28} &= -e_{13} + e_{24} - e_{57} + e_{68}\\
\omega_{34} &= -e_{18} + e_{27} - e_{36} + e_{45}\\
\omega_{35} &= e_{17} + e_{28} - e_{35} - e_{46}\\
\omega_{36} &= e_{16} - e_{25} - e_{38} + e_{47}\\
\omega_{37} &= -e_{15} - e_{26} - e_{37} - e_{48}\\
\omega_{38} &= e_{14} + e_{23} - e_{58} - e_{67}\\
\omega_{45} &= e_{12} - e_{34} + e_{56} - e_{78}\\
\omega_{46} &= -e_{13} - e_{24} - e_{57} - e_{68}\\
\omega_{47} &= -e_{14} + e_{23} + e_{58} - e_{67}\\
\omega_{48} &= -e_{15} + e_{26} + e_{37} - e_{48}\\
\omega_{56} &= -e_{14} + e_{23} - e_{58} + e_{67}\\
\omega_{57} &= e_{13} + e_{24} - e_{57} - e_{68}
\end{aligned}$$

$$\begin{aligned}\omega_{58} &= e_{16} + e_{25} + e_{38} + e_{47}\\ \omega_{67} &= e_{12} - e_{34} - e_{56} + e_{78}\\ \omega_{68} &= -e_{17} + e_{28} - e_{35} + e_{46}\\ \omega_{78} &= e_{18} + e_{27} - e_{36} - e_{45}\end{aligned}$$

The way we number this set is as follows. The ω_{i8}s are the seven 2-forms obtained from Corrigan's equations (we could start with any such set)[8]. Then start with for example ω_{18} and find all the 2-forms that constitute a linear space together with ω_{18}, these will be placed to the first row and eight column. Actually given one row one can construct the matrix uniquely from the following requirement: given for example the first row, ω_{23} has to form a linear space together both with ω_{13} and ω_{12}, the structure of these spaces are discussed in [3], looking at intersections, we determine the matrix completely.

The properties of products of forms belonging to the same linear subspace, as discussed above, implies that F^3 is proportional to $*F$.

Note that the Bianchi identities are linear in the connection A. Actually, the connection of the base manifold also come into play in $d\omega_{ij}$, since

$$dF_{ij} + F_{ik} \wedge A_{kj} - A_{ik} \wedge F_{kj} = dc_{ij} \wedge \omega_{ij} + c_{ij} d\omega_{ij} + c_{ik}\omega_{ik} \wedge A_{kj} - c_{kj} A_{ik} \wedge \omega_{kj}$$

We start by the general case, with arbitrary c_{ij}. We solve the components of the dc_{ij} and the components of the connection of the manifold from the Bianchi identities. Then, the remaining components of the Bianchi identities are linear homogeneous equations for the components of the connection on the bundle. We have shown that, generically, this coefficient matrix of this homogeneous system is nonsingular, hence the connection of the bundle is trivial. On the other extreme, if we assume all c_{ij}'s are equal to each other, then the coefficient matrix is identically zero. It follows that the connection of the base manifold is determined by the connection on the bundle (or vice versa).

After this stage we assume that the base manifold is conformally flat. This amounts to assuming that $e_i = e^p dx_i$, hence $de_i = p_i e_i$, where p_i is the partial derivative of p, in the direction i. We can incorporate the common multiplicative function in F into the conformal factor. Finally we use the Coulomb gauge condition, $\nabla \cdot A = 0$ and we solve $F = dA - A \wedge A$ for A, in terms of the derivatives of p. The second derivatives of p satisfy $p_{ij} = 0$ for $i \neq j$ and

$$p_{ii} = -1 - \tfrac{1}{2}\left[p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 + p_7^2 + p_8^2\right],$$

fixing the base manifold completely.

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