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Equations of spatial hydrodynamic interaction of weakly nonspherical gas bubbles in liquid in an acoustic field

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Abstract. A mathematical model of spatial hydrodynamic interaction of gas bubbles in liquid in an acoustic field taking into account small deformations of their surfaces is proposed. It is a system of ordinary differential equations of the second order in radii of the bubbles, the position vectors of their centers and the amplitudes of deviation of their shape from the spherical one in the form of spherical harmonics. The equations derived are of the first order of accuracy in A / R and of the fourth order in R / D, where R is the characteristic radius of the bubbles, A is the amplitude of characteristic deviation of their surface from the spherical one in the form of spherical harmonics, D is the characteristic distance between bubbles. The derivation of the equations is carried out by the method of spherical functions with the use of the Bernoulli integral, the kinematic and dynamic boundary conditions on the surface of the bubbles. The effects of viscosity and compressibility of the liquid are considered approximately, the gas in the bubbles is assumed homobaric.

1. Introduction

Studying of hydrodynamic interaction of bubbles in liquid is of interest to various applications in chemistry, power, medicine, etc. In literature at theoretical studying of interaction of bubbles analytic-numerical methods are widely applied, in which the governing relations are a system of ordinary differential equations of the second order [1, 2]. And if the spatial interaction is considered, due to complexity of the task, the bubbles are, as a rule, assumed spherical. However as a result of interaction the surfaces of the bubbles can be deformed. At deformation of any bubbles in a group the property of the group can significantly change. In such cases deformation of the bubbles should be considered, otherwise the obtained predictions of their behavior can be incorrect. In the present paper the equations of spatial hydrodynamic interaction of gas bubbles in liquid in an acoustic field taking into account their small deformations are derived.

2. Problem definition

It is supposed that the bubbles are in the antinode of intensive standing wave with the pressure changing with respect to the harmonious law

$$p_\infty = p_0 - \Delta p \sin \omega t\,,$$

where t is time, Δp, ω are the amplitude and frequency of oscillations, p_0 is the static pressure of the liquid.

The gas in the bubbles is assumed ideal homobaric with the pressure changing according to the adiabatic law

$$p_k = \left(p_0 + \frac{2\sigma}{R_{0k}} \right)\left(\frac{R_{0k}}{R_k} \right)^{3\kappa},$$

where σ is the coefficient of the surface tension, κ is the isentropic exponent, R_k, $R_{0\,k}$ are the current and initial radii of the k-th bubble, $k = 1, 2..., K, K$ being the number of the bubbles.

The effects of viscosity and compressibility of liquid are assumed small. Therefore they are taken into account by means of corrections to the equations of interaction of the bubbles in the assumption that the liquid is ideal incompressible. The dynamics of the liquid in terms of the velocity potential Φ is described by the equations

$$\nabla^2\Phi = 0, \quad \frac{\partial \Phi}{\partial t} - \mathbf{w}_k \cdot \nabla\Phi + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + \frac{p - p_\infty}{\rho_0} = 0, \qquad (1a, b)$$

where ρ_0 is the density of the liquid, p is the pressure, $\mathbf{w}_k = \dot{x}_k\,\mathbf{i} + \dot{y}_k\,\mathbf{j} + \dot{z}_k\,\mathbf{k}$, x_k, y_k, z_k are the coordinates of the center of the k-th bubble, **i**, **j**, **k** are the unit vectors of the Cartesian coordinate system, the point from above means differentiation with respect to time. The effects of viscosity and compressibility of liquid are assumed small and considered by means of corrections. On the surface of the bubbles the following kinematic and dynamic contact conditions are posed

$$\frac{\partial F_k}{\partial t} + \nabla\Phi\cdot\nabla F_k = 0, \quad p = p_k - \frac{2\sigma}{R_k}. \qquad (2a, b)$$

Here $F_k(r_k, \theta_k, \varphi_k, t) = 0$ is the equation of the surface of the k-th bubble in its local spherical coordinate system r_k, θ_k, φ_k with the origin in its center.

3. Equations of interaction of bubbles

The liquid velocity potential Φ is presented in the form of a series

$$\Phi = \sum_{k=1}^{K}\sum_{\gamma=0}^{\infty}\sum_{\varsigma=-\gamma}^{\gamma} B_{\gamma k}^{\varsigma}(t)\,Y_\gamma^\varsigma(\theta_k,\varphi_k)\,r_k^{-\gamma-1}, \qquad (3)$$

and the equation of the surface of the k-th bubble is written as

$$F(r_k,\theta_k,\varphi_k,t) = r_k - R_k(t) - \sum_{n=2}^{N}\sum_{m=-n}^{n} a_{nk}^m(t)\,Y_n^m(\theta_k,\varphi_k) = 0, \qquad (4)$$

where $Y_\gamma^\varsigma(\theta_k,\varphi_k) = P_\gamma^{|\varsigma|}(\cos\theta_k)\,e^{i\varsigma\varphi_k}$ are the spherical functions, $P_\gamma^{|\varsigma|}$ are the associated Legendre polynomials of degree γ and order $|\varsigma|$, i is the imaginary unit, $e^{i\varsigma\varphi_k} = \cos\varsigma\varphi_k + i\sin\varsigma\varphi_k$, N is the number of the harmonics used in representation of the surface of the bubbles, a_{nk}^m is the amplitude of deviation of the surface of the k-th bubble from the spherical one ($r_k = R_k$) in the form of the surface harmonic Y_n^m. In the present paper the distortions of the spherical shape of the bubbles, characterized by the parameter $\varepsilon = \max\limits_{n,m,k}\left|\varepsilon_{nk}^m\right|$ ($\varepsilon_{nk}^m = a_{nk}^m / R_k$), are assumed small (powers ε^2 and above can be neglected in comparison with 1). Moreover, it is assumed that the bubbles are not very close to each other so that the quantity δ^5 can also be neglected in comparison with 1. Here $\delta = \max\limits_{k,j}\left[\left(R_k + R_j\right)/d_{kj}\right]$, $k, j = 1, 2..., K$, $k \neq j$, d_{kj} is the distance between the centers of the k-th and j-th bubbles.

The coefficients $B_{\gamma k}^{\varsigma}$ of the potential are found by substituting (3) and (4) in kinematic boundary conditions (2a) with allowing for the orthogonality of the spherical functions. Further, substitution in Bernoulli integral (1b) (with taking into account dynamic boundary conditions (2b)) of potential expressions (3) with the found coefficients $B_{\gamma k}^{\varsigma}$ and a number of rather bulky transformations with

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allowing for the orthogonality of the spherical functions, give the following system of equations in radii of the bubbles R_k, the position vectors of their centers $\mathbf{p}_k$ and the amplitudes of deviation of their shape from the spherical one a_{nk}^m

$$R_k\ddot{R}_k+\frac{3\dot{R}_k^2}{2}-\frac{\dot{\mathbf{p}}_k^2}{4}-\frac{p_k-p_\infty}{\rho_0}+\psi_{0k}+\Delta_k=\sum_{j=1,\,j\neq k}^{K}\left[\frac{\dot{B}_{0j}}{d_{kj}}-\frac{R_j^2\,\mathbf{p}_{kj}\cdot\left(R_j\ddot{\mathbf{p}}_j+\dot{R}_j\dot{\mathbf{p}}_k+5\dot{R}_j\dot{\mathbf{p}}_j\right)}{2d_{kj}^3}+\right.$$

$$+\frac{R_j^3\dot{\mathbf{p}}_j\cdot\left(\dot{\mathbf{p}}_k+2\dot{\mathbf{p}}_j\right)}{4d_{kj}^3}-\frac{3R_j^3\mathbf{p}_{kj}\cdot\dot{\mathbf{p}}_j\mathbf{p}_{kj}\cdot\left(\dot{\mathbf{p}}_k+2\dot{\mathbf{p}}_j\right)}{4d_{kj}^5}+\sum_{l=1,l\neq k}^{K}\frac{3B_{0j}B_{0l}\mathbf{p}_{kj}\cdot\mathbf{p}_{kl}}{4d_{kj}^3d_{kl}^3}-$$

$$\left.-\sum_{l=1,l\neq j}^{K}\frac{\left(R_j^3B_{0l}\right)'\mathbf{p}_{jl}\cdot\mathbf{p}_{kj}}{2d_{kj}^3d_{jl}^3}-\frac{3C_{10}^{\gamma 0}\Theta_{211}^{s\varsigma-\gamma,2}\left(R_j^2a_{2j}^sY_{1kj}^{\gamma}\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_j^*\right)'}{4d_{kj}^2}\right],$$

$$\frac{R_k\ddot{\mathbf{p}}_k}{3}+\dot{R}_k\dot{\mathbf{p}}_k+\left(\frac{2\Theta_{211}^{s,\varsigma,-s-\varsigma,0}}{3}\dot{R}_k\varepsilon_{2k}^s-\frac{3\alpha_{211}^{s,\varsigma,-s-\varsigma}}{2}\dot{a}_{2k}^s\right)\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*-\frac{7\alpha_{211}^{s,\varsigma,-s-\varsigma}}{6}R_k\boldsymbol{\delta}_\varsigma\cdot\ddot{\mathbf{p}}_k^*\varepsilon_{2k}^s+$$

$$+\left(\Theta_{n12}^{m,\varsigma,-s,2}\Theta_{211}^{s,\varsigma,-s-\varsigma,0}-3\Theta_{n111}^{m,\varsigma,\gamma,-m-\varsigma-\gamma,0}\right)\frac{\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*\boldsymbol{\delta}_\gamma\cdot\dot{\mathbf{p}}_k^*\varepsilon_{nk}^m}{2}+\boldsymbol{\psi}_{1k}=\sum_{j=1,\,j\neq k}^{K}\left\{-\frac{\left(R_kB_{0j}\right)'\mathbf{p}_{kj}}{d_{kj}^3}-\right.$$

$$-\frac{\left(R_kR_j^3\dot{\mathbf{p}}_j\right)'-2R_kB_{0j}\dot{\mathbf{p}}_j}{2d_{kj}^3}-\frac{3R_j^3R_k\left(\dot{\mathbf{p}}_j\cdot\dot{\mathbf{p}}_j\mathbf{p}_{kj}+2\mathbf{p}_{kj}\cdot\dot{\mathbf{p}}_j\dot{\mathbf{p}}_j\right)}{2d_{kj}^5}+\frac{3\mathbf{p}_{kj}\cdot\left[\left(R_kR_j^3\dot{\mathbf{p}}_j\right)'-2R_kB_{0j}\dot{\mathbf{p}}_j\right]\mathbf{p}_{kj}}{2d_{kj}^5}+$$

$$+\frac{15R_j^3R_k\left(\mathbf{p}_{kj}\cdot\dot{\mathbf{p}}_j\right)^2\mathbf{p}_{kj}}{2d_{kj}^7}+C_{01}^{0\varsigma}\left[\frac{\Theta_{211}^{s,\varsigma,-s-\varsigma,2}\left(B_{0j}Y_{1kj}^{-\varsigma}a_{2k}^s\right)'}{2d_{kj}^2}+\frac{\Theta_{211}^{s,\varsigma,-s-\varsigma,0}B_{0j}Y_{1kj}^{-\varsigma}\left(\dot{a}_{2k}^s+2\dot{R}_k\varepsilon_{2k}^s\right)}{3d_{kj}^2}\right]+$$

$$\left.+C_{01}^{0\gamma}\left[\left(\Theta_{n12}^{m,\varsigma,-s,2}\Theta_{211}^{s,\gamma,-s-\gamma,0}+\Theta_{n12}^{m,\gamma,-s,2}\Theta_{211}^{s,\varsigma,-s-\varsigma,0}\right)-6\Theta_{n111}^{m,\varsigma,\gamma,-m-\varsigma-\gamma,0}\right]\frac{B_{0j}Y_{1kj}^{-\gamma}\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*\varepsilon_{nk}^m}{2d_{kj}^2}\right\},$$

$$\frac{R_k\ddot{a}_{\mathrm{c}\,k}^{-\sigma}}{\mathrm{c}+1}+\frac{3\dot{R}_k\dot{a}_{\mathrm{c},k}^{-\sigma}}{\mathrm{c}+1}-\frac{(\mathrm{c}-1)\ddot{R}_ka_{\mathrm{c},k}^{-\sigma}}{\mathrm{c}+1}-\frac{3\Upsilon_{n1\mathrm{c}}^{m,\varsigma,\sigma,2(\mathrm{c}-2)/3}}{2\left(1+\mathrm{c}\right)}R_k\boldsymbol{\delta}_\varsigma\cdot\ddot{\mathbf{p}}_k^*\varepsilon_{nk}^m+\frac{3\Theta_{n1\mathrm{c}}^{m,\varsigma,\sigma,2}}{2\left(1+\mathrm{c}\right)}\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*\dot{a}_{nk}^m+$$

$$+\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*\left(\frac{9\alpha_{11\mathrm{c}}^{\varsigma,\gamma,\sigma}}{8}\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*+\frac{9}{4}\left(\frac{\Theta_{n1\zeta}^{m,\varsigma,-\xi,2}\Theta_{\zeta 1\mathrm{c}}^{\xi,\varsigma,\sigma,0}}{\zeta+1}-\Theta_{n11\mathrm{c}}^{m,\varsigma,\gamma,\sigma,0}\right)\boldsymbol{\delta}_\gamma\cdot\dot{\mathbf{p}}_k^*\varepsilon_{nk}^m+\frac{3\Theta_{n1\mathrm{c}}^{m,\varsigma,\sigma,0}}{2(n+1)}\left(\dot{a}_{nk}^m+2\dot{R}_k\varepsilon_{nk}^m\right)\right)=$$

$$=\sum_{j=1,\,j\neq k}^{K}\left\{\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*\left(\frac{9C_{01}^{0\gamma}\Theta_{11\mathrm{c}}^{\varsigma,\gamma,\sigma,0}B_{0j}Y_{1kj}^{-\gamma}}{4d_{kj}^2}-\frac{9R_j^3C_{11}^{\gamma\zeta}Y_{2kj}^{\gamma-\zeta}\Theta_{11\mathrm{c}}^{\varsigma,\zeta,\sigma,0}\boldsymbol{\delta}_\gamma\cdot\dot{\mathbf{p}}_j^*}{8d_{kj}^3}+\frac{5C_{02}^{0s}\Theta_{21\mathrm{c}}^{s,\varsigma,\sigma,0}B_{0j}R_kY_{2kj}^{-s}}{2d_{kj}^3}\right)+\right.$$

$$+\frac{5C_{02}^{0-\sigma}\delta_{\mathrm{c}}^2\left(B_{0j}R_k^2Y_{2kj}^{\sigma}\right)'}{3d_{kj}^3}-\frac{5C_{12}^{\gamma s}\Theta_{21\mathrm{c}}^{s,\varsigma,\sigma,0}R_j^3R_kY_{3kj}^{\gamma-s}\boldsymbol{\delta}_\gamma\cdot\dot{\mathbf{p}}_j^*\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*}{4d_{kj}^4}+\frac{21B_{0j}R_k^2C_{03}^{0\xi}Y_{3kj}^{-\xi}\Theta_{31\mathrm{c}}^{\xi,\varsigma,\sigma,0}\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_k^*}{8d_{kj}^4}-$$

$$-\frac{5\delta_{\mathrm{c}}^2\left(C_{12}^{\varsigma-\sigma}\left(R_j^3R_k^2Y_{3kj}^{\sigma+\varsigma}\boldsymbol{\delta}_\varsigma\cdot\dot{\mathbf{p}}_j^*\right)'+6B_{0j}R_k^2C_{02}^{0-\sigma}\dot{d}_{kj}Y_{2kj}^{\sigma}\right)}{6d_{kj}^4}+\frac{7C_{03}^{0-\sigma}\delta_{\mathrm{c}}^3\left(B_{0j}R_k^3Y_{3kj}^{\sigma}\right)'}{4d_{kj}^4}-$$

$$-\frac{9C_{01}^{0\varsigma}C_{01}^{0\gamma}\Theta_{11\mathrm{c}}^{\varsigma\gamma\sigma,0}B_{0j}B_{0l}Y_{1kj}^{-\varsigma}Y_{1kl}^{-\gamma}}{8d_{kj}^2d_{kl}^2}+\frac{3C_{01}^{0\varsigma}\Theta_{n1\mathrm{c}}^{m,\varsigma,\sigma,2}\left(B_{0j}Y_{1kj}^{-\varsigma}a_{nk}^m\right)'}{2(\mathrm{c}+1)d_{kj}^2}+$$

$$+\frac{9C_{01}^{0\gamma}\left(\Theta_{211}^{s,\varsigma,-\zeta,2}\Theta_{11c}^{\gamma,\zeta,\sigma,0}+\Theta_{211}^{s,\gamma,-\zeta,2}\Theta_{11c}^{\varsigma,\zeta,\sigma,0}\right)B_{0\,j}Y_{1k\,j}^{-\gamma}\boldsymbol{\delta}_{\varsigma}\cdot\dot{\mathbf{p}}_k^{*}\varepsilon_{2k}^{s}}{8d_{k\,j}^{2}}+\frac{3C_{01}^{0\varsigma}\Theta_{n1c}^{m,\varsigma,\sigma,0}B_{0\,j}Y_{1k\,j}^{-\varsigma}\left(\dot{a}_{nk}^{m}+2\dot{R}_k\varepsilon_{nk}^{m}\right)}{2(n+1)d_{k\,j}^{2}}+$$

$$\left.+\frac{9C_{01}^{0\gamma}\left(\Theta_{n1p}^{m,\varsigma,-q,2}\Theta_{p1c}^{q,\gamma,\sigma,0}+\Theta_{n1p}^{m,\gamma,-q,2}\Theta_{p1c}^{q,\varsigma,\sigma,0}-2(p+1)\Theta_{n11c}^{m,\varsigma,\gamma,\sigma,0}\right)B_{0\,j}Y_{1k\,j}^{-\gamma}\varepsilon_{nk}^{m}\boldsymbol{\delta}_{\varsigma}\cdot\dot{\mathbf{p}}_k^{*}}{4(p+1)d_{k\,j}^{2}}\right\}.$$

Here the points from above and the primes mean differentiation with respect to time, $k = 1, 2, \ldots, K$ (K is the number of the interacting bubbles), $c = 2, 3, \ldots, N$ (N is the number of harmonics), $s = -c, -c+1, \ldots, c-1, c$, summation is supposed over the repeating indexes: from –1 to 1 over γ, ς, ζ, from –2 to 2 over s, from –3 to 3 over ξ, from 2 to N over n and p, from $-n$ to n over m, from $-p$ to p over q; $B_{0k} = -R_k^2\dot{R}_k$, $\boldsymbol{\delta}_{\varsigma} = \left(\delta_{\varsigma}^{1}, \delta_{\varsigma}^{-1}, \delta_{\varsigma}^{0}\right)$, $\delta_{\varsigma}^{\gamma}$ is the Kronecker delta, $\mathbf{p}_k = \left(x_k, y_k, z_k\right)$, $\mathbf{p}_k^{*} = \left(\frac{x_k - i\,y_k}{2}, \frac{x_k + i\,y_k}{2}, z_k\right)$, x_k, y_k, z_k are the coordinates of the center of the k-th bubble in the system of the global Cartesian coordinates, $\mathbf{p}_{k\,j} = \mathbf{p}_k - \mathbf{p}_j$, $d_{kj} = |\mathbf{p}_{kj}|$ is the distance between the centers of the k-th and j-th bubbles, ψ_{0k}, $\boldsymbol{\psi}_{1\,k}$, Δ_k are the corrections for viscosity and compressibility of the liquid [2]. Numbers $Y_{\gamma k\,j}^{\varsigma}$, $C_{\gamma\varsigma}^{\gamma'\varsigma'}$, $\Theta_{n_1n_2n_3}^{m_1,m_2,m_3,k}$, $\Upsilon_{n_1n_2n_3}^{m_1,m_2,m_3,k}$ and $\alpha_{n_1n_2n_3}^{m_1m_2m_3}$ are defined as follows

$$Y_{\gamma k\,j}^{\varsigma} = Y_{\gamma}^{\varsigma}\left(\theta_{k\,j}, \varphi_{k\,j}\right),\; C_{\gamma\varsigma}^{\gamma'\varsigma'} = (-1)^{\gamma+\left(|\gamma'-\varsigma'|-|\gamma'|-|\varsigma'|\right)/2}\left(\gamma+\varsigma-|\gamma'-\varsigma'|\right)!\Big/\left[\left(\gamma-|\gamma'|\right)!\left(\varsigma+|\varsigma'|\right)!\right],$$

$$\Theta_{n_1n_2n_3}^{m_1,m_2,m_3,k} = \frac{2k - n_1\left(n_1+1\right) - n_2\left(n_2+1\right) + n_3\left(n_3+1\right)}{2}\alpha_{n_1n_2n_3}^{m_1m_2m_3},$$

$$\Upsilon_{n_1n_2n_3}^{m_1,m_2,m_3,k} = \frac{2k + n_1\left(n_1+1\right) + n_2\left(n_2+1\right) - n_3\left(n_3+1\right)}{2}\alpha_{n_1n_2n_3}^{m_1m_2m_3},$$

$$\alpha_{n_1n_2n_3}^{m_1m_2m_3} = \frac{2n_3+1}{4\pi}\frac{\left(n_3-|m_3|\right)!}{\left(n_3+|m_3|\right)!}\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{n_1}^{m_1}(\theta,\varphi)Y_{n_2}^{m_2}(\theta,\varphi)Y_{n_3}^{m_3}(\theta,\varphi)\,d\theta d\varphi .$$

4. Conclusion

The system of ordinary differential equations of the second order in radii of bubbles, the position vectors of their centers and the amplitudes of deviation of their shape from the spherical one, describing the spatial hydrodynamic interaction of weakly nonspherical gas bubbles in an acoustic field, is derived. The derivation of the equations is carried out by the method of spherical functions with the use of Bernoulli integral, the kinematic and dynamic boundary conditions on the surfaces of the bubbles. The effects of viscosity and compressibility of the liquid are taken into account approximately, the gas in the bubbles is assumed homobaric.

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