Journal of Physics: Conference Series **662** (2015) 012027 doi:10.1088/1742-6596/662/1/012027

Initial value method for general singular perturbation problems

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Abstract. In general, the numerical solution of a boundary value problem will be more difficult than the numerical solution of the corresponding initial value problem. Hence, we prefer to convert the given second order problem into a first order problem. In this paper we present an initial value method. It is distinguished by the following fact: the original second order problem is replaced by an asymptotically equivalent first order problem and then solved as an initial value problem. Classical RungeKutta method is used to solve the first order equation. The method is first described for solving problems with left end boundary layer. This is extended for solving singular perturbation problems with right end, internal and terminal layers. We solve one problem to demonstrate the applicability of the method. The numerical results are compared with the exact solution.

1. Introduction

Singular perturbation problems containing a small positive parameter ϵ have appeared in many fields such as fluid mechanics, chemical kinetics, elasticity, aerodynamics, plasma dynamics and magneto hydrodynamics. A few notable examples are boundary layer problems, WKB problems. For small values of ϵ, it is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the parameter value, i.e. methods that are uniformly convergent. A wide variety of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention C.M Bender, and S.A.Orszag [1], E.Issacson, H.B.Keller [2], P.W.Hemker and J.J.H Miller [3], J.Kevorkian and Cole [4], Nayfeh [5], O'Malley [6], Y.N.Reddy [7], M.K.Kadalbajoo and Y.N.Reddy[8] ,R.Vulanovic [9], F.Mazzia and D.Trigiante[10] Van Dyke [11], J.J.H.Miller [12], [13]. Usually, the numerical solution of a boundary value problem will be more difficult than that of initial value problem. Hence, we prefer to convert the given second order problem into a first order problem.In this paper we present an initial value method. It is distinguished by the following fact: the original second order problem is replaced by an asymptotically equivalent first order problem and then solved as an initial value problem. Classical RungeKutta method is used to solve the first order equation. The method is first described for solving problems with left end boundary layer. This is extended for solving singular perturbation problems with right end, internal and terminal layers. We solve one problem to demonstrate the applicability of the method. The numerical results are compared with the exact solution.

2. Left end boundary layer problems

Let us consider a class of linear singularly perturbed two point boundary value problems of the form

$$\epsilon y''(x) + [a(x)y(x)]' + b(x)y(x) = h(x) \quad x \in [0,1], \tag{1}$$

$$\text{with } \; y(0) = \alpha, \tag{2.a}$$

$$\text{and } \; y(1) = \beta. \tag{2.b}$$

where ϵ is a small positive parameter $(0 < \epsilon \leq 1)$, α, β are given constants, $a(x)$, $b(x)$, and $h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[0,1]$ where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$. The initial value method consists of the following steps.

Step 1: Preliminary step we obtain the reduced problem of (1) by setting $\epsilon = 0$ in equation (1) and solve it for the solution with appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of $(1-2)(i.e)$

$$[a(x)y_0(x)]' + b(x)y_0(x) = h(x), \tag{3}$$

$$\text{and } \; y_0(1) = \beta. \tag{3.a}$$

Step 2: Set up the approximate equation to equation (1) as follows

$$\epsilon y''(x) + [a(x)y(x)]' + b(x)y_0(x) = h(x), \tag{4}$$

where we have simply replaced the $y(x)$ term by $y_0(x)$, the solution of the reduced problem $(3-3a)$.

Step 3: Replace the approximated second order problem $(4-2)$ by an asymptotically equivalent first order problem as follows:

Integrate equation (4) to get

$$\epsilon y'(x) + a(x)y(x) = f(x) + k, \tag{5}$$

where $f(x) = \int (h(x) - b(x)y_0(x))dx$ and k is a constant to be determined.

Step4: To determine the constant k, we introduce the condition that the reduced equation of (5) should satisfy the boundary condition $y(1) = \beta$ $(i.e)$ $y(1) = \frac{(f(1)+k)}{(a(1))} = \beta$,

$$k = a(1)\beta - f(1). \tag{6}$$

Remark 1: This choice of k ensures that the solution of the reduced problem of $(1-2)$ satisfies the reduced equation of (5).

Step 5: We now adjoin the condition (which we drop, whenever we formulate the reduced problem of the equation $(1-2)$, $y(0) = \alpha$ to the equation (5) to obtain first order problem as follows

$$\epsilon y'(x) + a(x)y(x) = f(x) + k \quad x \in [0,1], \tag{7}$$

$$\text{with } \; y(0) = \alpha, \tag{7.a}$$

k is given by (6). Thus in a manner of speaking we have replaced the original second order problem $(1-2)$ with the asymptotically equivalent first order problem $(7-7a)$. We solve this

initial value problem to obtain the solutions over the interval $0 \leq x \leq 1$. There now exist a number of efficient methods for the solution of initial value problems. In order to solve the initial value problems in our numerical experimentation we make use of the classical fourth order Runge-Kutta method. In fact any standard analytical or numerical methods can be used.
Remark 2: For the case $b(x) = 0$ we do not require step (1)&(2) because we can integrate directly the given equation. Detailed discussion and numerical examples can be found in Kadalbajoo and Reddy [8].

3. Right end boundary layer problems

We now describe this initial value method for the singularly perturbed two point boundary value problems with the right end boundary layer of the underlying interval. To be specific we consider a class of linear singularly perturbed two point boundary value problems of the form

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad x \in [0,1], \tag{8}$$

$$\text{with} \;\; y(0) = \alpha, \tag{9.a}$$

$$\text{and} \;\; y(1) = \beta, \tag{9.b}$$

where ϵ is a small positive parameter $(0 < \epsilon \leq 1)$, α, β are given constants, $a(x)$, $b(x)$, and $h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0, 1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[0, 1]$ where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$. The initial value method consists of the following steps.
Step 1: Preliminary step:
We obtain the reduced problem by setting $\epsilon = 0$ in equation (8) and solve it for the solution with appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of $(8-9)(i.e)$

$$[a(x)y_0(x)]^{'} + b(x)y_0(x) = h(x), \tag{10}$$

$$\text{and} \;\; y_0(1) = \alpha. \tag{10.a}$$

Step 2: Set up the approximate equation to equation (8) as follows

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y_0(x) = h(x), \tag{11}$$

where we have simply replaced the $y(x)$ term by $y_0(x)$, the solution of the reduced problem $(10 - 10a)$.
Step 3: Replace the approximated second order problem $(11-9)$ by an asymptotically equivalent first order problem as follows:
Integrate equation (11) to get

$$\epsilon y^{'}(x) + a(x)y(x) = f(x) + k, \tag{12}$$

where $f(x) = \int(h(x) - b(x)y_0(x))dx$ and k is a constant to be determined.
Step 4: To determine the constant k, we introduce the condition that the reduced equation of (12) should satisfy the boundary condition $y(0) = \alpha$

$$y(0) = \frac{1}{(a(0))}[f(0) + k] = \alpha,$$

$$k = a(0)\alpha - f(0). \tag{13}$$

Remark 3: This choice of k ensures that the solution of the reduced problem of $(8-9)$ satisfies the reduced equation of (12).
Step 5: We now adjoin the condition (which we drop, whenever we formulate the reduced problem of equation $8-9$, $y(1)=\beta$ to (12) to obtain the first order problem as follows.

$$\epsilon y^{'}(x) + a(x)y(x) = f(x) + k \quad x \in [0,1], \tag{14}$$

$$\text{with} \ \ y(1) = \beta, \tag{14.a}$$

where k is a constant given by (13)

Thus in a manner of speaking we have replaced the original second order problem $(8-9)$ with the asymptotically equivalent first order problem $(14-14a)$. We solve this initial value problem to obtain the solution over the interval $0 \leq x \leq 1$.There now exist a number of efficient methods for the solution of initial value problem. In our numerical experimentation, we make use of classical fourth order Runge - Kutta method. In fact any standard analytical or numerical method can be used.
Remark 4: For the case $b(x)=0$, we do not require the preliminary step, because we can directly integrate the given equation.

4. Internal layer problems

We will now extend the initial value method to singular perturbation problem with an internal layer of the underlying interval. In this case $a(x)$ changes sign in the domain of interest. Without loss of generality we can take $a(0)=0$ and the interval to be $[-1,1]$. To describe the method, we again consider a class of linear singularly perturbed two point boundary value problems of the form

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad x \in [0,1], \tag{15}$$

$$\text{with} \ \ y(-1) = \alpha, \tag{16.a}$$

$$\text{and} \ \ y(1) = \beta. \tag{16.b}$$

where ϵ is a small positive parameter $(0 < \epsilon << 1)$, α, β are given constants $a(x),\ b(x),\ h(x)$ are assumed to be sufficiently continuously differentiable functions in $[-1,1]$. Furthermore we assume that $a(x) \leq M \leq 0$ in $[-1,0]$ where M is some negative constant and $a(x) \geq M > 0$ in $[0,1]$ where M is some positive constant. This assumption implies that the boundary layer is in the nbd of $x=0$. We now proceed as follows.
Step1: We first find the approximate solution at $x=0$. Without loss of generality we can take $a(0)=0$. At $x=0$ equation (15) becomes

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad \text{at} \ \ x = 0. \tag{17}$$

The reduced problem of (17) obtained by putting $\epsilon = 0$ gives us the approximation to $y(0)$.

$$b(0)y(0) = h(0), \tag{18}$$

is the reduced problem. Therefore

$$y(0) = \frac{h(0)}{b(0)} = \gamma. \tag{19}$$

Step 2: We now divide the interval $[-1,1]$ into two subintervals $[-1,0]$ and $[0,1]$ so that equation (15) has a right layer in $[-1,0]$ and a left layer in $[0,1]$.

Step 3: We now use initial value method for right end boundary layer as described in section (3) in the interval $[-1, 0]$. We have

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad x \in [-1, 0], \tag{20}$$

$$\text{with} \;\; y(-1) = \alpha, \;\; \text{and} \;\; y(0) = \gamma. \tag{21}$$

The corresponding initial value problem is

$$\epsilon y^{'}(x) + a(x)y(x) = f(x) + k; \quad x \in [-1, 0], \tag{22}$$

$$\text{with} \;\; y(0) = \gamma. \tag{22.a}$$

where k is a constant given by

$$k = a(-1)\alpha - f(-1). \tag{23}$$

Step 4: We use initial value method for left end boundary layer for the other half interval $[0, 1]$ as described in section(2). We have

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad x \in [0, 1], \tag{24}$$

$$\text{with} \;\; y(0) = \gamma, \tag{25.a}$$

$$\text{and} \;\; y(1) = \beta. \tag{25.b}$$

The corresponding initial value problem is

$$\epsilon y^{'}(x) + a(x)y(x) = f(x) + k; \quad x \in [0, 1], \tag{26}$$

$$\text{with} \;\; y(0) = \gamma. \tag{26.a}$$

where k is a constant given by

$$k = a(1)\beta - f(1). \tag{27}$$

Thus, in a manner of speaking we have replaced the original second order problem $(15 - 16)$ with asymptotically equivalent first order problems $(22 - 22a)$ and $(26 - 26a)$. We solve these initial value problems to obtain solution over $[-1, 0]$ and $[0, 1]$ respectively.

5. Numerical example

With the help of one model example, we shall demonstrate the applicability of initial value method for internal layer problems.

Example 5.1: Consider the following SPP

$$\epsilon y^{''}(x) + xy^{'}(x) - y(x) = 0; \quad x \in [-1, 1],$$

$$\text{with} \;\; y(-1) = 1, \text{and} \;\; y(1) = 2.$$

For this example we have $a(x) = x$, $b(x) = -1$ and $f(x) = 0$. Further we have an internal layer of width $o(\sqrt{\epsilon})$ at $x = 0$ (for details, see O'Malley $[6, pp168 - 172, eq8.1case(i)]$ and Kevorkian and Cole $[4, pp41 - 43$, eqs (2.3.76) and (2.3.77)]

The given equation can be written as $\epsilon y^{''}(x) + (xy)^{'} - 2y(x) = 0$;

Here $a(x) = x$; $b(x) = -2$; $h(x) = 0$

Step1: $\gamma = 0$

Step 2: In the interval $[-1, 0]$ we have a right layer. The solution of the reduced problem

$(xy)^{'} - 2y(x) = 0$ with $y(-1) = 1$ is $y_0(x) = -x$, $k = 0$, $f(x) = -x^2$
The initial value problem is

$$\epsilon y^{'}(x) + xy(x) = -x^2; \quad x \in [0,1]$$
$$\text{with} \;\; y(0) = 0.$$

Step 3: In the interval $[0,1]$ we have a right layer. The solution of the reduced problem $(xy)^{'} - 2y = 0$ with $y(1) = 2$ is $y_0(x) = 2x$, $k = 0$, $f(x) = 2x^2$. We solve these initial value problems using classical RungeKutta method. The numerical results are presented in **Table 1** for $\epsilon = 10^{-3}$

Table 1. Computational results for Example 5.1 with $\epsilon = 10^{-3}$ and $h = 0.01$.

x	y(x)	Exact Solution
-1.000	1.0000000	1.0000000
-0.500	0.4979919	0.4999964
-0.100	0.0884604	0.1032475
-0.080	0.0642969	0.0866667
-0.060	0.0384793	0.0727863
-0.040	0.0152044	0.0629293
0.000	0.0000000	0.0606750
0.020	0.0049300	0.0785011
0.040	0.0316510	0.1029288
0.060	0.0777414	0.1327857
0.080	0.1287349	0.1666600
0.100	0.1768589	0.2032466
0.500	0.9959838	0.9999939
1.000	2.0000000	2.0000000

6. Two boundary layers problems

The suggestions given for internal layers problems can be extended mutatis mutandis to problems with two boundary layers. Consider the class of linear singularly perturbed problems of the form

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad x \in [-1,1], \tag{28}$$
$$\text{with} \;\; y(-1) = \alpha, \tag{29.a}$$
$$\text{and} \;\; y(1) = \beta. \tag{29.b}$$

where ϵ is a small positive parameter $(0 < \epsilon \leq 1)$ α, β are given constants, $a(x)$, $b(x)$, $h(x)$ are assumed to be sufficiently continuously differentiable function in$[1,1]$. Furthermore we assume $a(x) \geq M > 0$ in $[-1,0]$ and $a(x) \leq M < 0$ in $[0,1]$. This assumption implies that the boundary layer is at $x = -1$ and $x = 1$. With out loss of generality we take $a(x) = 0$ at $x = 0$ since $a(x)$ changes sign in the domain of interest. We now proceed as follows:
Step1: We first find the approximate solution at $x = 0$. Equation (28) is now

$$\epsilon y^{''}(x) + b(x)y(x) = h(x). \tag{30}$$

The reduced problem of (28) gives us the approximation to $y(0)$

$$\text{where} \qquad y(0) = \frac{h(0)}{b(0)} = \gamma. \tag{31}$$

Step 2: We now divide the interval $[-1,1]$ into two subintervals $[-1,0]$ and $[0,1]$ so that equation (28) has a left layer in $[-1,0]$ and a right layer in $[0,1]$.
Step 3: We now use initial value technique for left end boundary layer as described in section (2) in the interval $[-1,0]$. We have

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x) \quad x \in [-1,0], \tag{32}$$
$$\text{with } \ y(-1) = \alpha, \tag{33.a}$$
$$\text{and } \ y(0) = \gamma. \tag{33.b}$$

The corresponding initial value problem is

$$\epsilon y^{'}(x) + a(x)y(x) = f(x) + k \quad x \in [-1,0], \tag{34}$$
$$\text{with } \ y(-1) = \alpha, \tag{34.a}$$

where k is constant given by

$$k = a(0)\gamma - f(0). \tag{35}$$

Step 4: We use initial value technique for right end boundary layer as described in section (3) above for the interval $[0,1]$. We have

$$\epsilon y^{''}(x) + [a(x)y(x)]^{'} + b(x)y(x) = h(x), \tag{36}$$
$$\text{with } \ y(0) = \gamma, \tag{37.a}$$
$$\text{and } \ y(1) = \beta. \tag{33.b}$$

The corresponding initial value problem is

$$\epsilon y^{'}(x) + a(x)y(x) = f(x) + k \quad x \in [0,1], \tag{38}$$
$$\text{with } \ y(1) = \beta, \tag{38.a}$$

where k is constant given by

$$k = a(0)\gamma - f(0). \tag{39}$$

Thus in a manner of speaking we have replaced the original second order problem $(28-29)$ with asymptotically equivalent first order problems (34-34a) and (38-38a). We solve these initial value problems to obtain solutions over $[-1,0]$ and $[0,1]$ respectively. There now exist a number of efficient methods for the solutions of initial value problems. We use classical Runge-Kutta method in our numerical experimentation. Any other method can also be used.

7. Numerical example

To demonstrate the applicability of the method we solve one problem.
Example 7.1: Consider the following SPP

$$\epsilon y^{''}(x) - xy^{'}(x) - y(x) = 0 \quad x \in [-1,1],$$
$$\text{with } \ y(-1) = 1\text{and } \ y(1) = 2.$$

For this example we have $a(x) = -x$, $b(x) = 1$ and $f(x) = 0$. Further we have two boundary layers one at $x = -1$ and one at $x = 1$ (for details, see O'Malley [6,pp168-173,eq 8.1 case (i)] The given equation can be written as

$$\epsilon y^{''}(x) - (xy)^{'}(x) = 0,$$

here $a(x) = -x$, $b(x) = 0$, and $h(x) = 0$.
Step 1: $\gamma = 0$.
Step 2: In the interval $[-1, 0]$ we have a left layer. The solution of the reduced problem $(xy)^{'} = 0$ with $y(0) = 0$ is $y_0(x) = 0$, $k = 0$, $f(x) = 0$.
The initial value problem is

$$\begin{aligned} \epsilon y^{'}(x) - xy(x) = 0 \quad x \in [-1, 0], \\ \text{with} \;\; y(-1) = 1. \end{aligned}$$

Step 3: In the interval $[0, 1]$ we have a right layer. The solution of the reduced problem $(xy)^{'} = 0$ with $y(0) = 0$ is $y_0(x) = 0$, $k = 0$, $f(x) = 0$.
The initial value problem is

$$\begin{aligned} \epsilon y^{'}(x) - xy(x) = 0 \quad x \in [0, 1], \\ \text{with} \;\; y(1) = 2. \end{aligned}$$

We solve these initial value problems using classical Runge-Kutta method. The numerical results are presented in **Table 2** for $\epsilon = 10^{-3}$.

Table 2. Computational results for Example 5.1 with $\epsilon = 10^{-3}$ and $h = 0.01$.

x	y(x)	Exact Solution
-1.000	1.0000000	1.0000000
-0.980	0.0000000	0.6393120
-0.960	0.0000000	0.0042141
-0.940	0.0000000	0.0002865
-0.920	0.0000000	0.0000201
-0.900	0.0000000	0.0000014
-0.700	0.0000000	0.0000000
-0.300	0.0000000	0.0000000
0.300	0.0000000	0.0000000
0.900	0.0000000	0.0000029
0.920	0.0000000	0.0000402
0.940	0.0000000	0.0005730
0.960	0.0000000	0.0084281
0.980	0.0000000	0.1278624
1.000	2.0000000	2.0000000

8. Discussion and conclusion

We have presented and illustrated an initial value method for solving singularly perturbed two point boundary value problem. We have replaced the original second order problem by an equivalent first order problem and then solved it as an initial value problem. We used Runge-Kutta method to solve the first order equations. We extended the method to solve problems with internal and terminal layers. We solved one problem each to demonstrate the applicability of the method. It is observed that the method compares well with the known method.

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