

# Buckling analysis of moderately thick rectangular plates using coupled displacement field method

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**Abstract.** A simple and efficient coupled displacement field method is developed to study the buckling load parameters of the moderately thick rectangular plates. This method has been successfully applied to study the same for the Timoshenko beams. A single term trigonometric admissible displacement field is assumed for one of the variables, say, the total rotations (in both  $X, Y$  directions). Making use of the coupling equations, the spatial variation for the remaining lateral displacement field is derived in terms of the total rotations. The coupled displacement field method makes the energy formulation to contains half the number of unknown independent coefficients, in the case of a rectangular plate, contrary to the conventional Rayleigh-Ritz method. The expressions for the non-dimensional buckling load parameters of the moderately thick rectangular plates with all the edges simply supported are derived. The numerical values of these parameters obtained using the coupled displacement field method match very well with open literature demonstrating the effectiveness of the coupled displacement field method.

## 1. Introduction

Knowledge of buckling load parameters of moderately thick plates is a necessary design input that has to be considered in the initial design phase. The energy methods provide a convenient means for computing the buckling load parameters and the solutions obtained using this approach are upper bounds and the accuracy of the solution depends on the admissible functions chosen for the lateral displacement and total rotations. The widely used energy method is the classical Rayleigh - Ritz (RR) method where in the displacement field of a structural member is generally approximated by simple trigonometric or algebraic admissible functions. Single or multi-term functions are chosen to satisfy the geometric boundary conditions involved in the problem.

The concept of the coupled displacement field(CDF) which was successfully used in the finite element (FE) structural members analysis [1-3]. A continuum analogue of the FE analysis with the CDF method is not received much attention except in the formulation of Zhou[4] where the two fields (displacement and rotation) are coupled through an equation which is dependent on the applied load. This approach different from the FE method based on the CDF.



A general CDF method is presented here, where the coupling equation is independent of the applied load, to study the buckling and free vibration behavior of uniform Timoshenko beams. The effectiveness of the proposed CDF method is demonstrated successfully by comparing the buckling load and the frequency parameters with those obtained from the RR method [5,6] for the short columns and beams. In the present study, the authors made an attempt to show the applicability of the CDF method to study the buckling of uniform moderately thick rectangular plates.

The proposed CDF method, if generalized to a  $n$  term admissible functions, will have  $n$  unknown coefficients because of the coupling equation used, where as, the RR method contains  $2n$  unknown coefficients for the square/rectangular plate [12]. On the other hand, if an accurate single term admissible function is used in the CDF method, then a one unknown coefficient problem has to be solved, where as two unknown coefficients are associated in the RR method. Thus the proposed CDF method significantly simplifies the formulation of the buckling problem of moderately thick rectangular plates. In this paper, the coupling equations used are taken from [7]. In general the CDF method reduces the complexity of the problem by a factor two compared to the RR method, as mentioned earlier. The practical utility of the CDF method is demonstrated by solving the buckling problem of the isotropic and uniform moderately thick rectangular plates for all edges simply supported boundary conditions, by using the CDF method. The solution procedure and the numerical results obtained speak for themselves about the simplicity of the CDF method applied to the buckling problem of the moderately thick rectangular plates compared to the RR method. The first order shear deformation theory is briefly given in the following section.

## 2. First order shear deformation theory of plates

The simplest shear deformation plate theory is the first order shear deformation plate theory (FSDT), also referred to as the Mindlin plate theory [8] where the displacements  $u, v$  and  $w$  are given by

$$u(x, y, z) = z\theta_x(x, y), \quad (1)$$

$$v(x, y, z) = z\theta_y(x, y), \quad (2)$$

$$w(x, y, z) = w(x, y), \quad (3)$$

where  $u$  and  $v$  are the inplane displacements in  $x$  and  $y$  directions,  $w$  is the transverse displacement along  $z$  direction,  $\theta_x$  and  $\theta_y$  denote rotations about the  $y$  and  $x$  axes respectively. In the FSDT, the shear correction factors are introduced to correct the discrepancy between the actual transverse shear stress distribution and those computed using the kinematic relations of the FSDT. The shear correction factor  $k$  depend not only on the geometric parameters, but also on the loading and boundary conditions of the plate. However, a value of  $k=5/6$ , the widely used value of the shear correction factor is used in the present study also.

$$U = \frac{D}{2} \int_0^a \int_0^b \left\{ \left( \frac{\partial \theta_x}{\partial x} \right)^2 + \left( \frac{\partial \theta_y}{\partial y} \right)^2 + 2\nu \left( \frac{\partial \theta_x}{\partial x} \right) \left( \frac{\partial \theta_y}{\partial y} \right) + 2(1 - \nu) \left( \frac{\partial \theta_x}{\partial y} \right) \left( \frac{\partial \theta_y}{\partial x} \right) \right\} dx dy$$

$$+ \frac{kGh}{2} \int_0^a \int_0^b \left\{ \left( \frac{dw}{dx} + \theta_x \right)^2 + \left( \frac{dw}{dy} + \theta_y \right)^2 \right\} dx dy, \quad (4)$$

$$W = \frac{N_x}{2} \int_0^a \int_0^b \left( \frac{\partial w}{\partial x} \right)^2 dx dy + \frac{N_y}{2} \int_0^a \int_0^b \left( \frac{\partial w}{\partial y} \right)^2 dx dy, \quad (5)$$

where  $h$  is the thickness of the plate,  $W$  is the work done because of the biaxial compressive load,  $N_x$  is the compressive load per unit length acting on the two edges perpendicular to  $x$  axis and  $N_y$  is the compressive load per unit length acting on the two edges perpendicular to  $y$  axis. Suitable admissible functions satisfying mainly the kinematic boundary conditions (some times the admissible functions may satisfy some or all of the natural boundary conditions and do not violate the variational principles) are assumed for  $w$  and  $\theta$ . In the present study for the sake of simplicity and clarity and easy understanding of the CDF method a single term admissible functions for  $w$  and  $\theta$  are chosen here and is shown that the single term trigonometric admissible functions for the two boundary conditions of the plates considered, given later, are found to be accurate enough for all the practical engineering purposes.

The detailed procedure for CDF method are discussed in this section for the evaluating the buckling load (biaxial and uniaxial compressive load per unit length) of a moderately thick uniform rectangular plate with all edges simply supported for which the exact buckling mode shape for the transverse displacement  $w$  is well known (sine waves in the  $x$  and  $y$  directions).

### 3. Coupled displacement field (CDF) method

In this method an admissible functions for  $\theta_x$  and  $\theta_y$  which satisfies all the geometric boundary conditions on the plate domain are assumed and the field for lateral displacement  $w$  is evaluated using the coupling equation, the derivation of which is briefly given below

The expression for the strain energy  $U$  is already given in Eq.(4) and the work done by the externally applied lateral loads  $W_b$  used in the static analysis is given as

$$W_b = \int_0^a \int_0^b p w dx dy, \quad (6)$$

where  $p$  is the lateral load distribution per unit area acting on the plate. Taking the variation of the total potential energy as

$$\delta(U - W_b) = 0, \quad (7)$$

The following static equilibrium equations independent of the externally applied load term [8] are obtained. Note that two coupling equations are obtained for the rectangular plate.

$$\frac{dw}{dx} = -\theta_x + \frac{h^2}{3.5} \left[ \frac{\partial^2 \theta_x}{\partial x^2} + \nu \frac{\partial^2 \theta_y}{\partial y \partial x} \right] + \frac{h^2}{10} \left[ \frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial y \partial x} \right], \quad (8)$$

$$\frac{dw}{dy} = -\theta_y + \frac{h^2}{3.5} \left[ \frac{\partial^2 \theta_y}{\partial y^2} + \nu \frac{\partial^2 \theta_x}{\partial y \partial x} \right] + \frac{h^2}{10} \left[ \frac{\partial^2 \theta_y}{\partial x^2} + \frac{\partial^2 \theta_x}{\partial y \partial x} \right]. \quad (9)$$

Eqs.(8) and (9) are called as the coupling equations in the CDF method and is used to couple the total rotations  $\theta_x$  and  $\theta_y$  and the transverse displacement  $w$ , so that the two independent unknown coefficients problem in the RR method reduces to a single unknown coefficient problem in the CDF method. The effectiveness of the CDF method is brought out in the following section.

Though admissible functions  $\theta_x$  and  $\theta_y$  can be written in a series form, here a single term admissible functions for  $\theta_x$  and  $\theta_y$  is chosen again with the same intention of simplicity and better understanding of the method as

$$\theta_x = \alpha f_1(x, y), \quad (10)$$

$$\theta_y = \alpha f_2(x, y), \quad (11)$$

where  $\alpha$  is the undetermined coefficient and  $f_i(x, y)$ ,  $i = 1, 2$  is the single term admissible function. Note that the functions for  $\theta_x$  and  $\theta_y$  are the same as, the rectangular plate is considered in the present study. Substituting the admissible functions for  $\theta_x$  and  $\theta_y$  as given in Eq.(10) and (11) in Eq.(8) and (9), the coupled displacement field for the lateral displacement  $w$ , after integration is obtained, as

$$w = \alpha f_3(x, y). \quad (12)$$

Note that because of the use of the coupling equation, the transverse displacement distribution  $w$  also contains the same undetermined coefficient  $\alpha$  as existing in the  $\theta$  distribution. The lowest buckling load parameter is obtained from the following equation.

$$\frac{d(U - W)}{d\alpha} = 0. \quad (13)$$

In the CDF method the admissible functions for  $\theta_x$  and  $\theta_y$  are assumed in the functional form, noting the similarity between  $\frac{dw}{dx}$ ,  $\theta_x$  and  $\frac{dw}{dy}$ ,  $\theta_y$ , as

$$\theta_x = \alpha \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (14)$$

$$\theta_y = \alpha \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}. \quad (15)$$

Substituting Eqs.(14) and (15) in the coupling Eqs.(8) and (9) and after simplification the slopes are obtained as

$$\frac{dw}{dx} = -\alpha \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left\{ 1 + 2.8198m^2 \left(\frac{h}{a}\right)^2 + n^2 \left(\frac{h}{b}\right)^2 (1.9739 + 2.8198v) \right\}, \quad (16)$$

$$\frac{dw}{dy} = -\alpha \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \left\{ 1 + 2.8198m^2 \left(\frac{h}{b}\right)^2 n^2 + m^2 \left(\frac{h}{a}\right)^2 (1.9739 + 2.8198v) \right\}. \quad (17)$$

After integration the lateral displacement field for  $w$ , as

$$w = -\alpha \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left\{ 1 + 2.8198m^2 \left(\frac{h}{a}\right)^2 + n^2 \left(\frac{h}{b}\right)^2 (1.9739 + 2.8198v) \right\}. \quad (18)$$

Substituting Eqs. (14), (15) and (18) in Eqs. (4) and (5) we get the expression for  $U$  and  $W$ , as

$$U = \frac{D}{2} \frac{ab}{4} \pi^4 \alpha^2 \left[ \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right]^2 + \frac{D}{2} \frac{ab}{4} \pi^4 \alpha^2 h^2 0.35462 \left\{ 7.9512 \left[ \left(\frac{m}{a}\right)^6 + \left(\frac{n}{b}\right)^6 \right] + \left[ \left(\frac{m}{a}\right)^4 \left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2 \left(\frac{n}{b}\right)^4 \right] (23.8542) \right\}. \quad (19)$$

and

$$W = \frac{N_x}{2} \alpha^2 \frac{ab}{4} \pi^2 \left( \left(\frac{m}{a}\right)^2 \left[ 1 + 2.8198m^2 \left(\frac{h}{a}\right)^2 + n^2 \left(\frac{h}{b}\right)^2 (2.81984) \right] \right)^2 + \frac{N_x}{2} \alpha^2 \frac{ab}{4} \pi^2 \left( \left(\frac{n}{b}\right)^2 \left[ 1 + 2.8198n^2 \left(\frac{h}{b}\right)^2 + m^2 \left(\frac{h}{a}\right)^2 (2.81984) \right] \right)^2, \quad (20)$$

where  $N_x = N_y$ , since the plate is under bi-axial compression load, by minimizing the total potential energy as  $(d(U - W)/d\alpha) = 0$ , the buckling load parameter is obtained, from one equation only as

$$\lambda = \frac{N_x b^2}{\pi^2 D} = \left\{ \frac{\left[ \left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2 \right]^2 + (1 - \nu) \left\{ 4.0280 \left[ \left(\frac{h}{a}\right)^2 \frac{m^6}{a^4} + \left(\frac{h}{b}\right)^2 \frac{n^6}{b^4} \right] + \left(\frac{h}{a}\right)^2 \left[ \frac{m^4 n^2}{a^2 b^2} + \frac{m^2 n^4}{b^4} \right] (12.0844) \right\}}{\left(\frac{m}{ab}\right)^2 \left\{ 1 + 2.8198 m^2 \left(\frac{h}{a}\right)^2 + n^2 \left(\frac{h}{b}\right)^2 (2.81984) \right\}^2} \right\}. \quad (21)$$

#### 4. Numerical results and discussion

To show the simplicity, effectiveness and ease of the proposed CDF method, buckling problem of the moderately thick uniform rectangular plates with all edges simply supported (Figure.1 and Figure.2) boundary conditions, for different aspect ratios and different loading conditions are given. Single term trigonometric functions are considered to study the influence of the thickness ratio  $h/b$  on the buckling load parameters. Table 1 shows values of non dimensional buckling loads for moderately thick square plates under uniaxial and bi axial compressive loads for thin plate. In the same table results obtained by the present method is compared with [7] and the match is very good. Table 2, Table 3 shows values of non dimensional buckling loads for moderately thick rectangular plates under uniaxial compressive loads for different aspect ratios. Table 2 also gives buckling load values for higher modes and for fundamental mode for various plate thickness ratios. Table 2 also shows results obtained by the present method is compared with [7] and the results are matching very closely. Table 4 shows values of non dimensional buckling loads for moderately thick rectangular plates under bi axial compressive loads for different aspect ratios. It is in general observed that non dimensional buckling loads are decreasing with increase of plate thickness ratio.

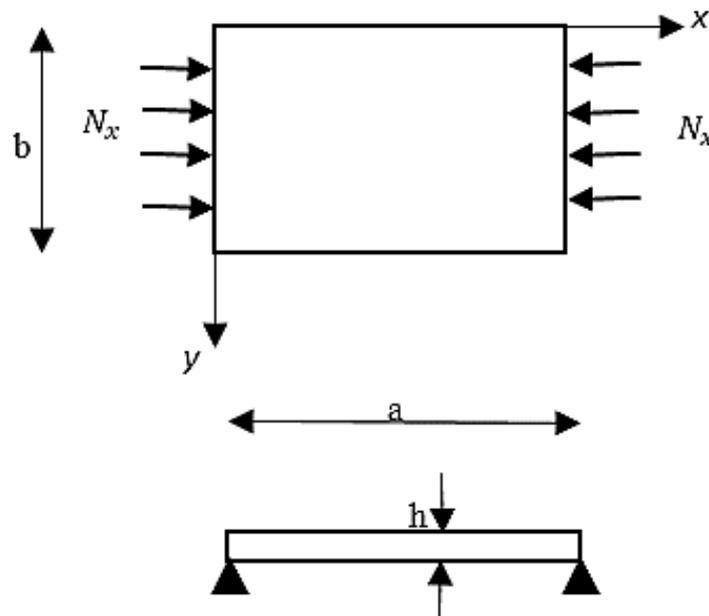
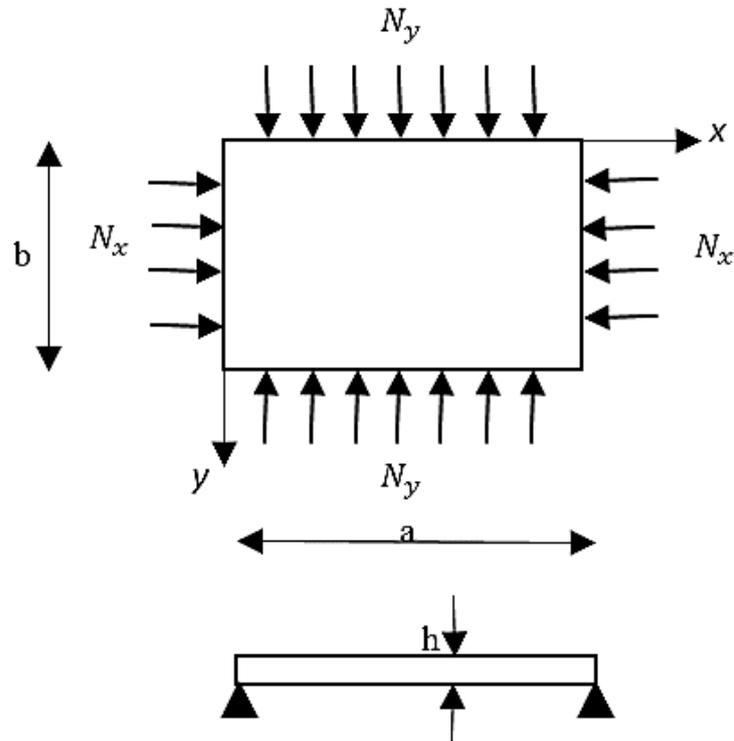


Figure 1. S-S-S-S rectangular plate under uniaxial compression



**Figure 2.** S-S-S-S rectangular plate under biaxial compression

**Table 1.** Values of non dimensional buckling load parameter  $\lambda$  with shear ( $h/b$ ) for all edges simply supported moderately thick square plate

$\gamma$	$a/b$	$h/b$	Present method	Ref[7]
0	1	0	4	4
		0.01	3.9977	3.998
		0.05	3.9443	3.948
		0.1	3.7864	3.8
1	1	0	2	2
		0.01	1.9988	1.9
		0.05	1.972	1.974
		0.1	1.8936	1.999

When  $\gamma = 0$  plate is under uniaxial compression,  $\gamma = 1$  plate is under biaxial compression.

**Table 2.** Values of non dimensional buckling load parameter  $\lambda$  with shear ( $h/b$ ) for all edges simply supported moderately thick rectangular plate under uniaxial compression

a/b	h/b	Present method	Ref[7]
0.5	0	6.25	6.25
	0.1	5.4776	5.523
	0.05	6.0369	6.051
	0.01	6.2413	6.25
1.5	0	4.3400 (2,1)	4.34
	0.1 (2,1)	4.0824	4.045
	0.05 (2,1)	4.2734	4.262
	0.01 (2,1)	4.3375	4.337
2	0	4.0000 (2,1)	4
2.5	0	4.1344 (3,1)	4.134
3	0	4.0000 (3,1)	4
	0.1 (3,1)	3.786	3.8
	0.05 (3,1)	3.9446	3.948
	0.01 (3,1)	3.9977	3.998

Numbers in the bracket( $m, n$ ) refers at which critical buckling load occurred; ( $m, n$ )=(1,1) for all other cases.

**Table 3.** Values of non dimensional buckling load parameter  $\lambda$  with shear ( $h/b$ ) for all edges simply supported moderately thick rectangular plate under uniaxial compression

a/b	h/b	Present method	Ref[10]		Ref[11 ]
			FSDT	HSDT	
0.2	0.5	1.3988	1.3988	1.6851	1.6851
	0.2	6.8755	6.8753	7.0529	7.0529
	0.1	15.6017	15.601	15.658	15.658
	0.05	22.8524	22.851	22.859	22.859
	0.02	26.2698	26.269	26.27	26.27
	0.01	26.8435	26.843	26.843	26.84
0.4	0.5	1.3761	1.3761	1.4455	1.4455
	0.2	4.6264	4.6264	4.6466	4.6466
	0.1	6.9826	6.9824	6.9853	6.9853
	0.05	8.0011	8.001	8.0012	8.0012
	0.02	8.3417	8.3417	8.3417	8.3417
	0.01	8.3934	8.3928	8.3928	8.3928
1	0.5	1.6598	1.6597	1.6759	1.6759
	0.2	3.2633	3.2636	3.2653	3.2653
	0.1	3.7864	3.7864	3.7865	3.7965
	0.05	3.9443	3.9443	3.9443	3.9443
	0.02	3.9909	3.9909	3.9909	3.9909
	0.01	3.9977	3.9977	3.9977	3.9977

**Table 4.** Values of non dimensional buckling load parameter  $\lambda$  with shear ( $h/b$ ) for all edges simply supported moderately thick rectangular plate under biaxial compression

a/b	h/b	Present method	Ref[7]
0.5	0.1	4.3821	4.418
	0.05	4.8301	4.841
	0.01	4.993	4.993
	0	5	5
1.5	0.1	1.3878	1.391
	0.05	1.4301	1.431
	0.01	1.4437	1.444
	0	1.4444	1.444
3	0.1	1.0772	1.079
	0.05	1.1025	1.103
	0.01	1.1111	1.111
	0	1.1111	1.111

## 5. Conclusions

In this paper, a simple and effective CDF method compared with other researchers is proposed to study the buckling of moderately thick rectangular plates. This method because of the use of the static equilibrium equations to couple the total rotations and the transverse displacement, becomes an  $n$  undetermined coefficient system when compared to the classical RR method[12] where  $2n$  undetermined coefficients are present, thus reducing the computational effort significantly. As a special case, if  $n=1$ , which means when a single term admissible function is used, the CDF method contains only one undetermined coefficient which simplifies the solution procedure as has been demonstrated. Numerical results, in terms of the buckling load parameters, for several thickness ratios are presented in this paper for the all edges simply supported boundary condition of the moderately thick rectangular plates show the simplicity of the CDF method when compared to the classical RR method. The results obtained by the other researchers are also compared with the present method wherever possible and are matching very closely.

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